Batyrev-Manin conjecture for K3 surfaces

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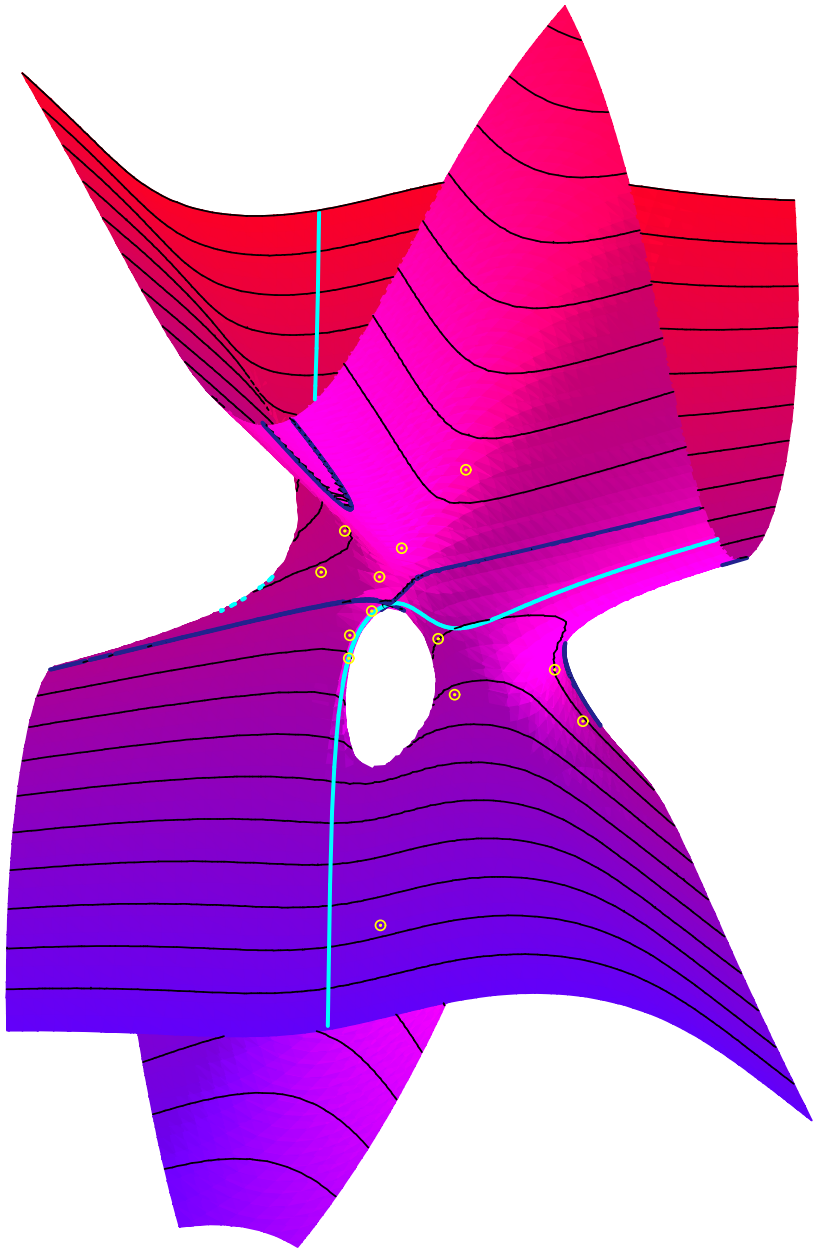
Banff



A K3 surface of
geometric Picard number 1
with two singular
(accumulating?)
curves of genus 0 and
several rational points

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Answer (Elsenhans, Jahnel, 2004):

$$1484801^4 + 2 \cdot 1203120^4 \;=\; 1169407^4 + 4 \cdot 1157520^4$$

Definition:

Let X be a K3 surface over a number field K.

Rational points on X are *potentially (Zariski) dense*,

if there is a finite field extension K'/K such that $X(K')$ is dense in X.

Theorem 1 (F. Bogomolov – Y. Tschinkel)**.** *Let X be a K3 surface over a number field. If either*

(a) $\rho(\overline{X}) = 2$ and $\overline{X}$ does not contain a (-2)-curve, or

(b) $\rho(\overline{X}) \geq 3$ (except 8 isom. classes of $\operatorname{Pic}\overline{X}$ as abstr. lattices),

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Theorem 2 (vL). *The quartic surface S in $\mathbb{P}^3(x,y,z,w)$ given by*

$$wf = 3pq - 2zg$$

with $f \in \mathbb{Z}[x,y,z,w]$ and $g,p,q \in \mathbb{Z}[x,y,z]$ equal to

$$\begin{aligned} f =& x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 \\ &+ y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \\ g =& xy^2 + xyz - xz^2 - yz^2 + z^3, \\ p =& z^2 + xy + yz, \\ q =& z^2 + xy \end{aligned}$$

has geometric Picard number 1.

Proof: (Bounding the Picard number from above)

Let X be a (smooth, projective, geometrically integral) surface over $\mathbb{Q}$.
Let $\mathcal{X}$ be an integral model of X over $\mathbb{Z}_{(p)}$ with good reduction at p.

From étale cohomology we get injections

$$\mathsf{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_l \hookrightarrow \mathsf{NS}(\mathcal{X}_{\overline{\mathbb{F}}_p}) \otimes \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1)).$$

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Corollary 3. *The rank $\rho(X_{\overline{\mathbb{Q}}})$ is at most the number of eigenvalues λ of Frobenius acting on $H^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$ for which λ is a root of unity.*

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This only allows us to find **even** upper bounds for $\rho(\overline{X})$...

Proof: (Lattice theory)

A *lattice* is a free $\mathbb{Z}$-module Λ of finite rank, together with a symmetric nondegenerate bilinear pairing $\Lambda \times \Lambda \to \mathbb{Q}$. A *sublattice* of Λ is a submodule Λ' of Λ such that the induced bilinear pairing on Λ' is nondegenerate.

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Lemma 4. *If Λ' is a sublattice of finite index of Λ, then we have*

$$\operatorname{disc} \Lambda' = [\Lambda : \Lambda']^2 \operatorname{disc} \Lambda.$$

This implies that $\operatorname{disc} \Lambda$ and $\operatorname{disc} \Lambda'$ have the same image in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

Proof: (Sketch)

We find finite-index sublattices M_2 and M_3 of $\mathrm{NS}(S_{\overline{\mathbb{F}}_2})$ and $\mathrm{NS}(S_{\overline{\mathbb{F}}_3})$. Both have rank 2, so $\operatorname{rk}\mathrm{NS}(S_{\overline{\mathbb{Q}}}) \leq 2$.

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The images of $\operatorname{disc} M_2$ and $\operatorname{disc} M_3$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ differ, so $\operatorname{rk}\mathrm{NS}(S_{\overline{\mathbb{Q}}}) \leq 1$.

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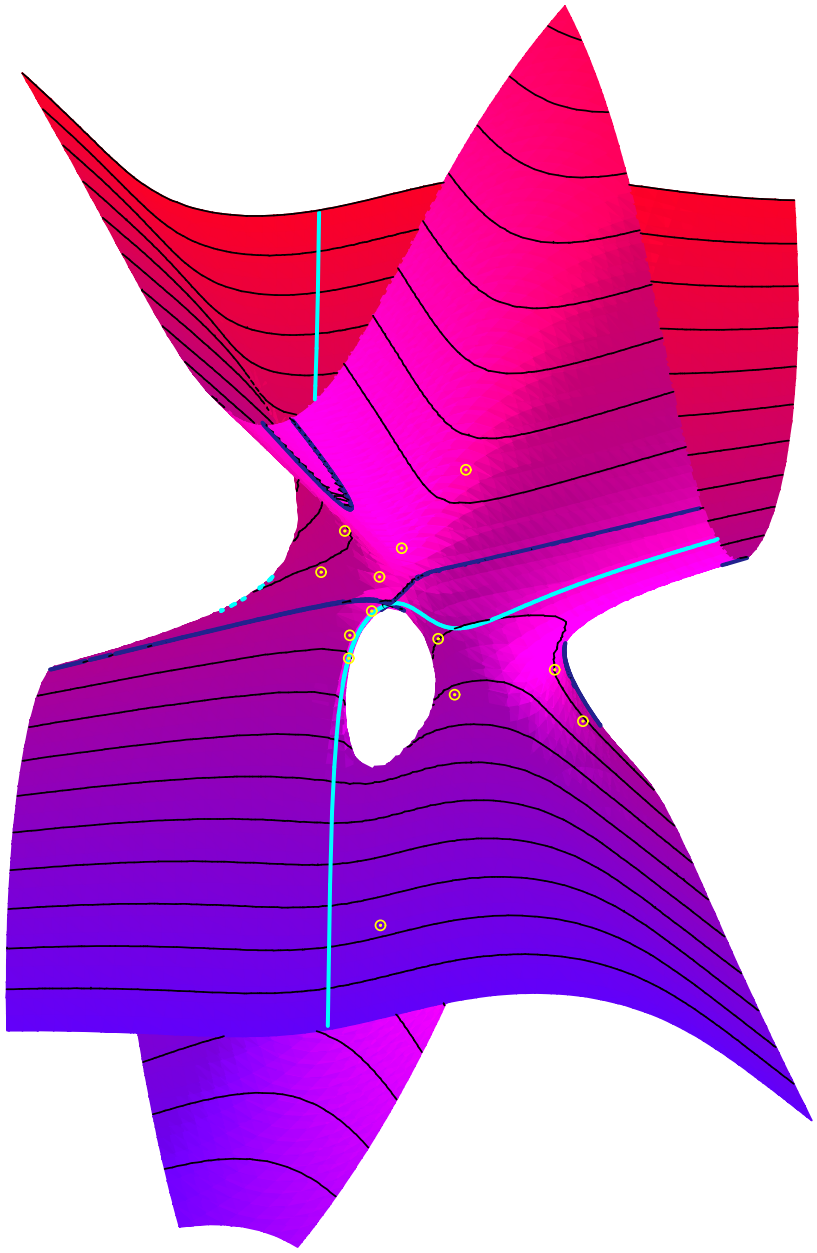
Indeed, recall $S : wf = 3pq - 2zg$
The reduction mod 2 contains the conic $w = p = 0$ ($\operatorname{disc} M_2 = -12$).
The reduction mod 3 contains the line $w = z = 0$ ($\operatorname{disc} M_3 = -9$).



The K3 surface S

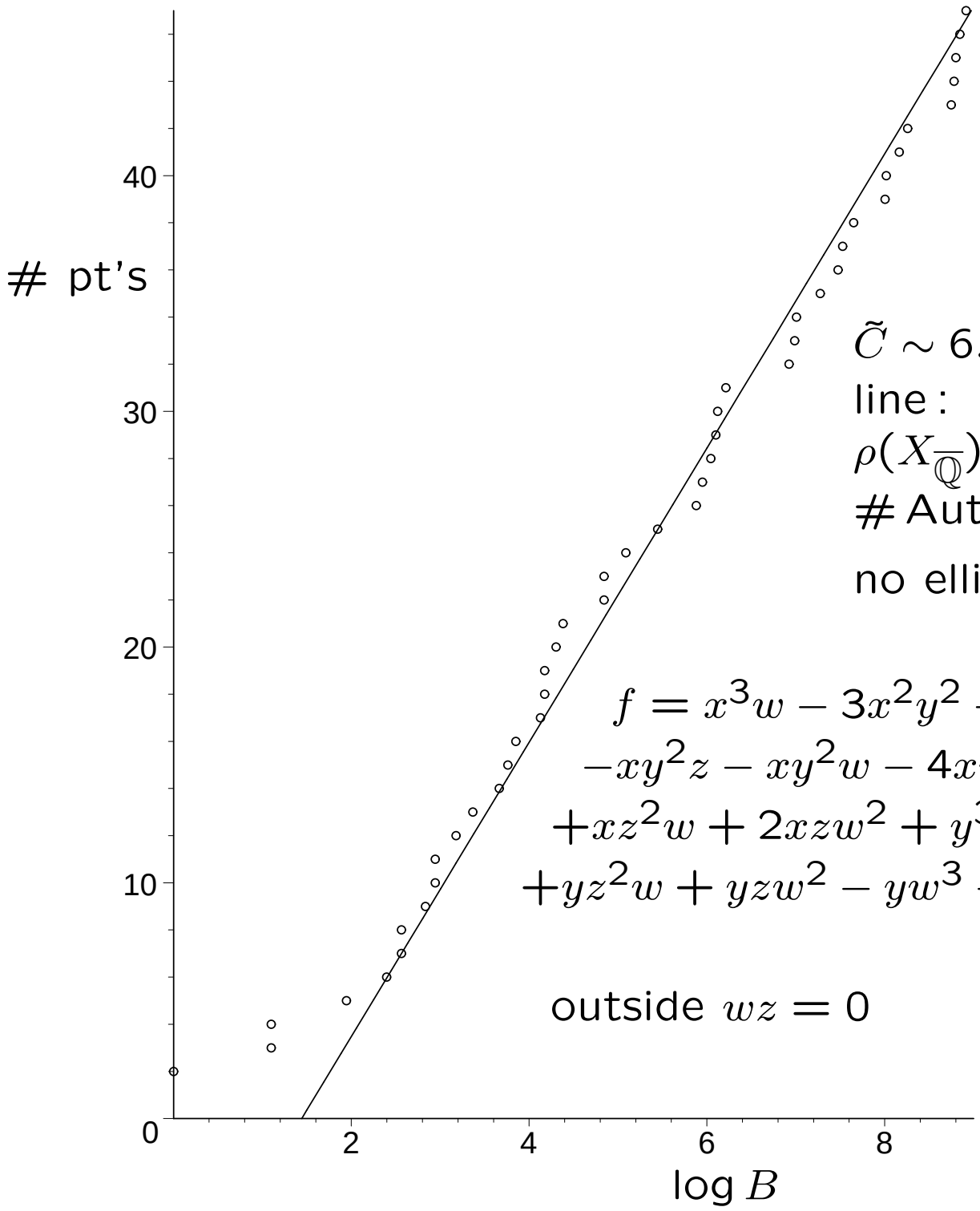
with 2 singular curves
of genus 0 and
several rational points.

There are 47 rational
points outside these curves
whose coordinates have
absolute value at most 10^4.

Look at the growth of the number of rational points of bounded height. Consider a surface X/K, choose a height H, and set

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) \ : \ H(x) \leq B\}$$

for any open subset U of X.



$\tilde{C} \sim 6.24$

line : $\tilde{C}(\log B) - 9$

$\rho(X_{\overline{\mathbb{Q}}}) = 1$

$\#\,\mathrm{Aut}\, X_{\overline{\mathbb{Q}}} < \infty$

no elliptic fibration

$$f = x^3w - 3x^2y^2 - x^2yw - x^2zw + x^2w^2$$
$$-xy^2z - xy^2w - 4xyz^2 - xyzw + 2xyw^2 - 2xz^3$$
$$+xz^2w + 2xzw^2 + y^3w + y^2zw - y^2w^2 - 5yz^3$$
$$+yz^2w + yzw^2 - yw^3 - z^4 + z^2w^2 + zw^3 + 2w^4$$

outside $wz = 0$

Heuristics

Let $X \subset \mathbb{P}^3_{\mathbb{Q}}$ be a smooth surface of degree d, given by $f = 0$.

There are $\sim \zeta(4)(2B+1)^4$ points of height at most B in $\mathbb{P}^3$,
so $\sim C_1 B^4$ points of height in $[B, 2B]$.

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We expect $\sim C_3 B^{4-d}$ points P on X with $H(P) \in [B, 2B]$ and

$$\sim \sum_{i=0}^{n} (2^i)^{4-d} \sim \begin{cases} (2^n)^{4-d} \text{ if } d < 4 \\ n \text{ if } d = 4 \\ C \text{ if } d > 4 \end{cases}$$

points P on X with $H(P) \leq 2^n$ for $n >> 0$.

Conjecture 1 (Batyrev, Manin). *Let X be a smooth, geometrically integral, projective variety over a number field K, and let D be an ample divisor. Assume that the canonical sheaf K_X satisfies $-K_X = aD$ for some $a > 0$. Then there exists a finite field extension L, a constant C, and an open subset $U \subset X$, such that with $b = \mathrm{rk}\,\mathrm{Pic}(X_L)$ we have*

$$N_{U_L,H}(B) \;\approx\; CB^a(\log B)^{b-1},$$

where H is the height associated to D.

(Both sides 0 **OR** neither for $B >> 0$ and quotient $\to 1$ as $B \to \infty$)

Example: del Pezzo surfaces (including cubics in $\mathbb{P}^3$) have $a = 1$.

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What if $a = 0$, in particular, if X is K3?
For generic K3's we have $\mathrm{rk}\,\mathrm{Pic}\,\overline{X} = 1$, so $N_U(B) \approx C$????

Conjecture 2 (Batyrev, Manin)**.** *Let X be a K3 surface defined over a number field K, and let D be an ample divisor. For all $\epsilon > 0$, we let $V = V(\epsilon, D)$ be the union of all K-rational curves on X of D-degree at most $2\epsilon^{-1}$. Then we have*

$$N_{X,H}(B) \;=\; N_{V,H}(B) + \mathcal{O}(B^{\epsilon}).$$

This means the K-rational curves are accumulating, as a rational curve of degree d has $\sim B^{2/d}$ rational points of height $< B$.

Theorem 5 (McKinnon)**.** *Let X be a K3 surface over a number field K that is the desingularisation of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over a singular $(4,4)$-curve. Assume that the exceptional curves E_i are irreducible. Let D be an ample divisor that is a $\mathbb{Q}$-linear combination of the E_i and the fibers F_1, F_2 of the maps $X \to \mathbb{P}^1$ and α an explicit expression in the coefficients. Set $L_{ij} = F_i - E_j$. If there is a curve E_i or component of L_{ij} with D-degree at most α^{-1}, then the first layer of the arithmetic stratification of X with respect to D is the union of all E_i and L_{ij}.*

Define the height-zeta function

$$Z(U,s) = \sum_{P \in U(K)} H(P)^{-s}.$$

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From

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s} = \begin{cases} 1 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases} \qquad (c > 0)$$

we get

$$\begin{aligned} N_U(x) &= \sum_{P \in U(K)} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(xH(P)^{-1}\right)^s \frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(U,s)\, x^s \frac{ds}{s} \qquad (c >> 0) \end{aligned}$$

Assuming $Z(U,s)$ is **nice**, including analytic on $\Re(s) > a - 2\epsilon$, except for a pole of order b at a, we can write

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The integral is smaller than the residue, the main term, which is

$$x^a p(\log x)$$

for some polynomial p of degree $\begin{cases} b-1 & \text{if } a \neq 0, \\ b & \text{if } a = 0. \end{cases}$

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More problems:

- infinitely many curves of genus 0 or 1,
- infinitely many automorphisms,
- Swinnerton-Dyer's surface has very slow growth,
- which height to use?

We ran experiments on quartic surfaces $X/\mathbb{Q}$ with regular differential ω, comparing exponents, and comparing the main coefficient C to the relatively naive heuristic coefficient

$$\tilde{C} = \int_{X(\mathbb{R})} \omega \times \prod_p (1 - p^{-1}) \int_{X(\mathbb{Q}_p)} \omega$$

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May scale ω so that for primes p of good reduction we just have $\int_{X(\mathbb{Q}_p)} \omega = \#X(\mathbb{F}_p)/p^2$.

For primes p of bad reduction we have $\int_{X(\mathbb{Q}_p)} \omega = \lim_{n\to\infty} \#X(\mathbb{Z}/p^n\mathbb{Z})/p^{2n}$ (depends on model of X, but so does $\int_{X(\mathbb{R})} \omega$; dependency cancels out).

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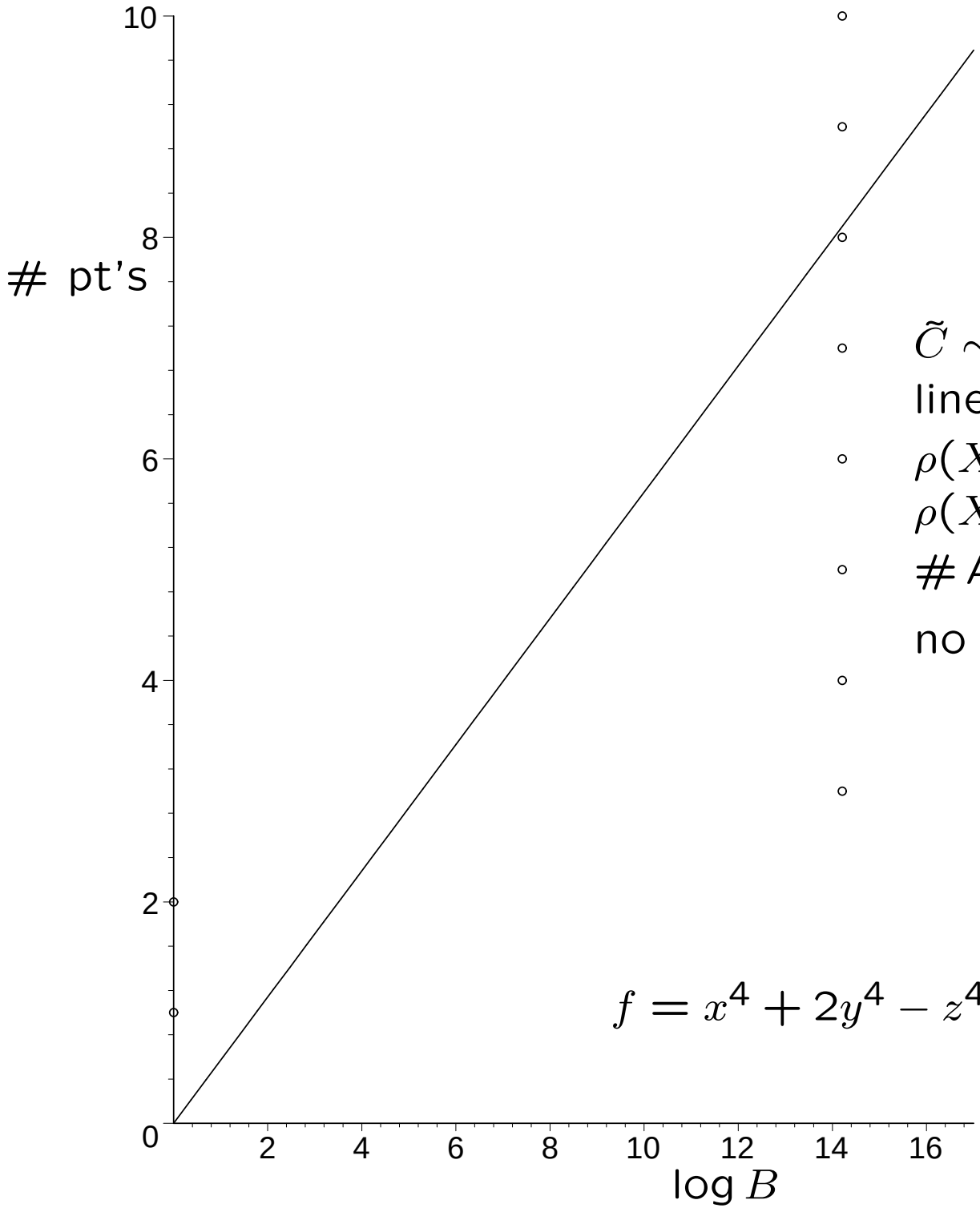
There are some subtle rational numbers that C and $\tilde{C}$ may be off by (for instance related to weak approximation).



$\tilde{C} \sim 0.57$

line : $\tilde{C}(\log B)$

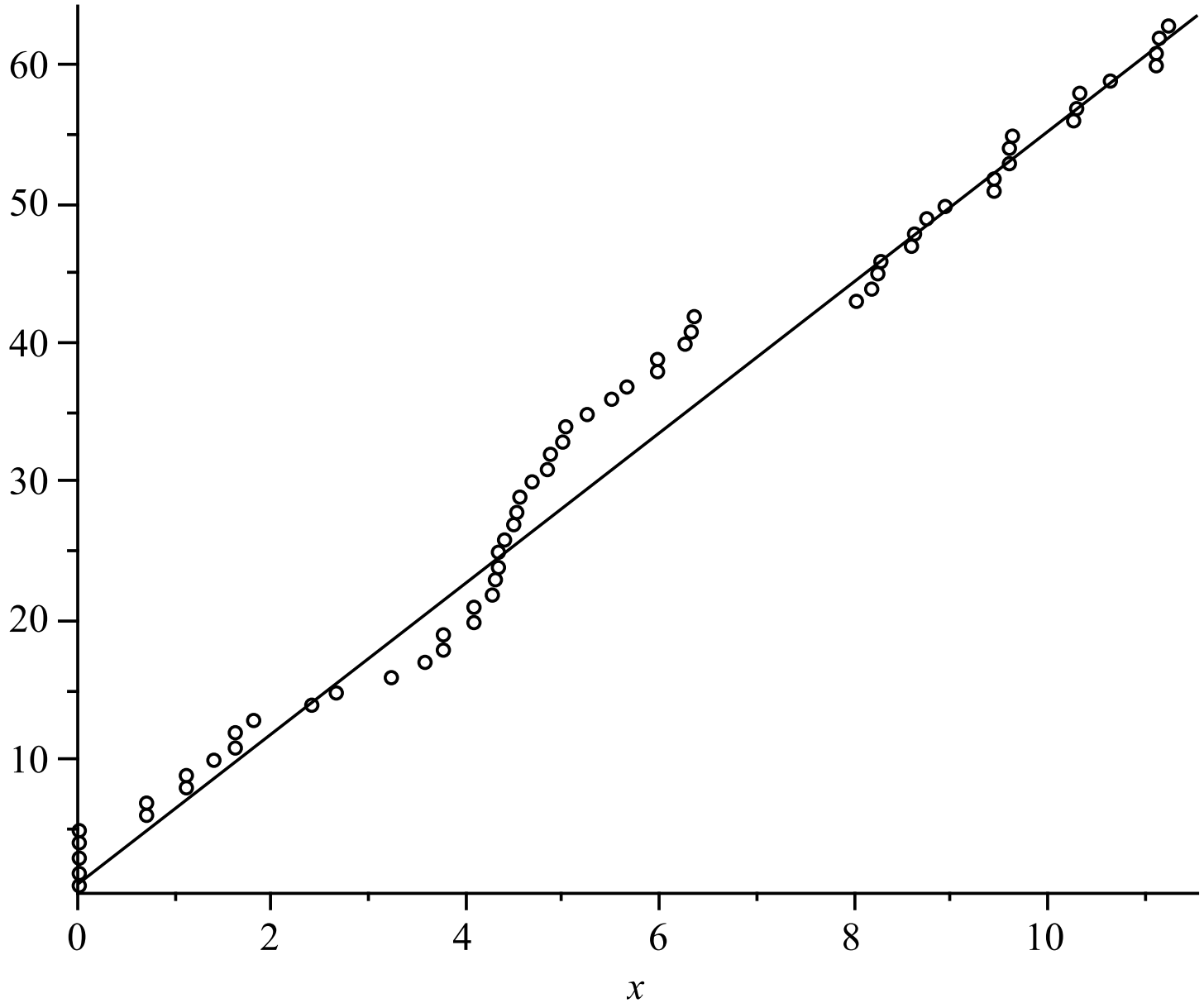
$\rho(X) = 1$

$\rho(X_{\overline{\mathbb{Q}}}) = 20$

$\#\,\mathrm{Aut}\, X_{\overline{\mathbb{Q}}} = \infty$

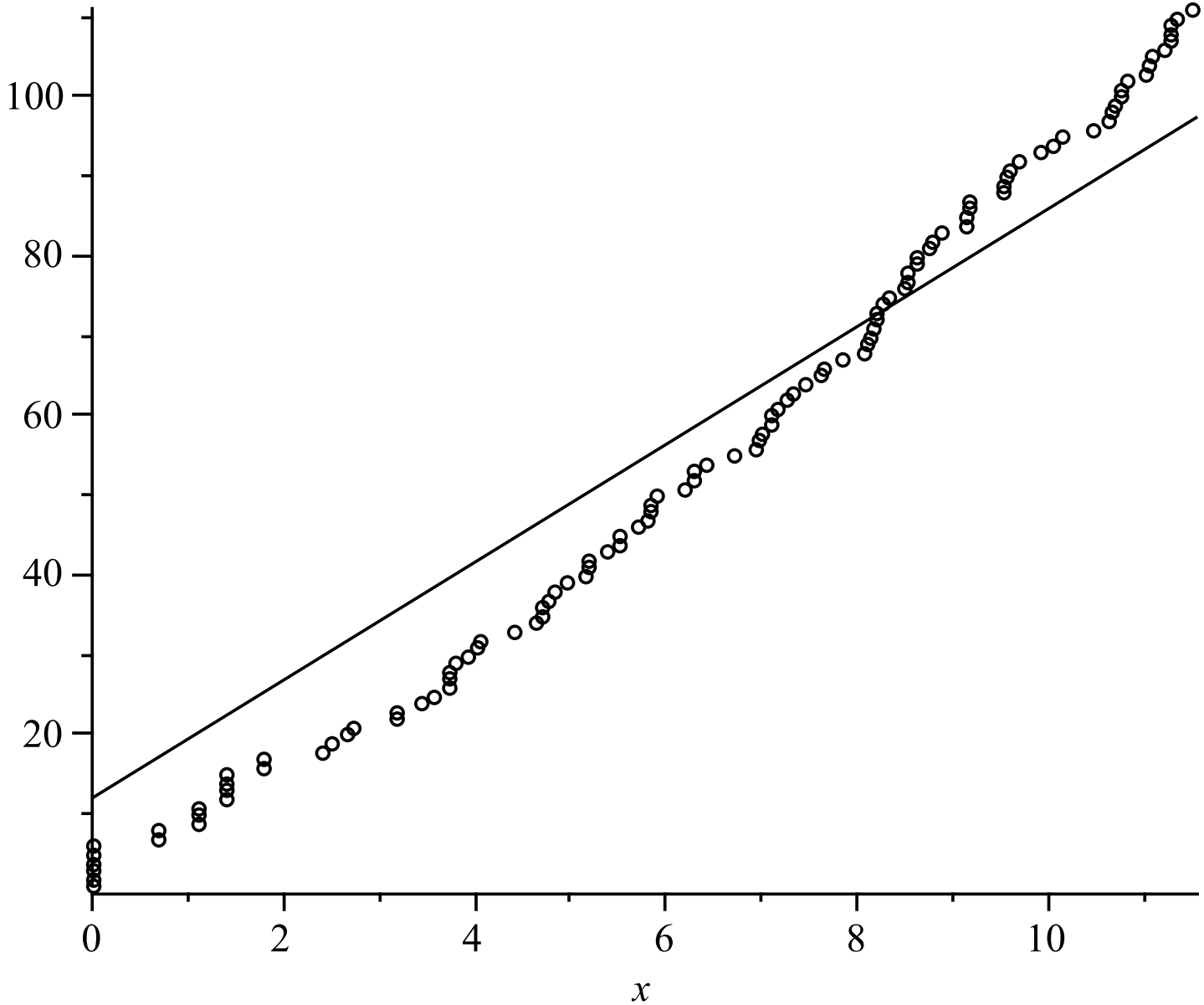
no elliptic fibration over $\mathbb{Q}$

$f = x^4 + 2y^4 - z^4 - 4w^4$



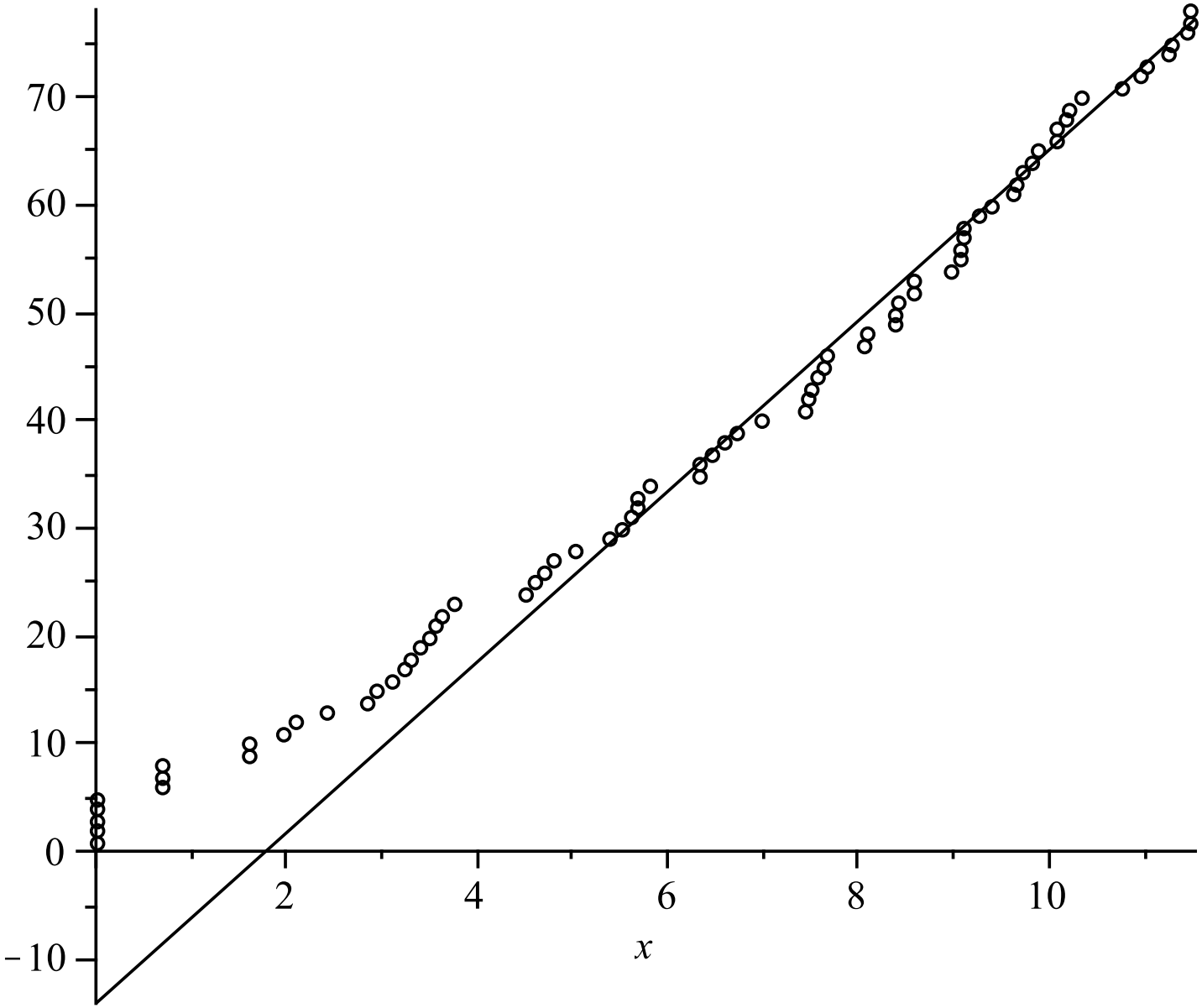
1*5.42*x+1

6x^3y+x^3w+3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+3x^2zw-xy^2z+2xy^2
xyz^2+2xyzw-2xyw^2-2xz^3+2xzw^2+xw^3+y^3w-
y^2z^2+2y^2zw-2yz^2w-2yzw^2+z^3w+2z^2w^2-2zw^3



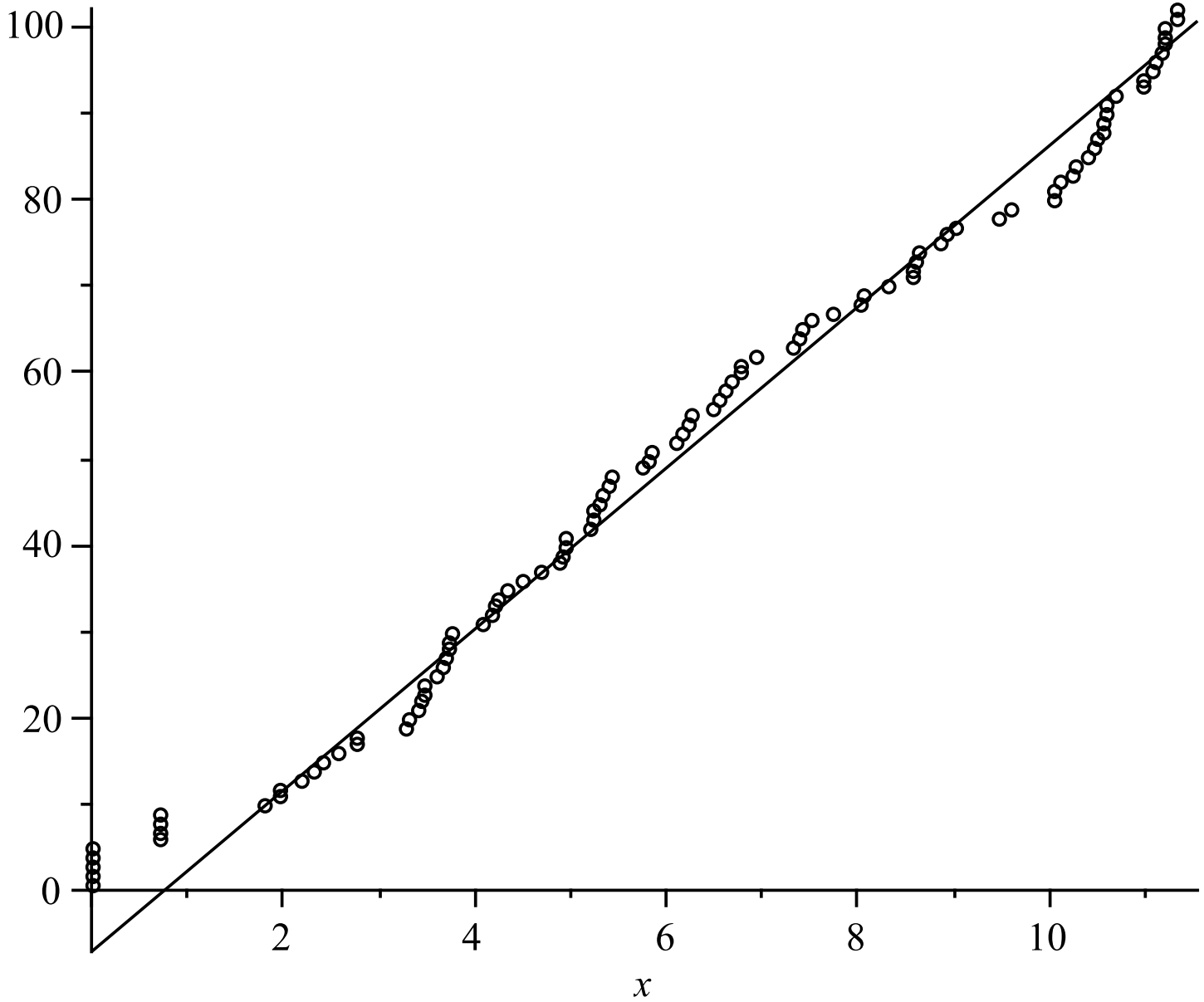
1*7.41*x+12

6x^3y+x^3w+3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+x^2zw-2x^2w^2-
xy^2z+2xy^2w+xyz^2+2xyzw-2xyw^2-2xz^3-2xz^2w+2xzw^2+3xw^
y^3w-y^2z^2-2y^2zw+2y^2w^2-z^3w+2zw^3-2w^4



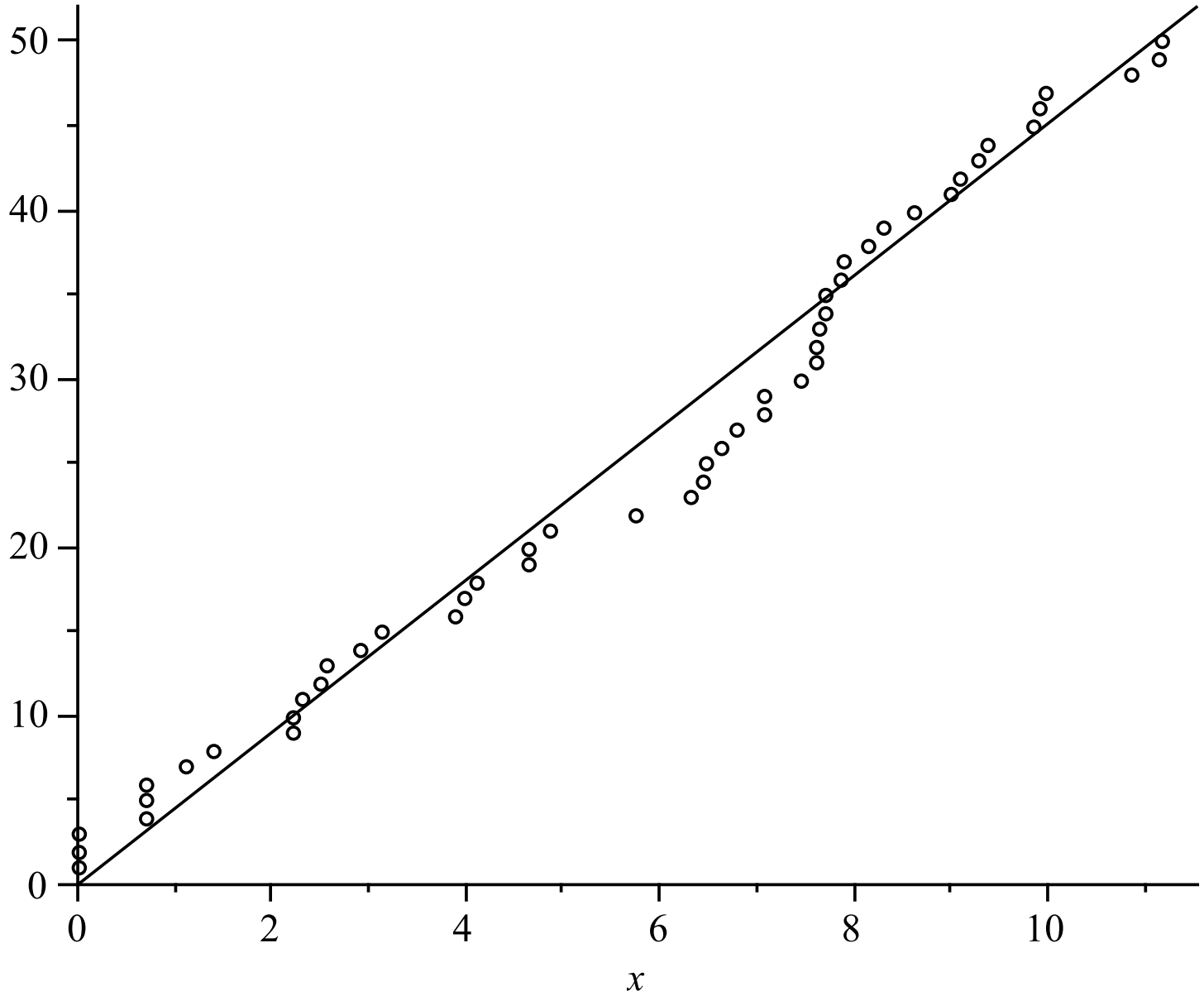
1*7.90*x-14

6x^3y+x^3w+3x^2y^2-2x^2yz+2x^2yw-x^2z^2-x^2zw+2x^2w^2+3xy^
xyz^2+2xyw^2-2xz^2w+3xw^3+2y^3z+y^3w+y^2z^2+2y^2zw-2yzw^
-2yw^3+z^3w-2z^2w^2-2zw^3+2w^4



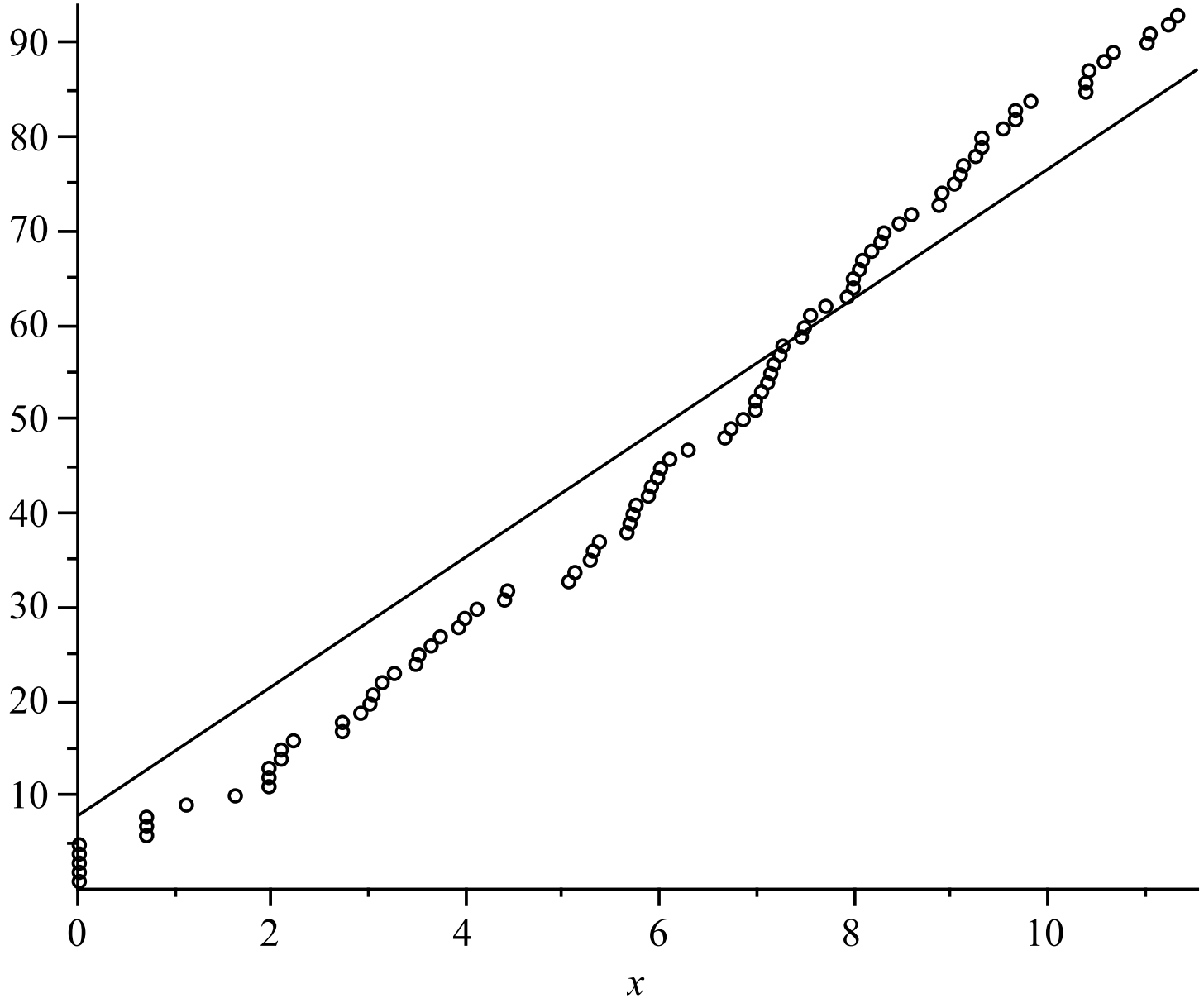
1*9.32*x-7

6x^3y+x^3w+3x^2y^2-2x^2yz+2x^2yw-x^2z^2-x^2zw+2x^2w^2+3xy^
xyz^2-2xyzw-2xyw^2+2xzw^2-xw^3+2y^3z+y^3w+
y^2z^2+2y^2zw+2yz^2w+2yzw^2-2yw^3+z^3w+2z^2w^2



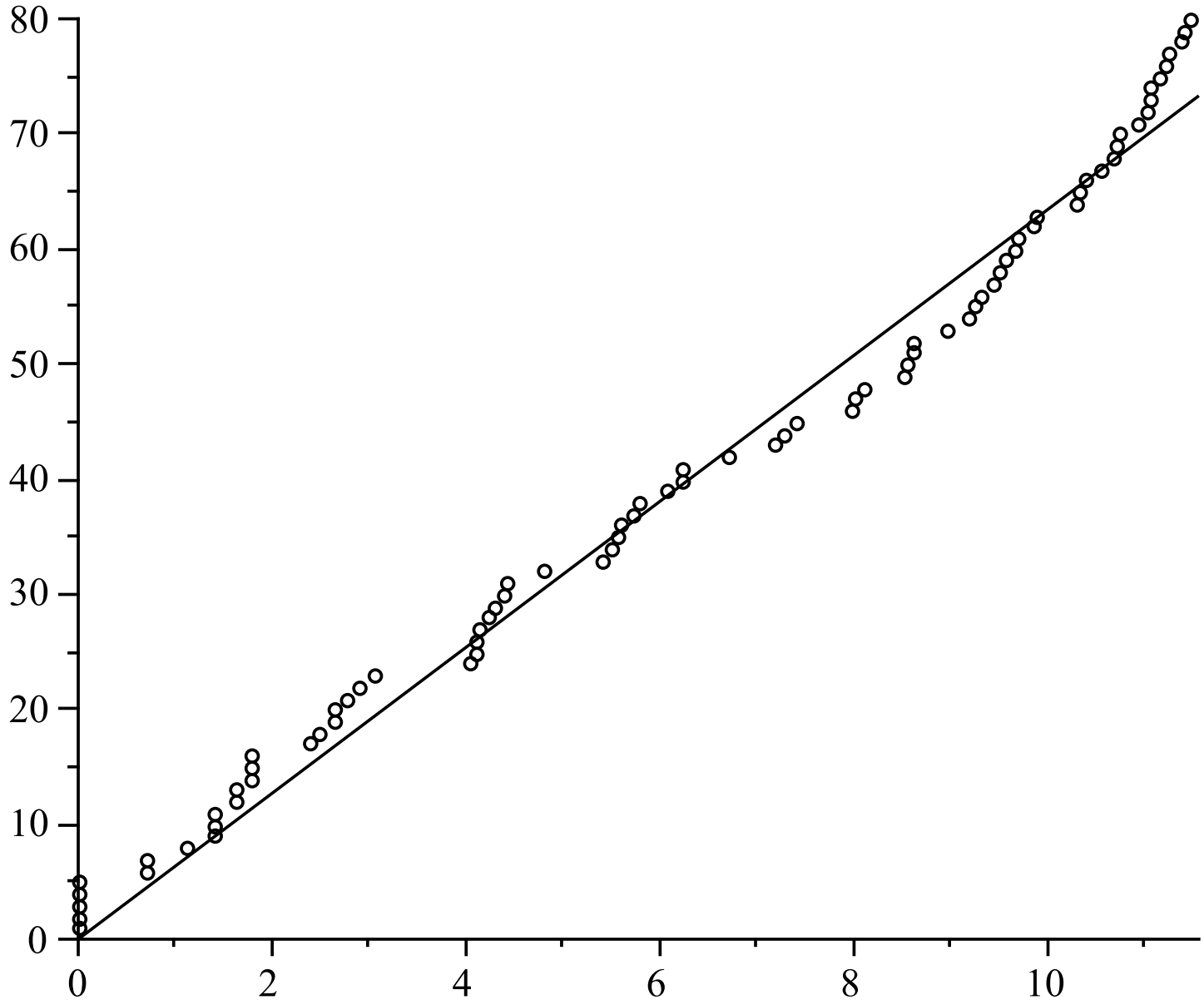
1*4.51*x

-6x^3y+x^3w+3x^2y^2-2x^2yz+2x^2yw+x^2z^2-x^2zw+xy^2z+2xy^2
xyz^2+2xyw^2-2xz^3-2xz^2w-2xzw^2+xw^3-y^3w-y^2z^2-2y^2w^2+2y
-2yz^2w-2yzw^2+2z^4+z^3w



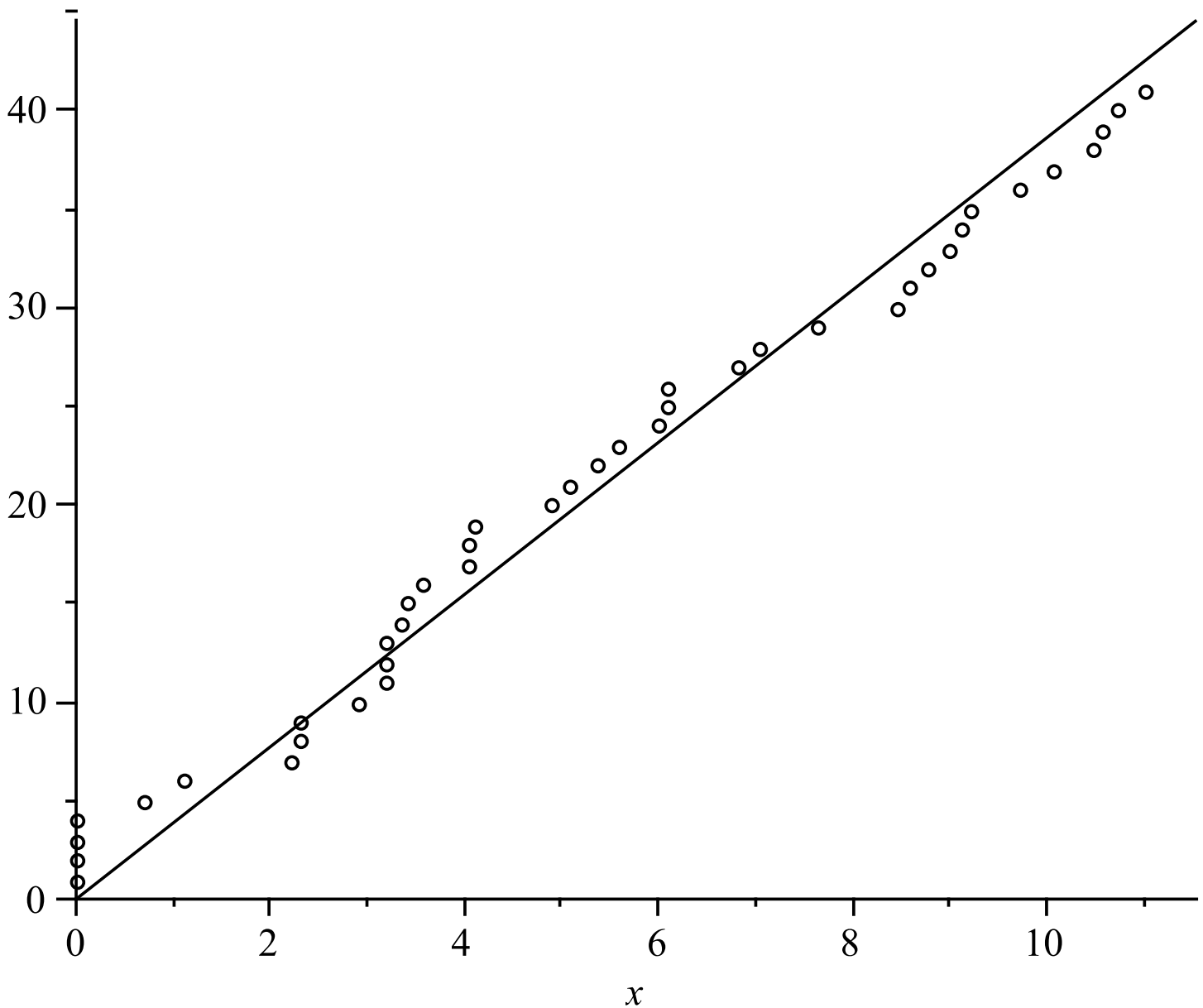
1*6.85*x+8

-6x^3y+x^3w+3x^2y^2-2x^2yz+3x^2z^2+3x^2zw-xy^2z+xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^2w+2yzw^2+
z^3w+2z^2w^2



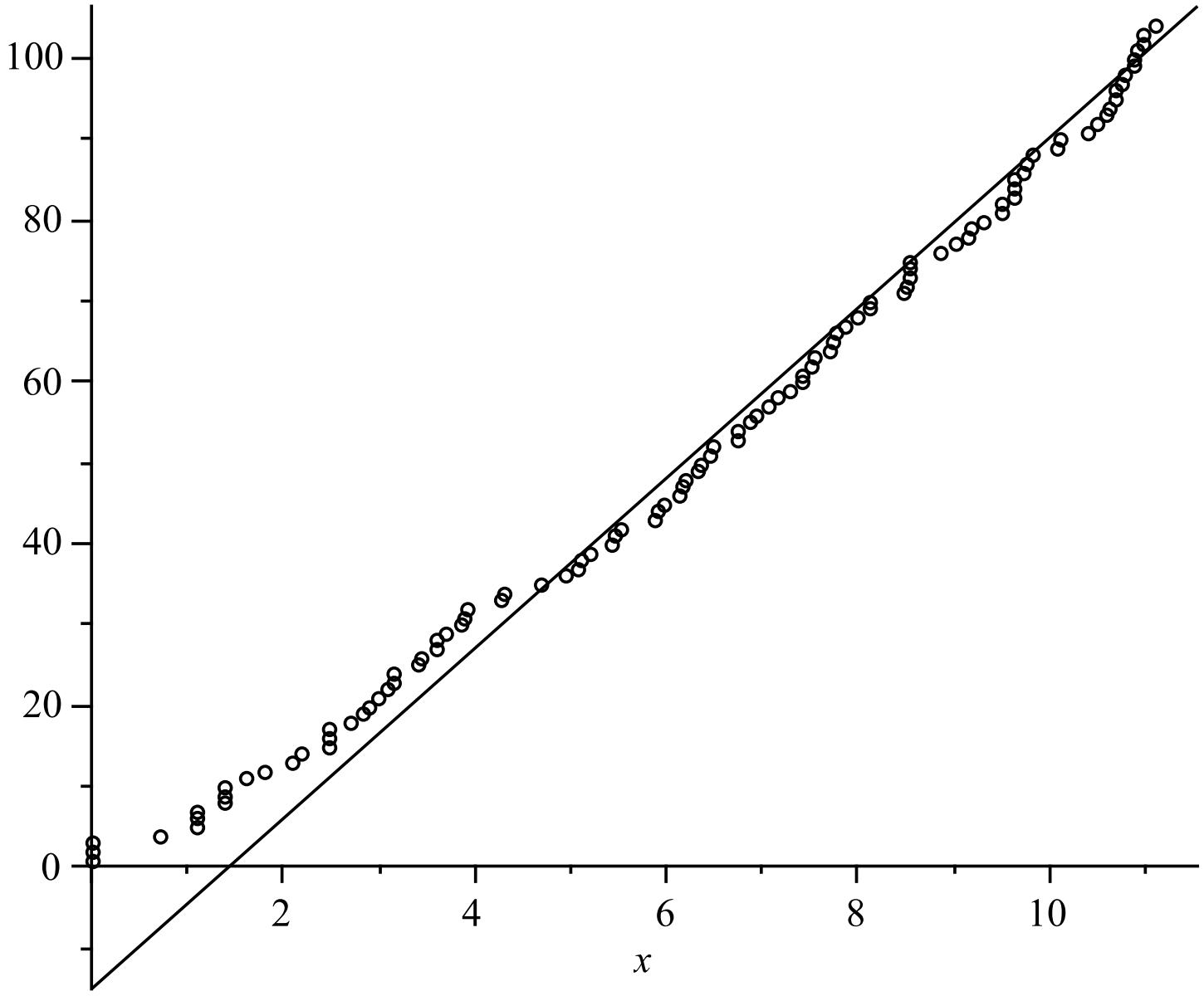
1*6.35*x

$$6x^3y+x^3w+3x^2y^2-2x^2yz+3x^2z^2+3x^2zw-xy^2z+xyz^2-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^2w+2yzw^2+z^3w+2z^2w^2$$



1*3.86*x

-6x^3y+x^3w+3x^2y^2-2x^2yz+3x^2z^2-x^2zw-xy^2z+xyz^2
-2xyzw-2xz^3+2xz^2w+2xzw^2+xw^3+y^3w-y^2z^2-2yz^2w-
z^3w+2z^2w^2+2zw^3-2w^4

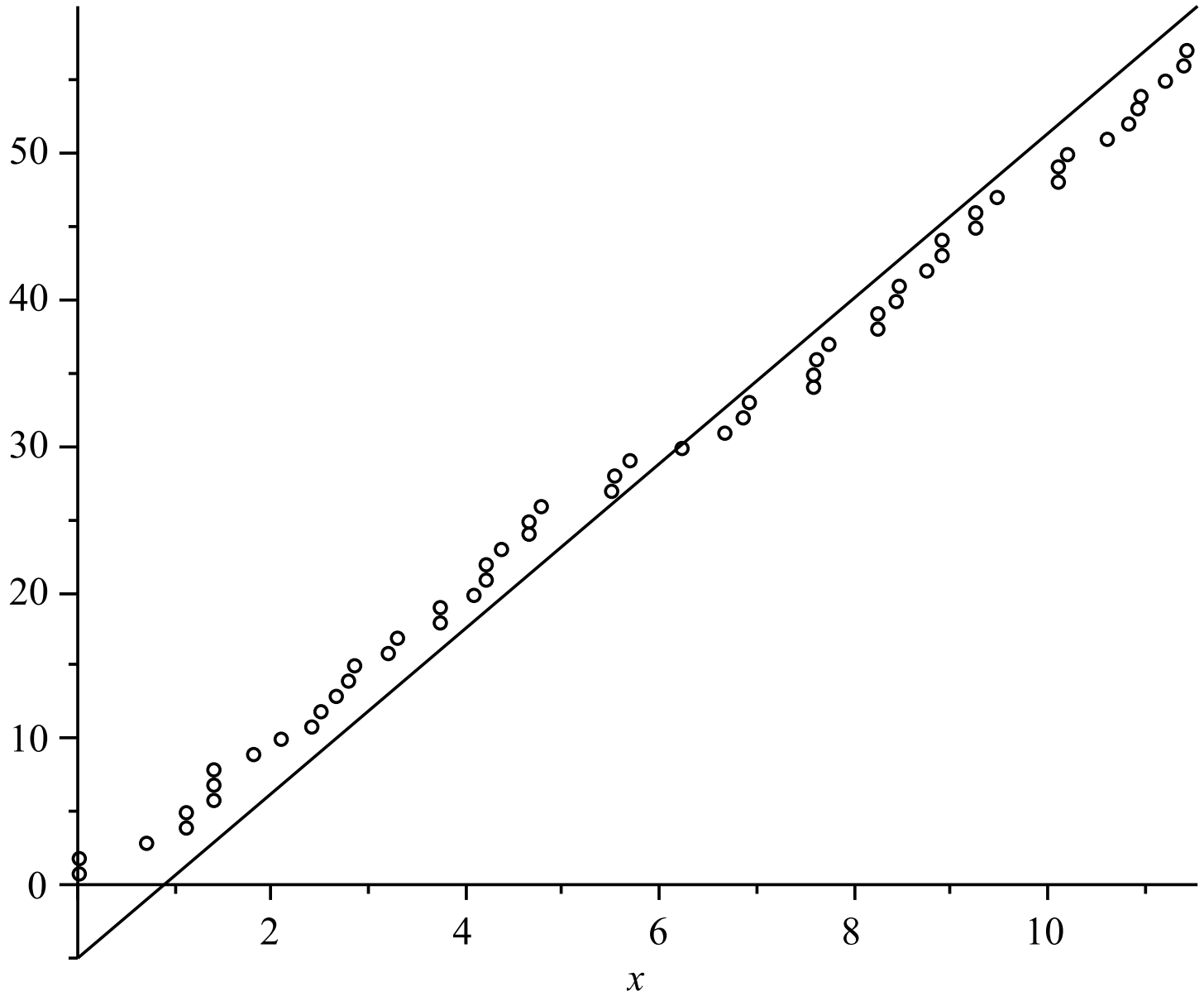


1*10.5*x-15

-6x^3y+x^3w+3x^2y^2-2x^2yz+3x^2z^2+x^2zw-xy^2z+xyz^2-2xz^

-2xz^2w+xw^3+y^3w-y^2z^2-z^3w+2zw^3

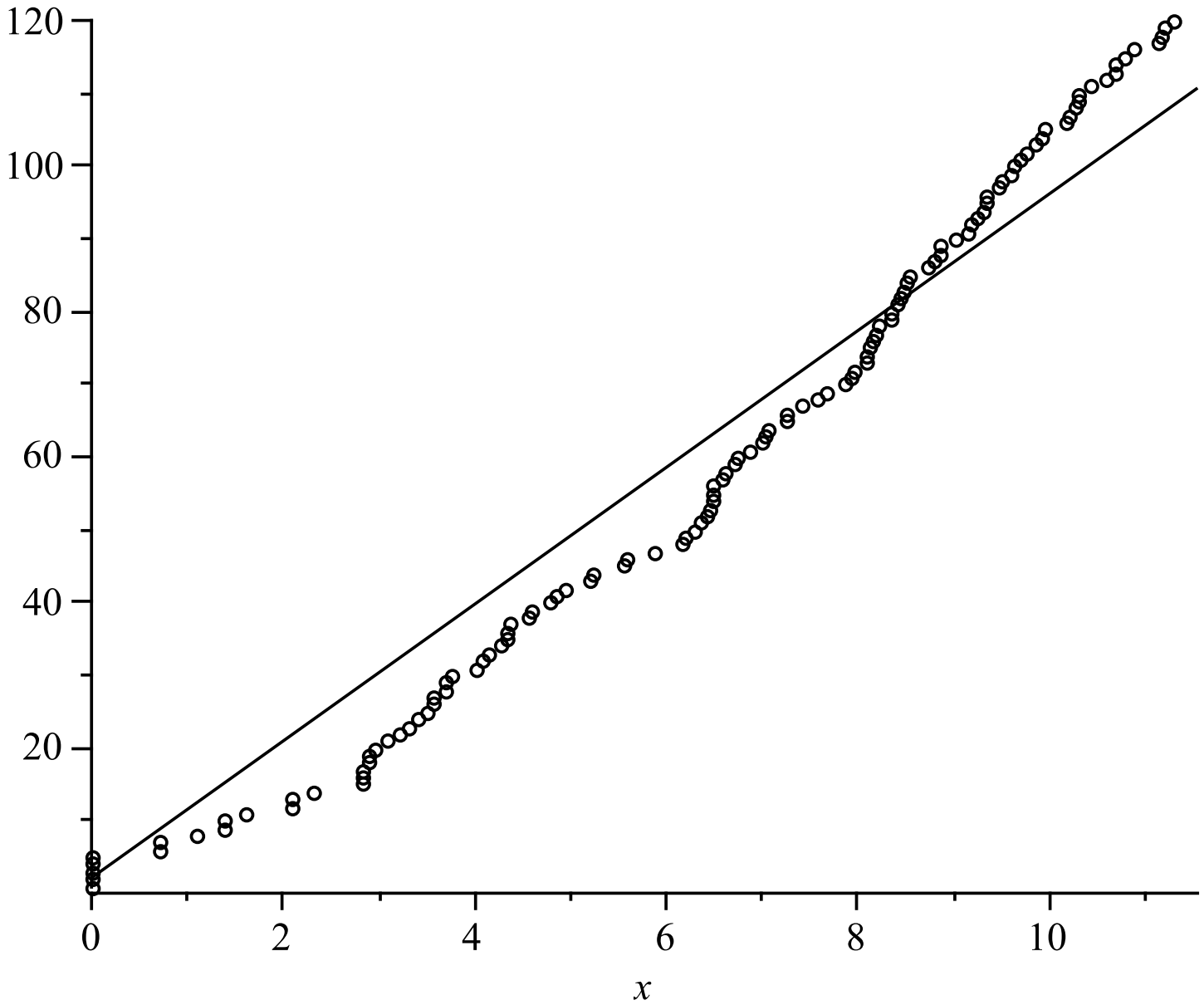
x=/=-2z



1*5.62*x-5

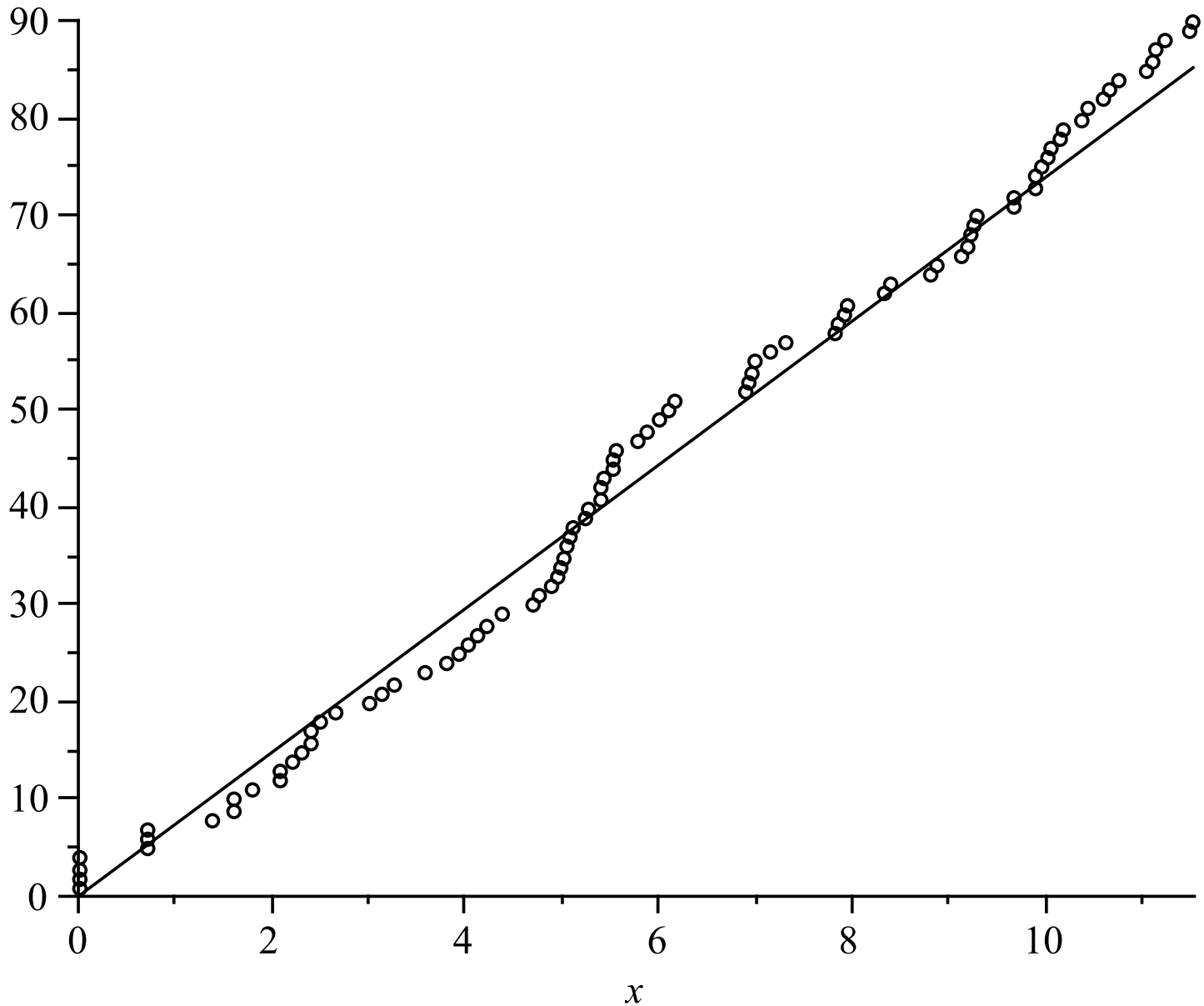
6x^3y+x^3w+3x^2y^2-2x^2yz+3x^2z^2+x^2zw-xy^2z+xyz^2-2xz^3
-2xz^2w+xw^3+y^3w-y^2z^2-z^3w+2zw^3

x =/= -2z



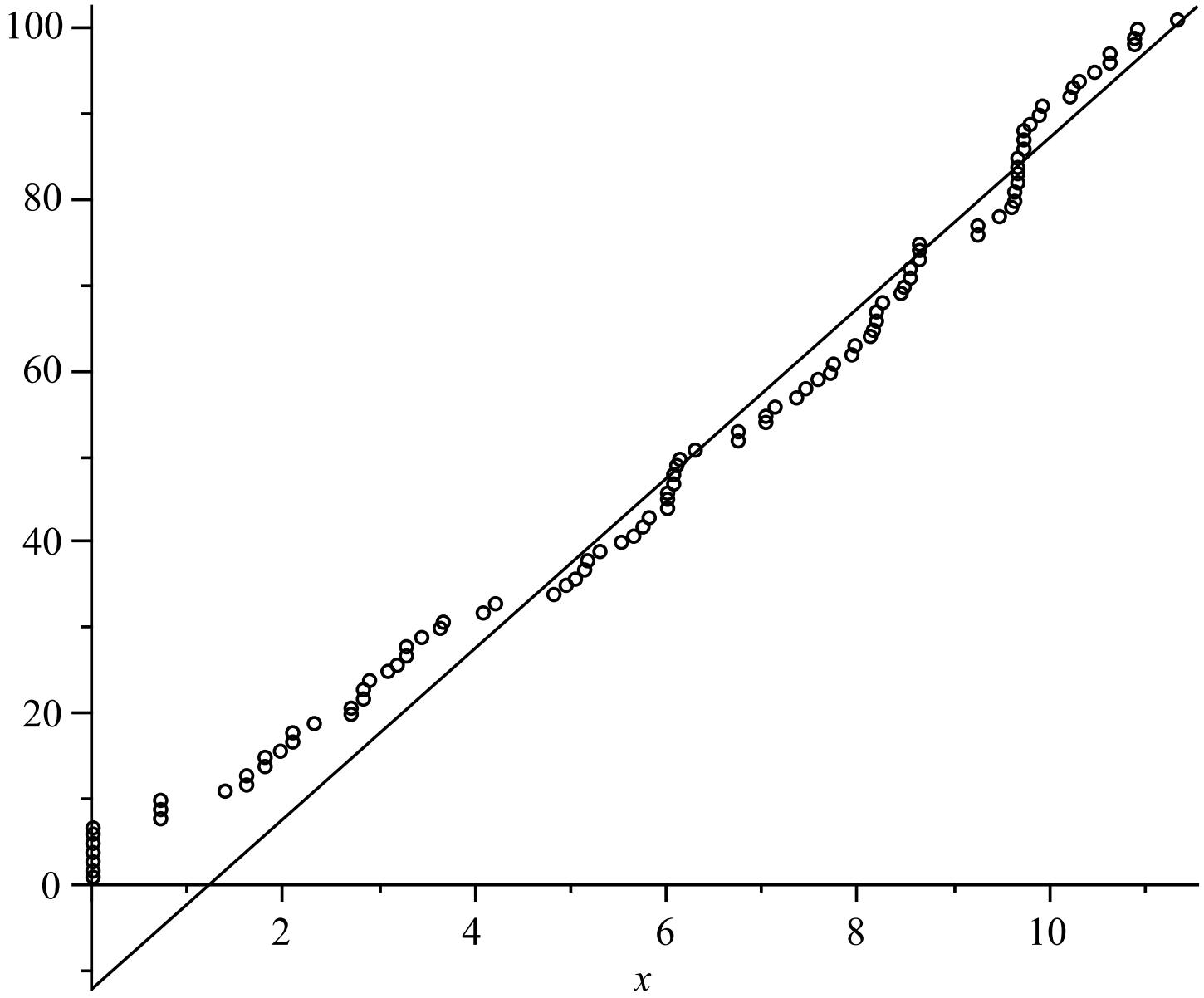
1*9.4*x+2

-6x^3y+x^3w+3x^2y^2-2x^2yz-x^2z^2+3x^2zw-
xy^2z+3xyz^2+2xz^3+2xzw^2+xw^3+y^3w-y^2z^2
-2y^2zw+2yz^3+2yz^2w+2yzw^2+2z^4+3z^3w



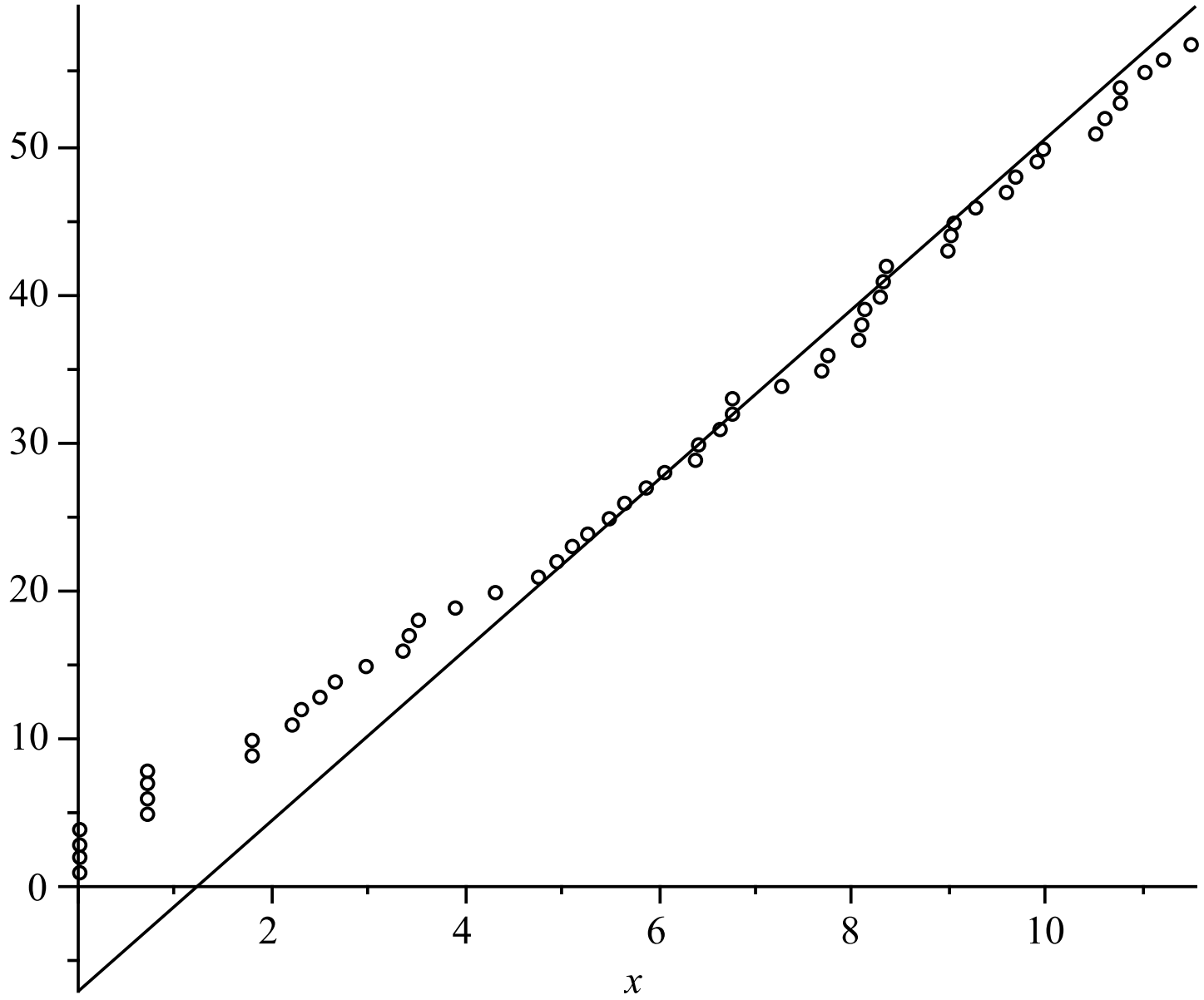
1*7.38*x

6x^3y+x^3w+3x^2y^2-2x^2yz+x^2z^2+3x^2zw-xy^2z-xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^3+2yzw^2+2z^4+
z^3w-2z^2w^2



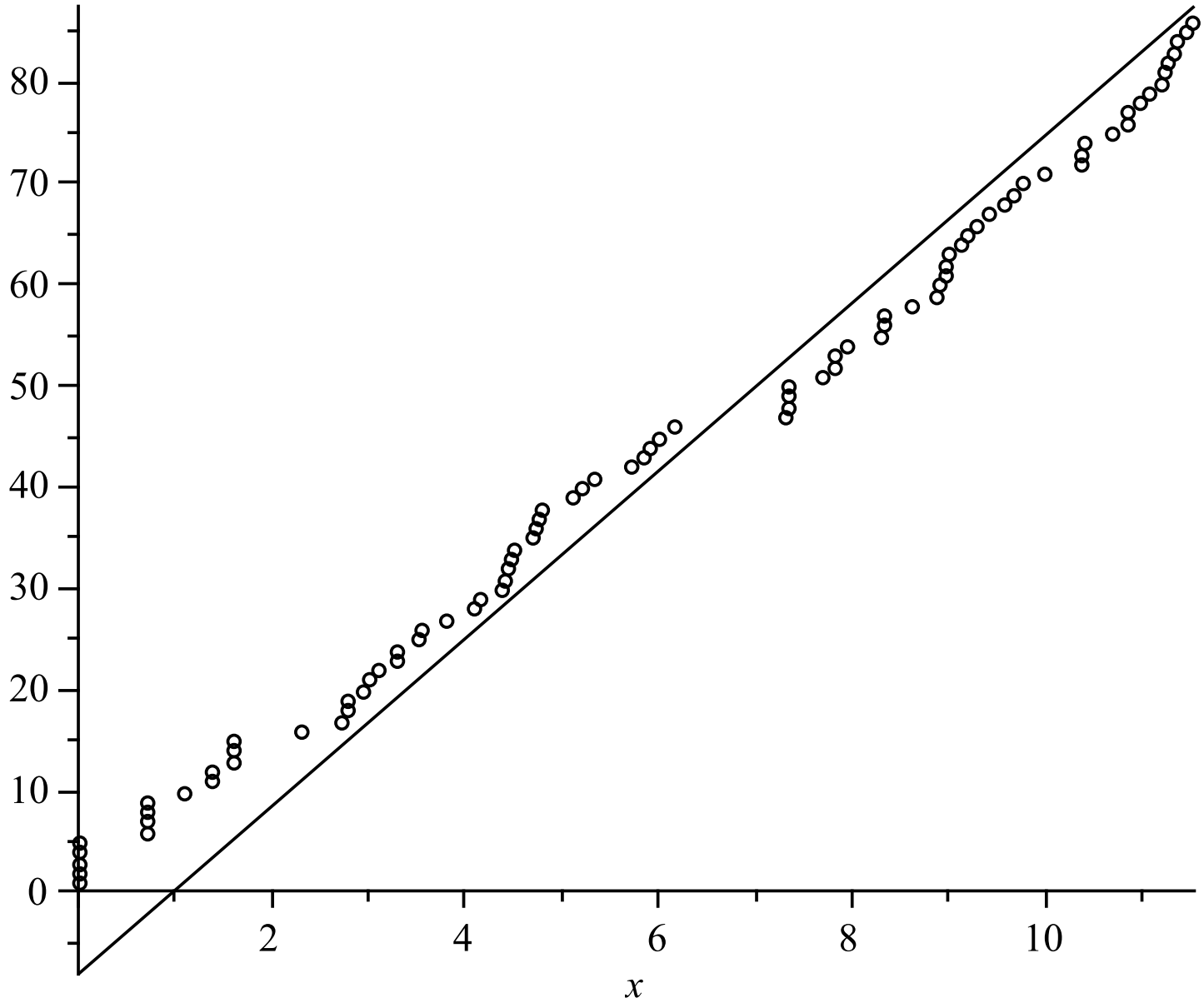
1*9.90*x-12

6x^3z+x^3w+3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+3x^2zw-xy^2z+2xy^2
xyz^2+2xyzw-2xyw^2-2xz^3+2xzw^2+xw^3+y^3w-
y^2z^2+2y^2zw-2yz^2w-2yzw^2+z^3w+2z^2w^2-2zw^3



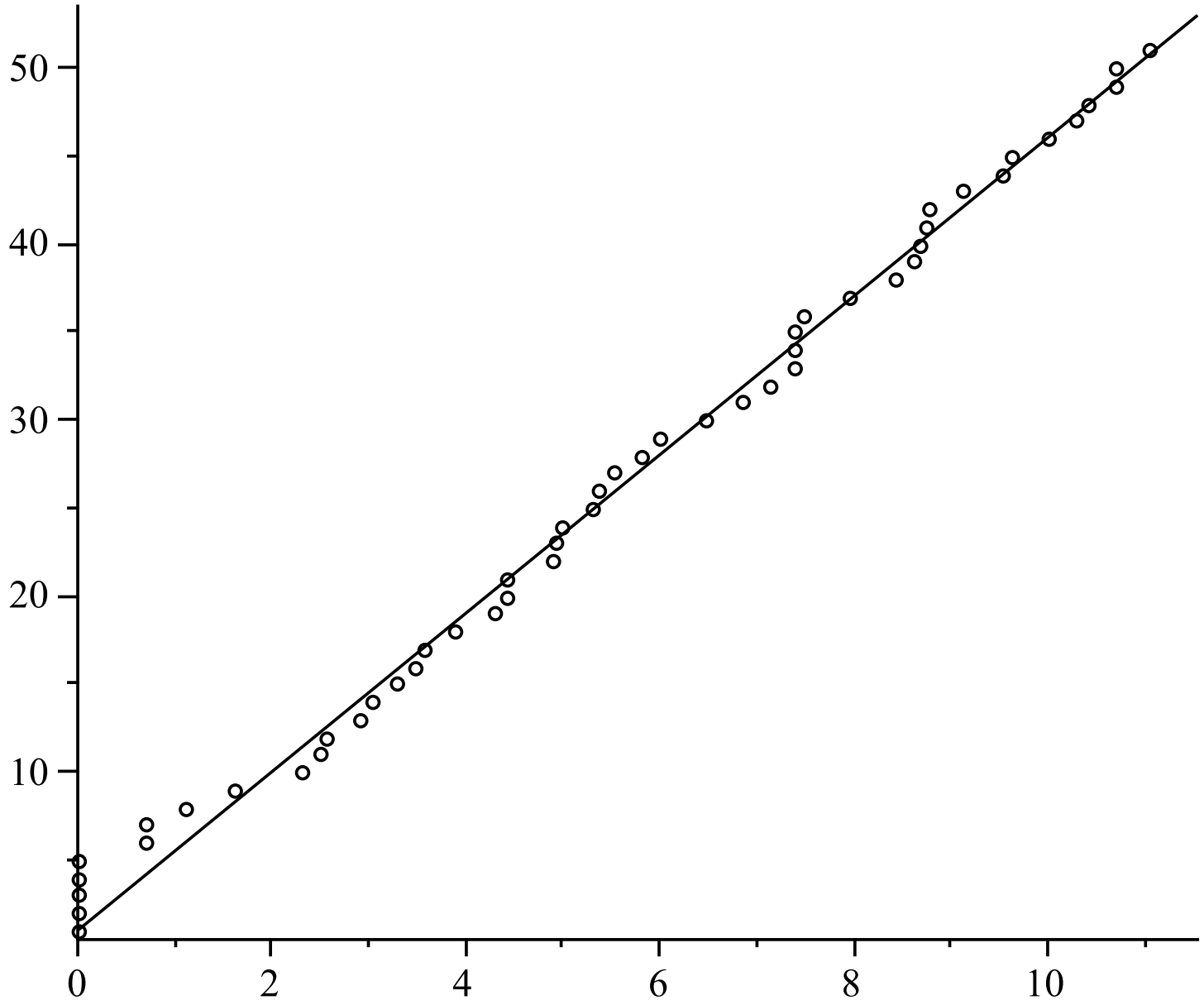
1*5.75*x-7

-6x^3z+x^3w+3x^2y^2-2x^2yz+2x^2yw-x^2z^2-x^2zw+2x^2w^2+3xy′
xyz^2-2xyzw-2xyw^2+2xzw^2-xw^3+2y^3z+y^3w+
y^2z^2+2y^2zw+2yz^2w+2yzw^2-2yw^3+z^3w+2z^2w^2



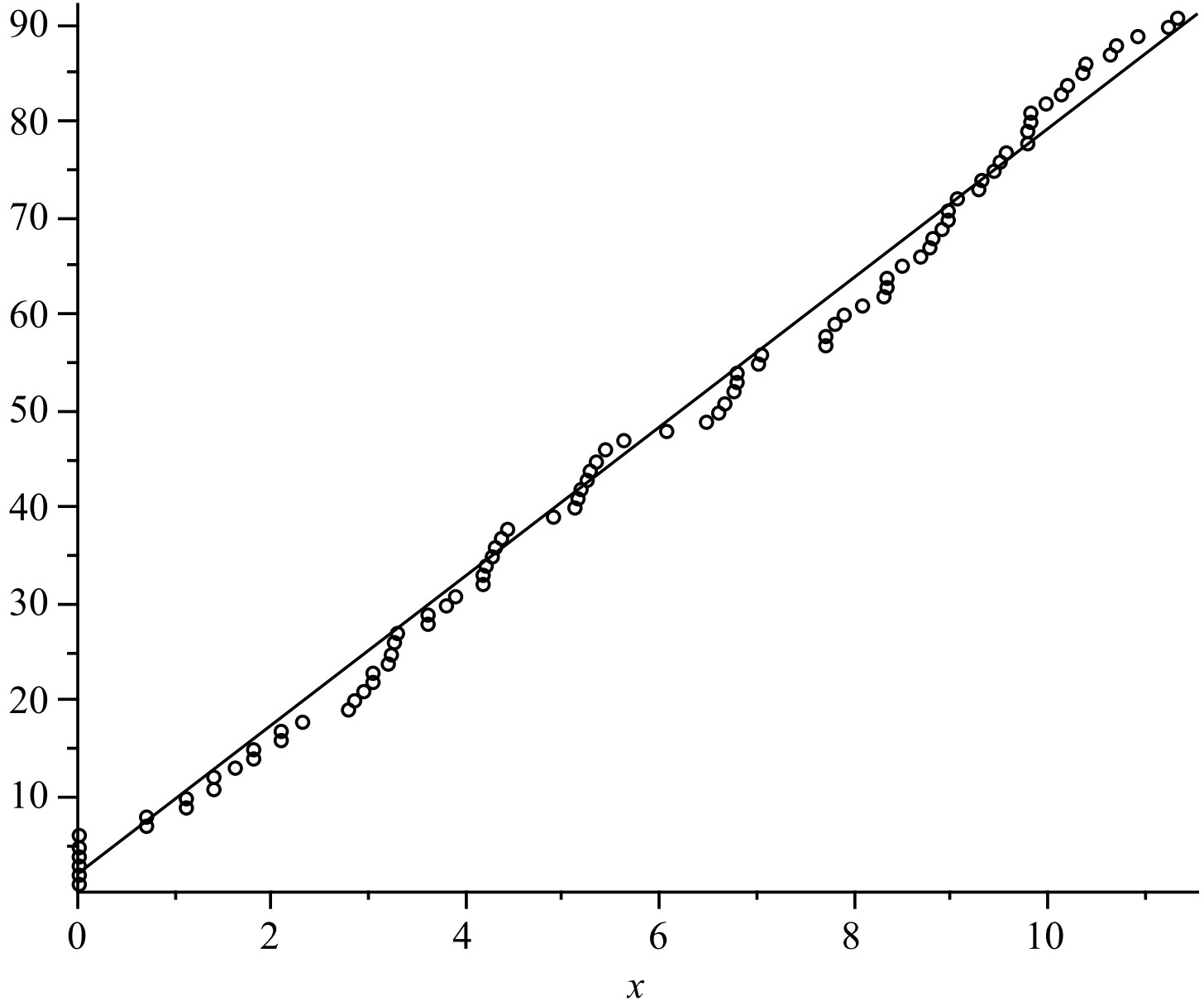
1*8.26*x-8

6x^3z+x^3w+3x^2y^2-2x^2yz-2x^2yw-x^2z^2-x^2zw-
xy^2z+2xy^2w+3xyz^2-2xyzw-2xyw^2+2xz^3-2xzw^2+xw^3+y^3w·
y^2z^2+2y^2zw+2yz^3-2yzw^2+2z^4+z^3w-2z^2w^2-2zw^3



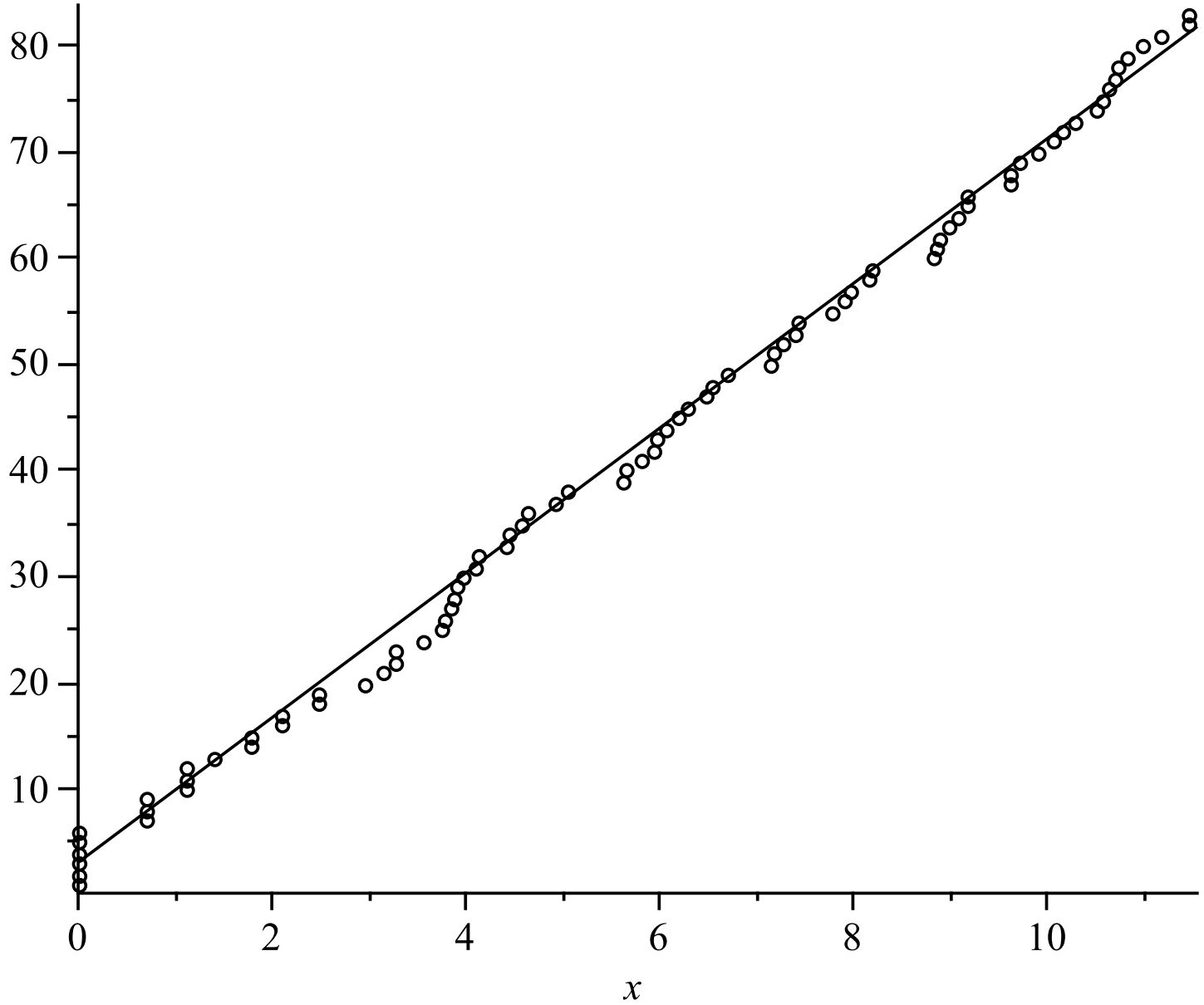
5/6*5.40*x+1

-6x^3z+x^3w+3x^2y^2-2x^2yz+3x^2z^2+3x^2zw-xy^2z+xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^2w+2yzw^2+
z^3w+2z^2w^2



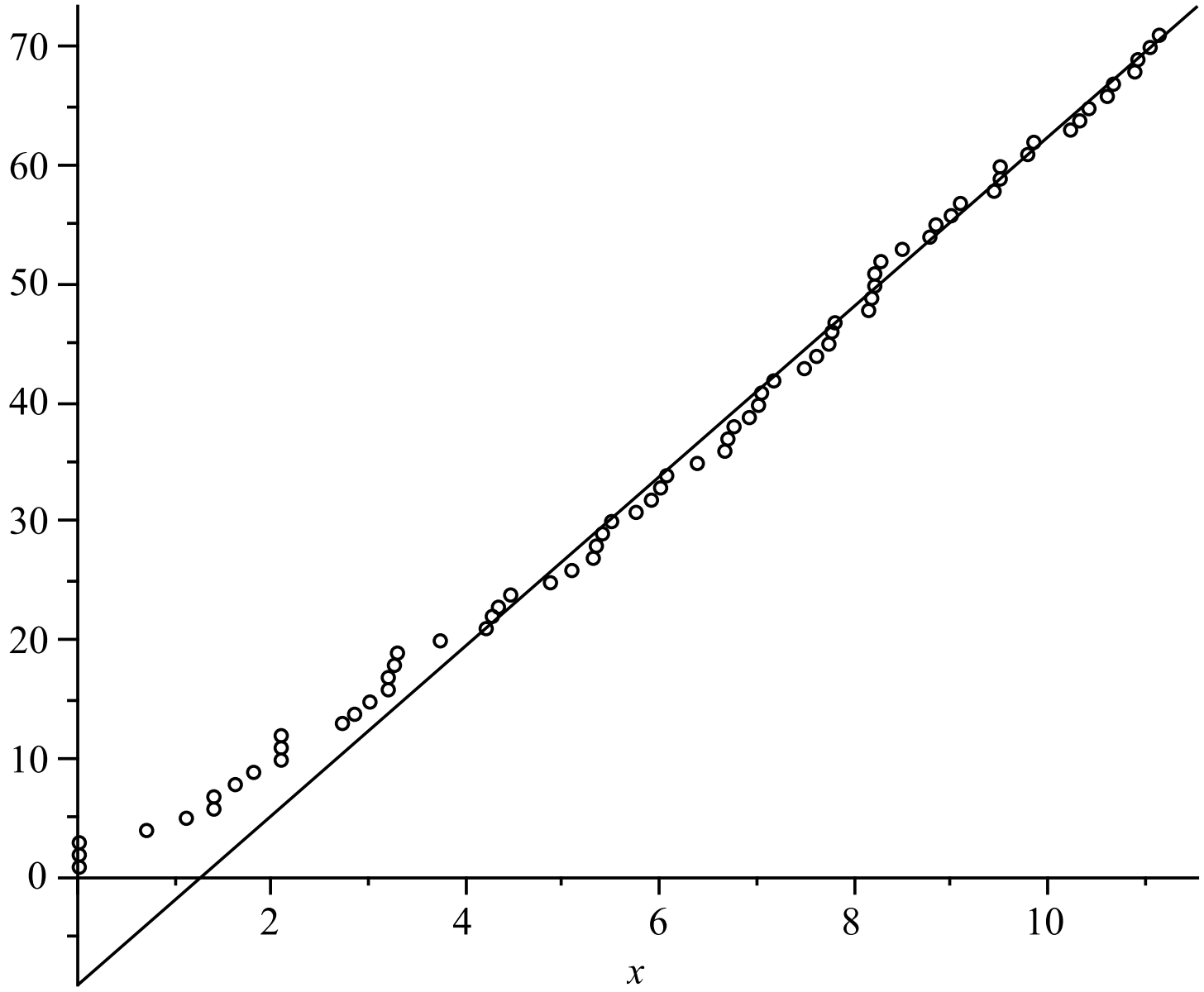
3/4*10.3*x+2

6x^3z+x^3w+3x^2y^2-2x^2yz+3x^2z^2+3x^2zw-xy^2z+xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^2w+2yzw^2+
z^3w+2z^2w^2



1*6.82*x+3

6x^3z+x^3w+3x^2y^2-2x^2yz+3x^2z^2-x^2zw-xy^2z+xyz^2
-2xyzw-2xz^3+2xz^2w+2xzw^2+xw^3+y^3w-y^2z^2-2yz^2w-
z^3w+2z^2w^2+2zw^3-2w^4

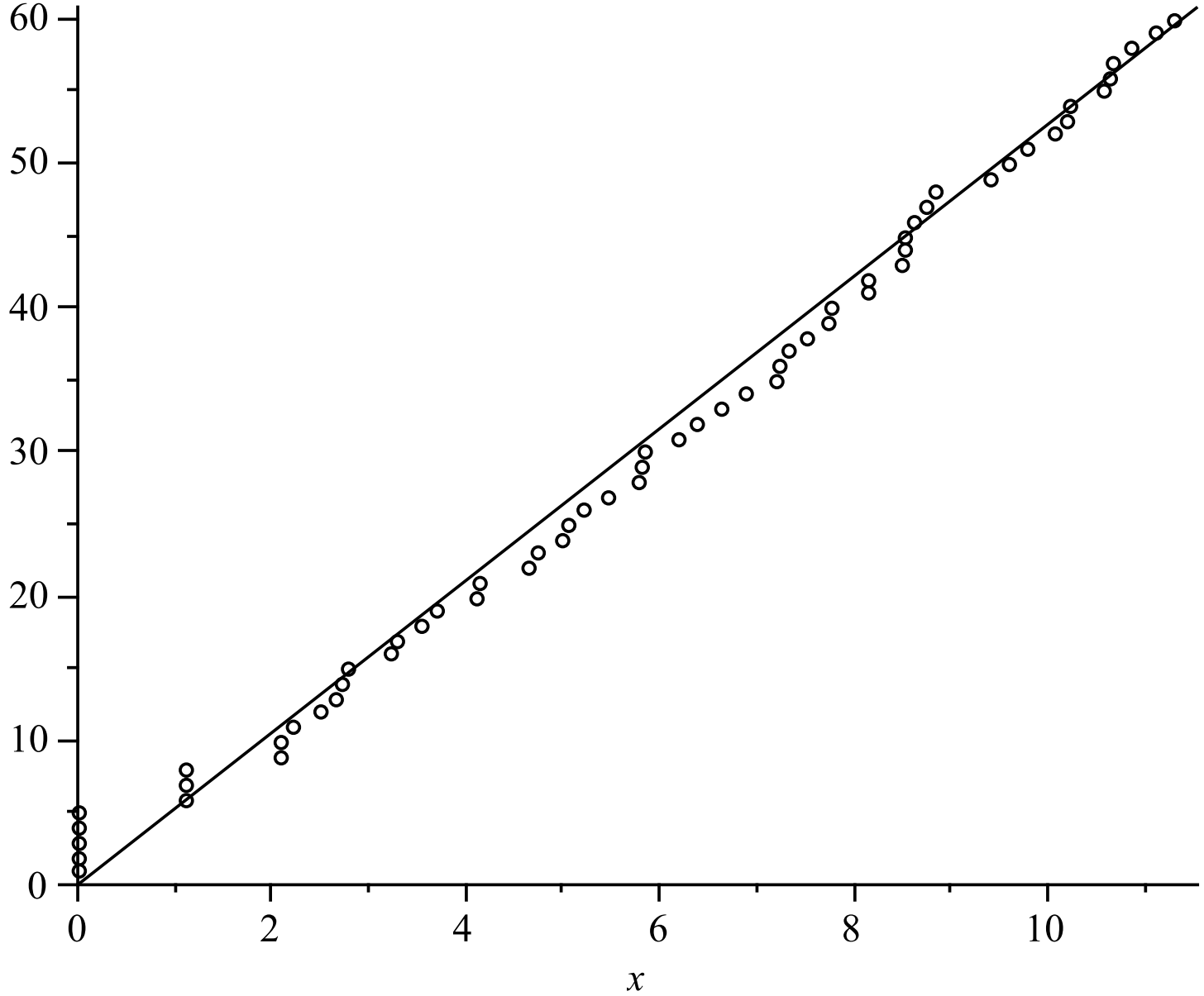


1*7.13*x-9

x+2z <>0

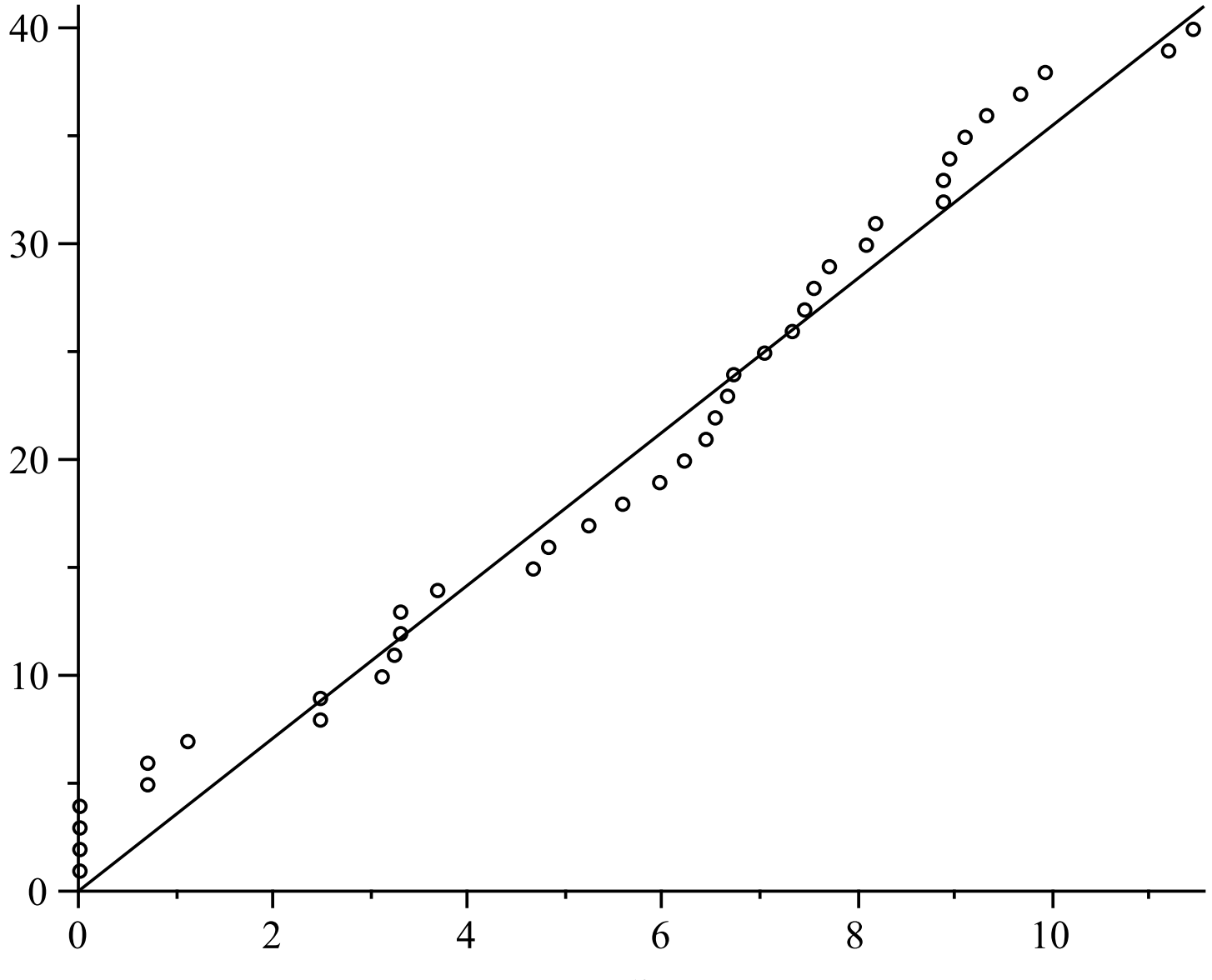
6x^3z+x^3w+3x^2y^2-2x^2yz+3x^2z^2+x^2zw-xy^2z+xyz^2-2xz^3

-2xz^2w+xw^3+y^3w-y^2z^2-z^3w+2zw^3



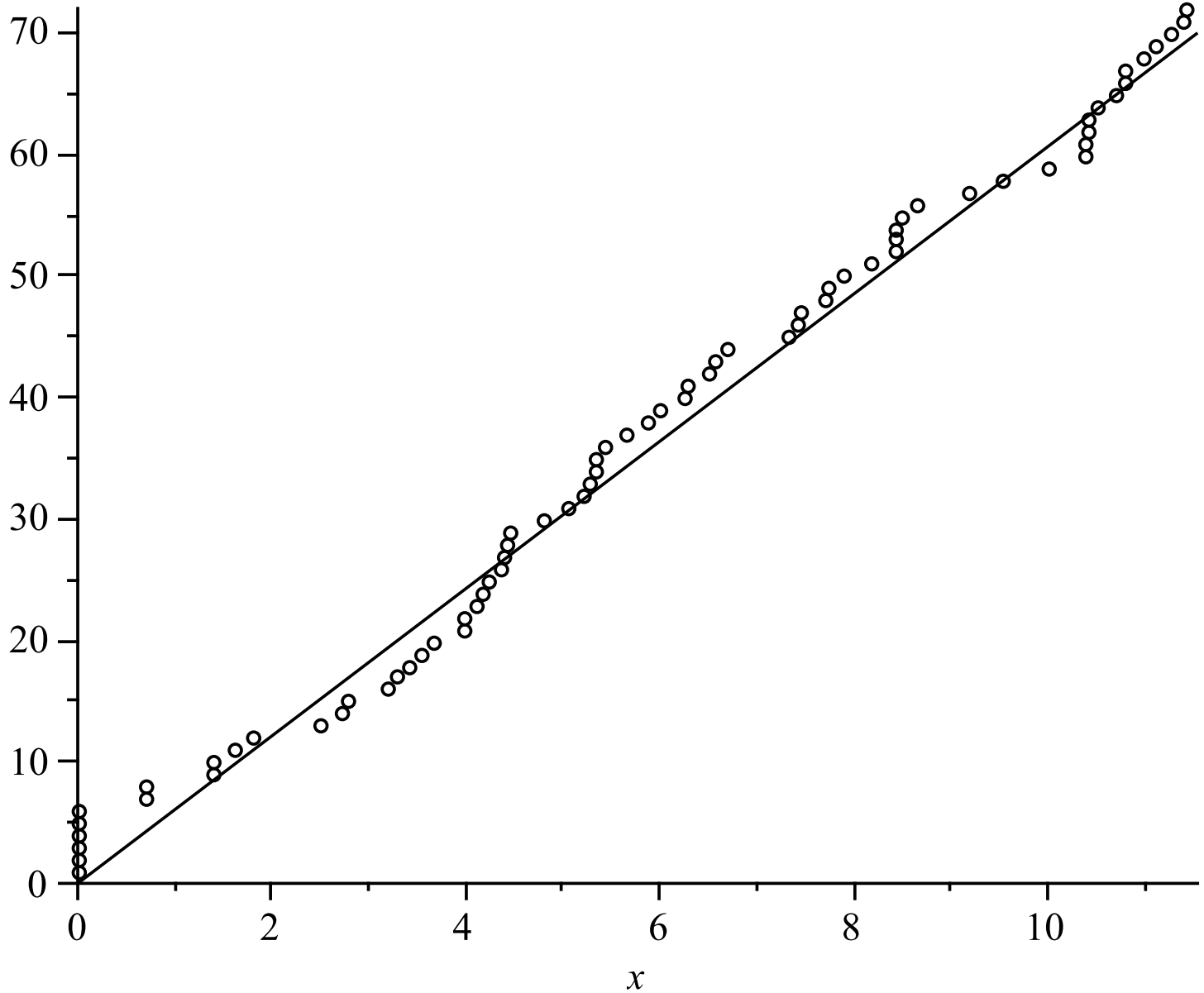
4/5*6.58*x

-6x^3z+x^3w+3x^2y^2-2x^2yz-x^2z^2+3x^2zw-
xy^2z+3xyz^2+2xz^3+2xzw^2+xw^3+y^3w-y^2z^2
-2y^2zw+2yz^3+2yz^2w+2yzw^2+2z^4+3z^3w



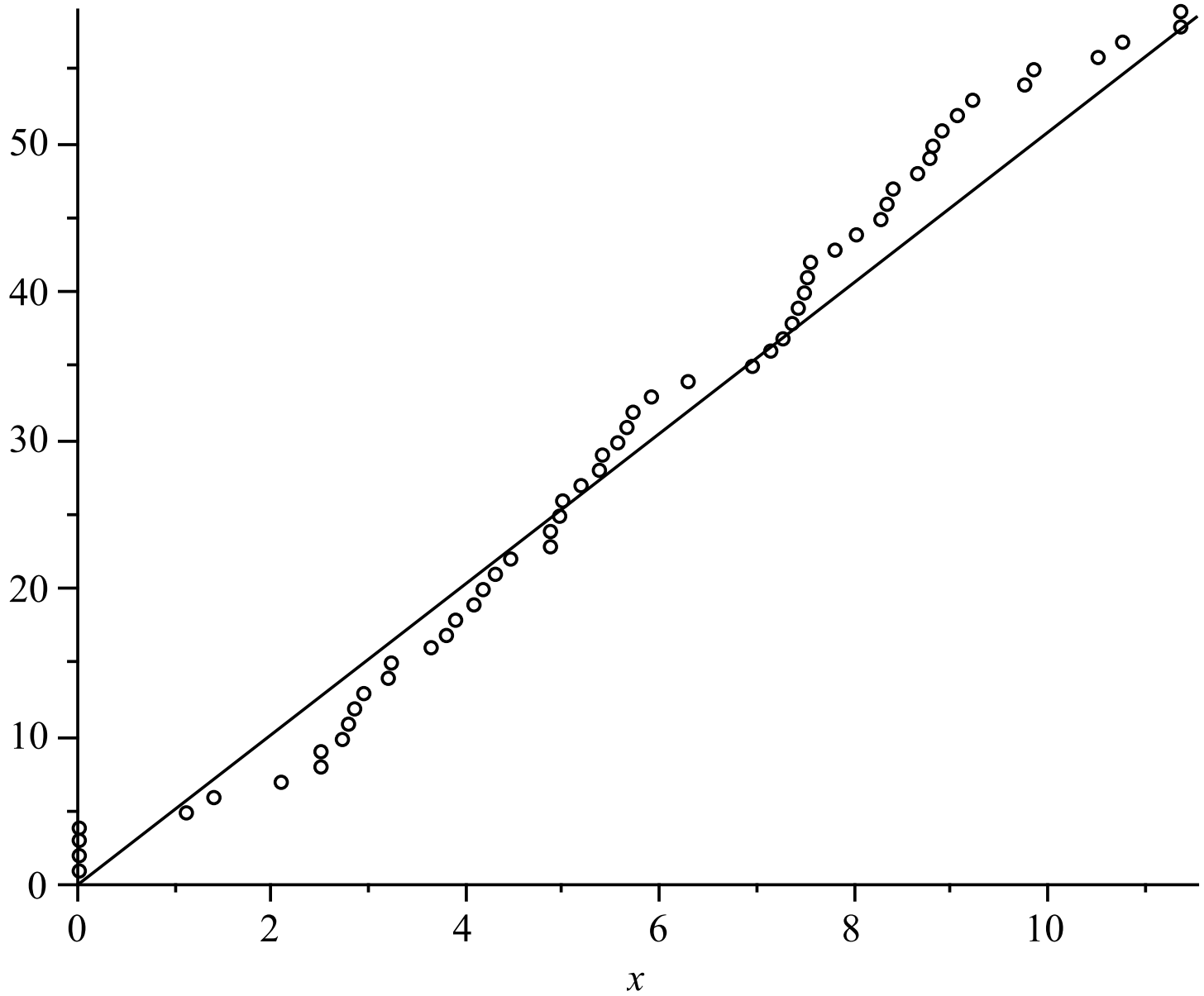
2/3*5.33*x

6x^3z+x^3w+3x^2y^2-2x^2yz-x^2z^2+3x^2zw-
xy^2z+3xyz^2+2xz^3+2xzw^2+xw^3+y^3w-y^2z^2
-2y^2zw+2yz^3+2yz^2w+2yzw^2+2z^4+3z^3w



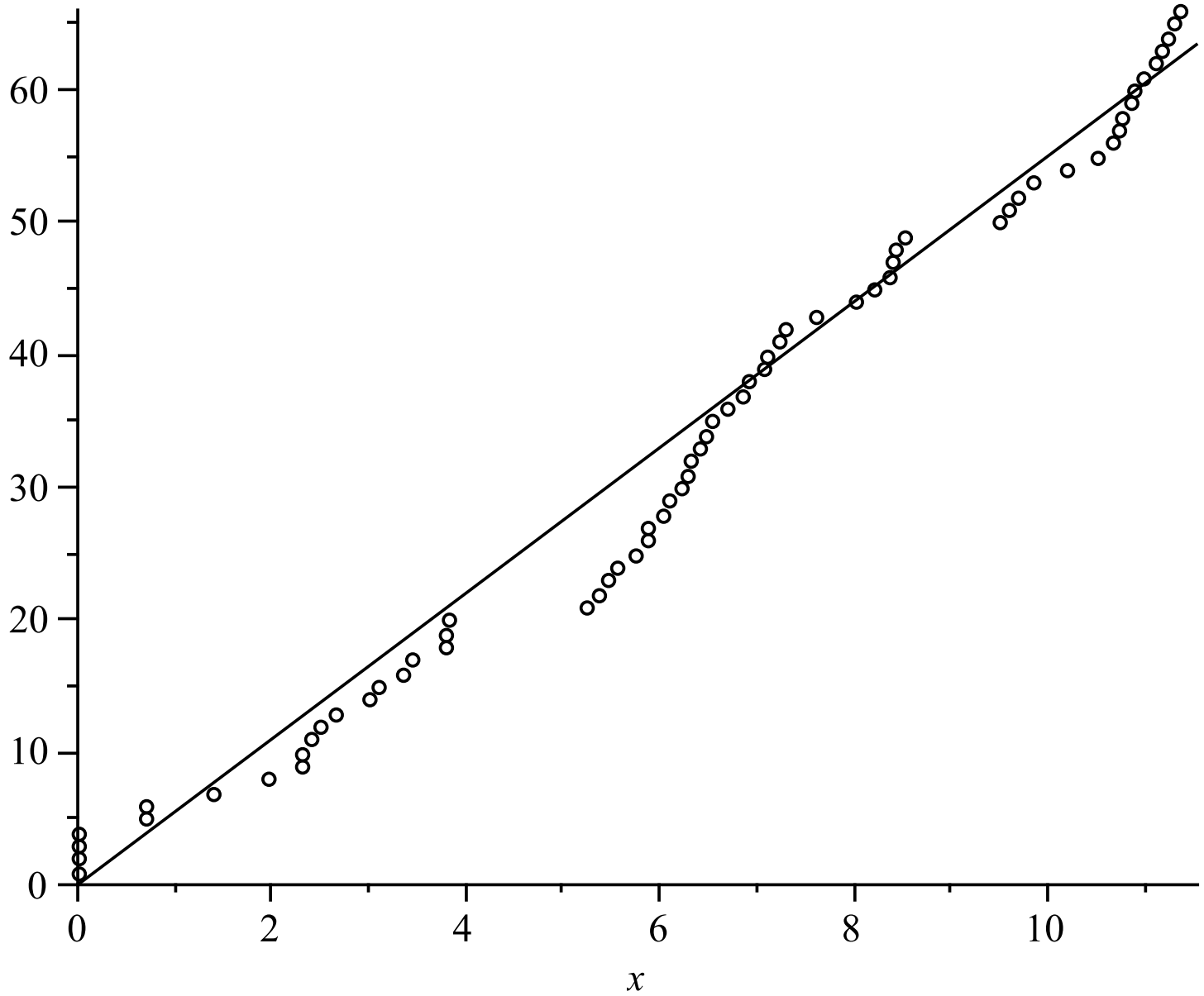
1*6.07*x

6x^3z+x^3w+3x^2y^2-2x^2yz+x^2z^2+3x^2zw-xy^2z-xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^3+2yzw^2+2z^4+
z^3w-2z^2w^2



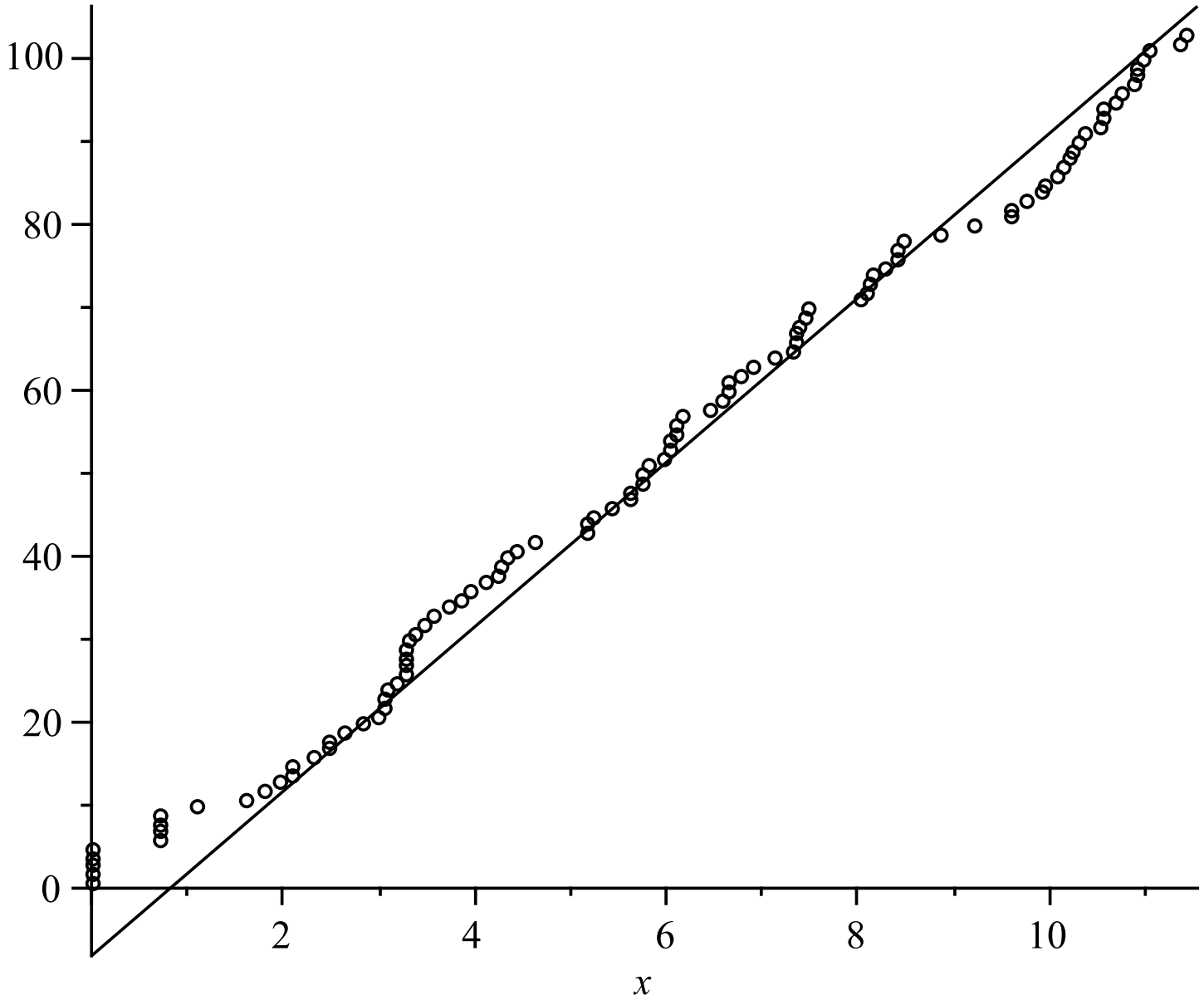
1*5.08*x

-6x^4+x^3w+3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+3x^2zw-xy^2z+2xy^2
xyz^2+2xyzw-2xyw^2-2xz^3+2xzw^2+xw^3+y^3w-
y^2z^2+2y^2zw-2yz^2w-2yzw^2+z^3w+2z^2w^2-2zw^3



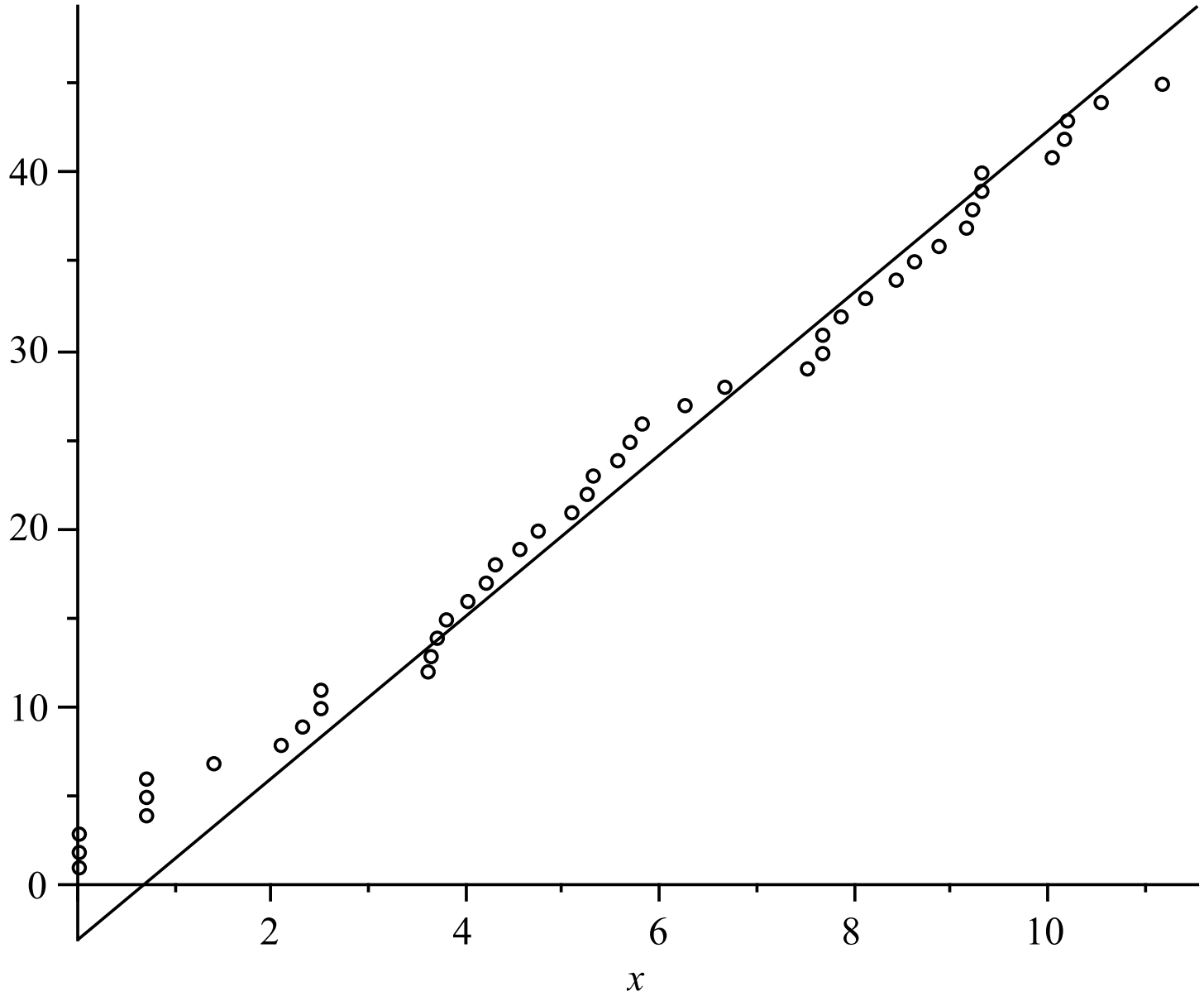
1*5.50*x

-6x^4+x^3w+3x^2y^2-2x^2yz+2x^2yw-x^2z^2-x^2zw+2x^2w^2+3xy^
xyz^2-2xyzw-2xyw^2+2xzw^2-xw^3+2y^3z+y^3w+
y^2z^2+2y^2zw+2yz^2w+2yzw^2-2yw^3+z^3w+2z^2w^2



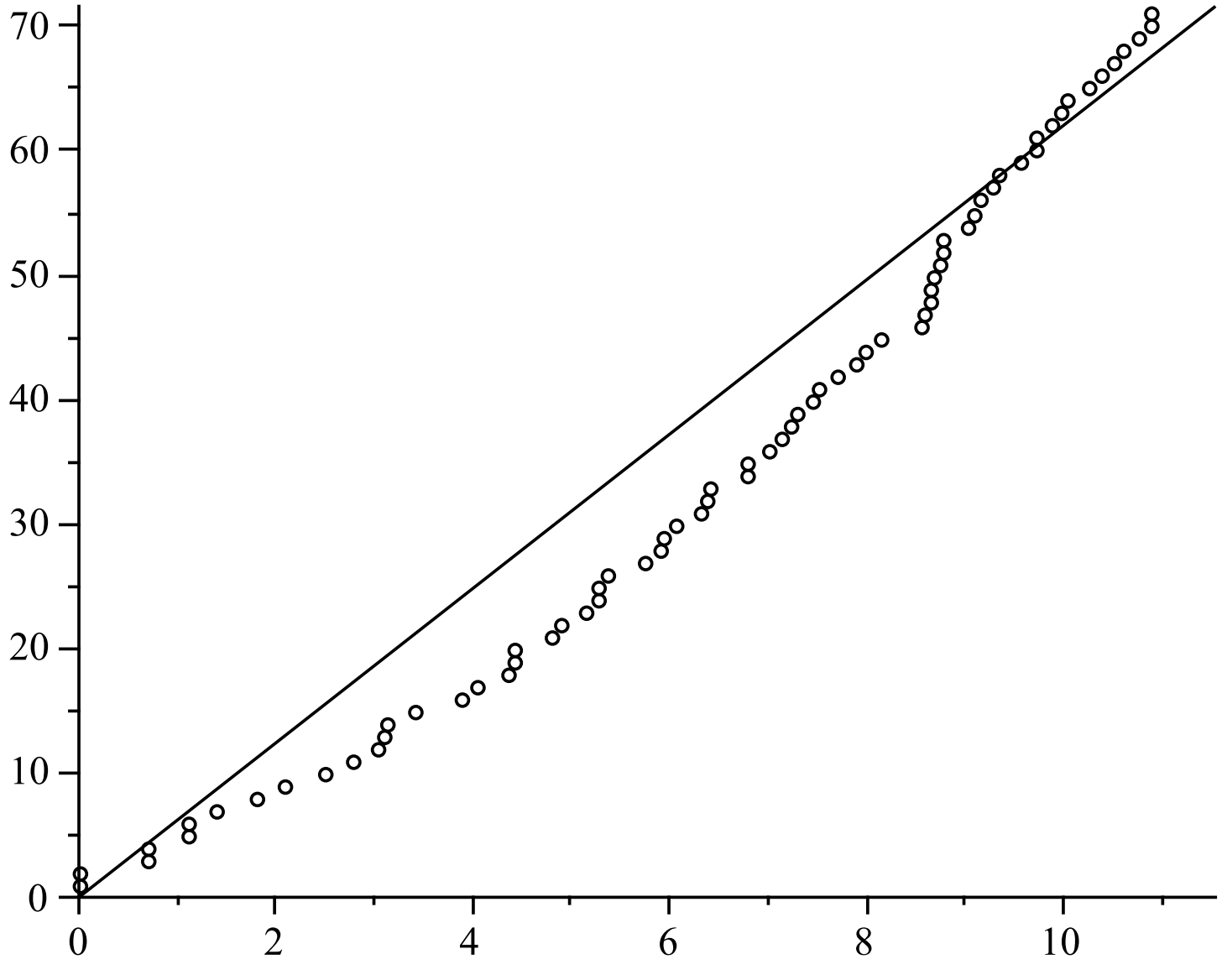
1*9.89*x-8

-6x^4+x^3w+3x^2y^2-2x^2yz+3x^2z^2+3x^2zw-xy^2z+xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^2w+2yzw^2+
z^3w+2z^2w^2



1*4.53*x-3

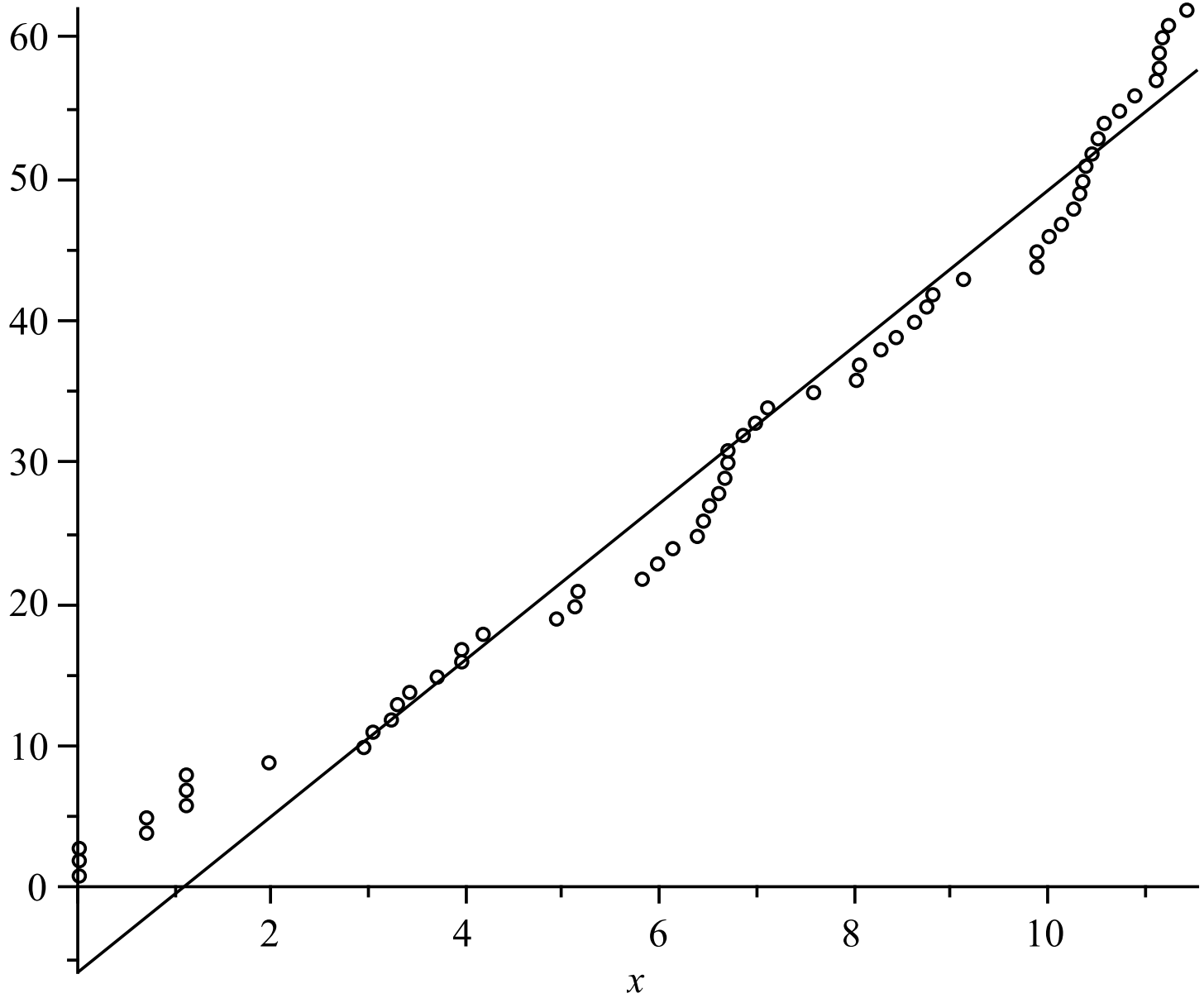
-6x^4+x^3w+3x^2y^2-2x^2yz+3x^2z^2-x^2zw-xy^2z+xyz^2
-2xyzw-2xz^3+2xz^2w+2xzw^2+xw^3+y^3w-y^2z^2-2yz^2w-
z^3w+2z^2w^2+2zw^3-2w^4



1*6.20*x

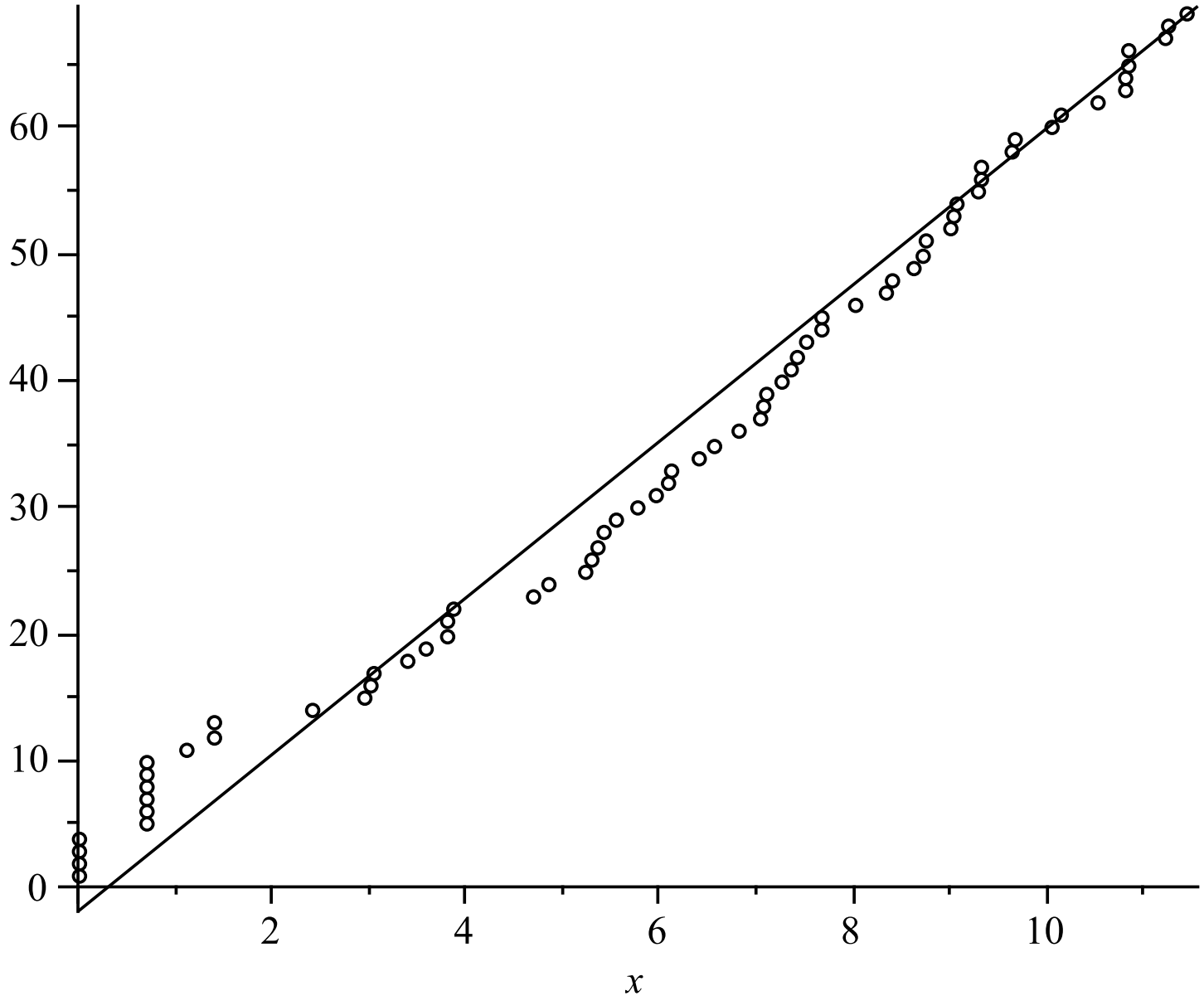
x <> -2z

-6x^4+x^3w+3x^2y^2-2x^2yz+3x^2z^2+x^2zw-xy^2z+xyz^2-2xz^3
-2xz^2w+xw^3+y^3w-y^2z^2-z^3w+2zw^3



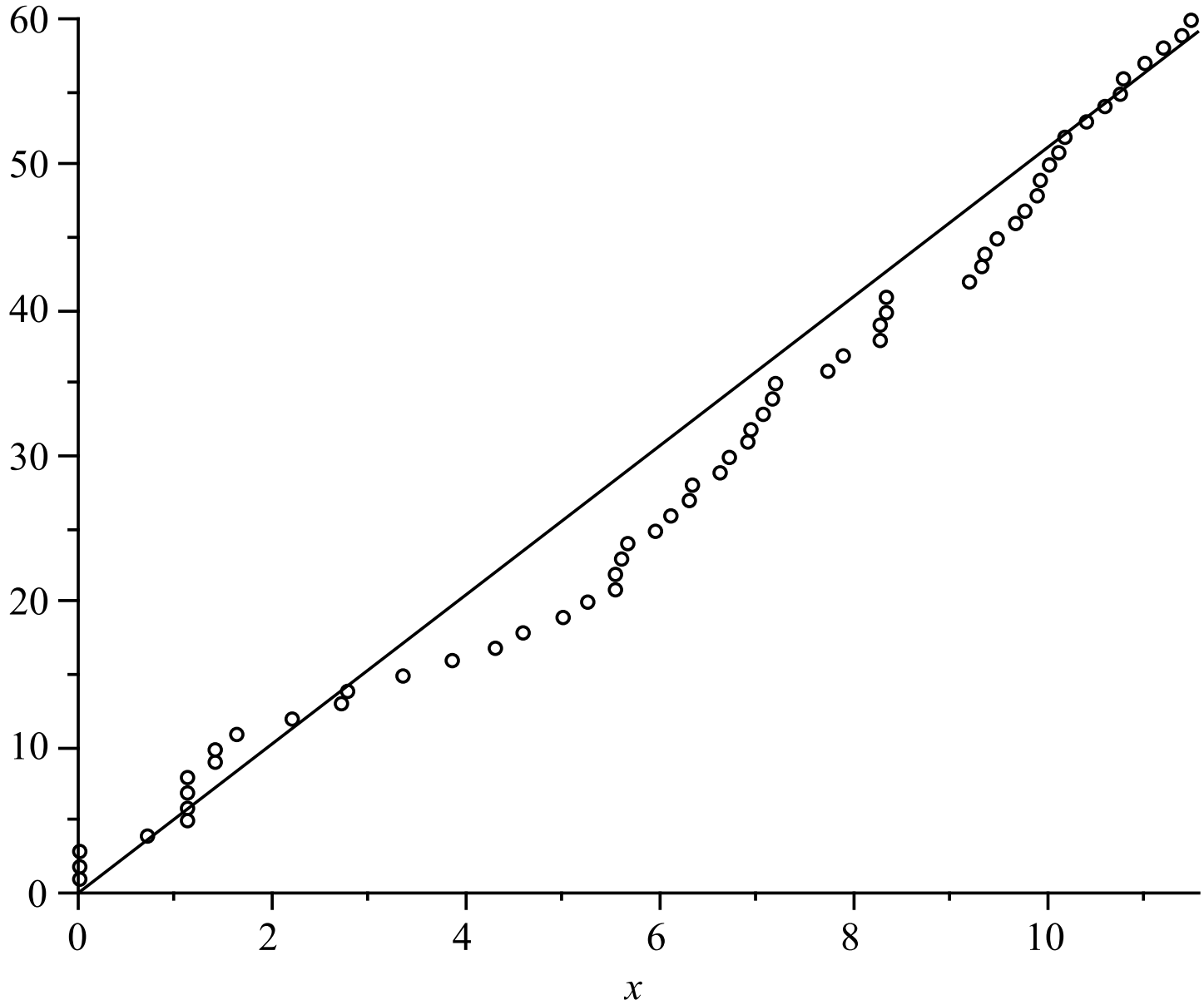
1*5.52*x-6

-6x^4+x^3w+3x^2y^2-2x^2yz-x^2z^2+3x^2zw-
xy^2z+3xyz^2+2xz^3+2xzw^2+xw^3+y^3w-y^2z^2
-2y^2zw+2yz^3+2yz^2w+2yzw^2+2z^4+3z^3w



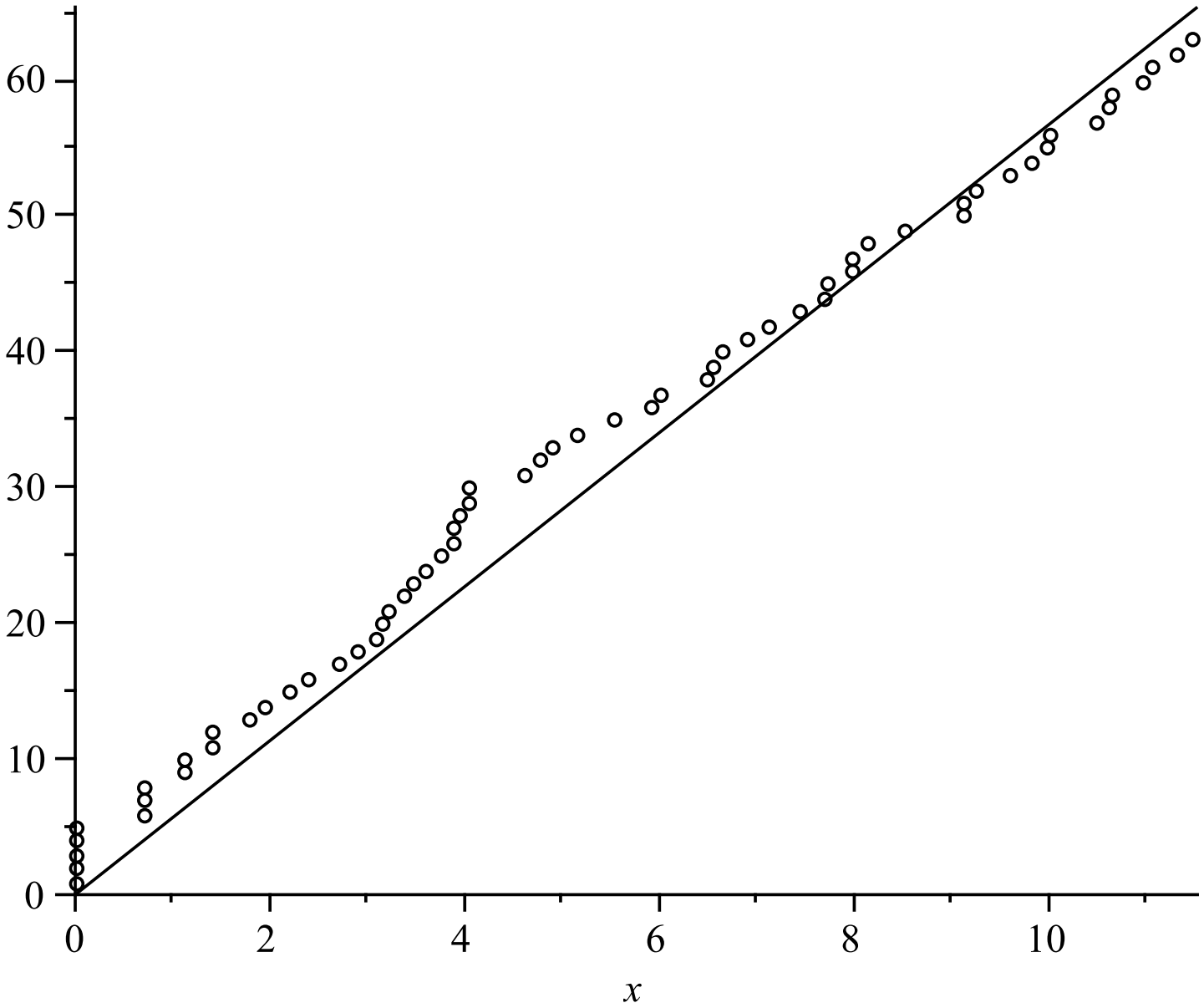
1*6.19*x-2

-6x^4+x^3w+3x^2y^2-2x^2yz+x^2z^2+3x^2zw-xy^2z-xyz^2
-2xz^3+2xzw^2+xw^3+y^3w-y^2z^2-2y^2zw-2yz^3+2yzw^2+2z^4+
z^3w-2z^2w^2



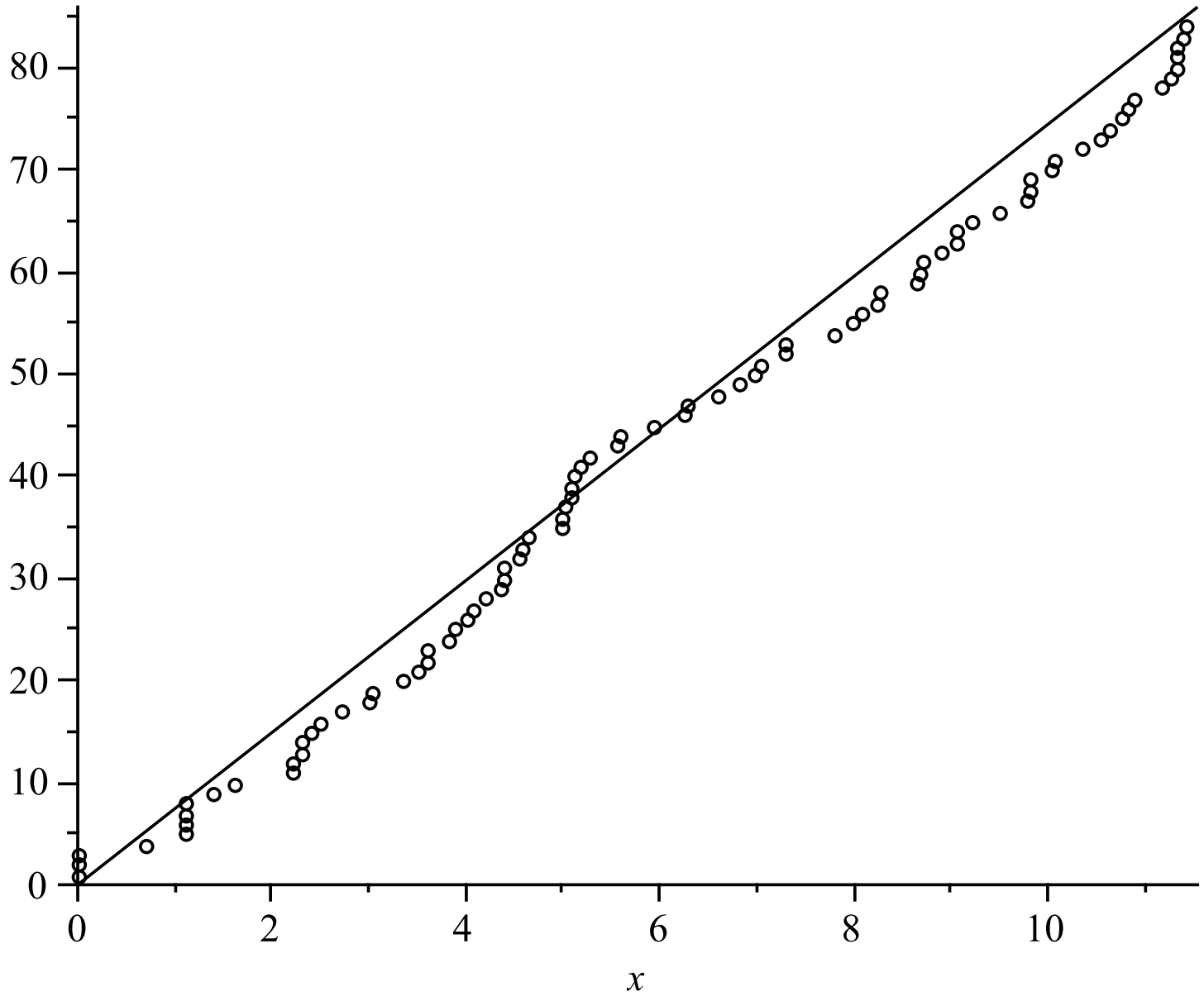
2/3*7.69*x

x^3w-3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+x^2zw-2x^2w^2+3xy^2z+3xy
-2xyzw-2xyw^2-2xz^3-xw^3+2y^3z-y^3w+y^2z^2+2y^2w^2-2yz^3
-2z^4+3z^3w-2zw^3



1*5.66*x

x^3w+3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+x^2zw-2x^2w^2+3xy^2z+3x
-2xyzw-2xyw^2-2xz^3-xw^3+2y^3z-y^3w+y^2z^2+2y^2w^2-2yz^3
-2z^4+3z^3w-2zw^3



1*7.44*x

x^3w+3x^2y^2-2x^2yz-2x^2yw+3x^2z^2+x^2zw-2x^2w^2+9xy^2z+3x
-2xyzw-2xyw^2-2xz^3-xw^3+2y^3z-y^3w+y^2z^2+2y^2w^2-2yz^3
-2z^4+3z^3w-2zw^3

Conjecture:

Let X be a K3 surface over a number field K, and let H be the height associated to an ample divisor. Suppose that X has geometric Picard number 1. Then there exist a finite field extension L/K, a countable (possibly finite) union V of curves of genus at most 1 on X_L, and a constant C such that

$$N_{X_L - V, H}(B) \approx C \log B.$$

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Let X be a K3 surface over a number field K, and let H be the height associated to an ample divisor. Suppose that X has geometric Picard number 1. Then there exist a finite field extension L/K, a countable (possibly finite) union V of curves of genus at most 1 on X_L, and a constant C such that

$$N_{X_L - V, H}(B) \approx C \log B.$$

Possible generalizations:

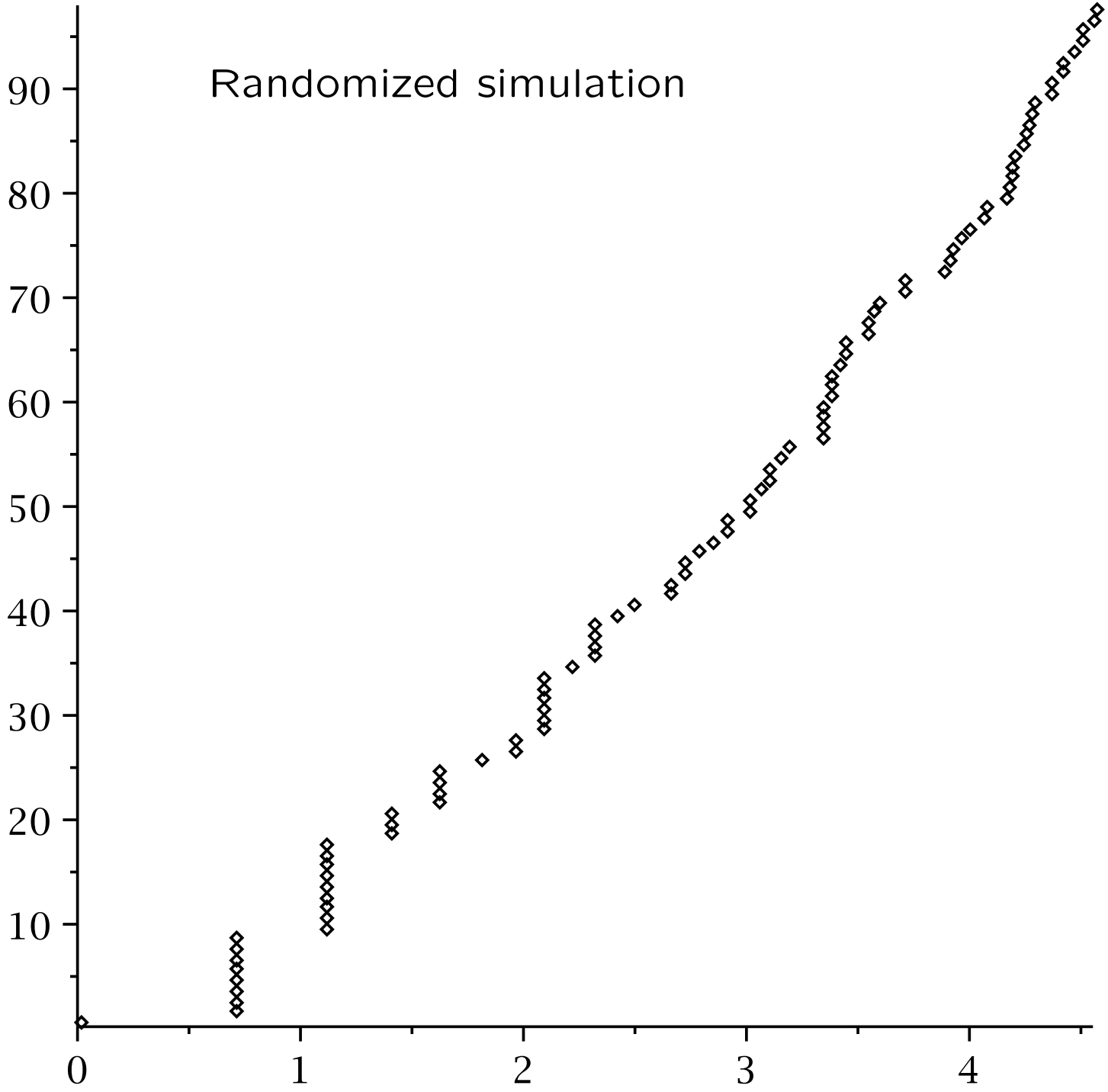
Omit field extension,

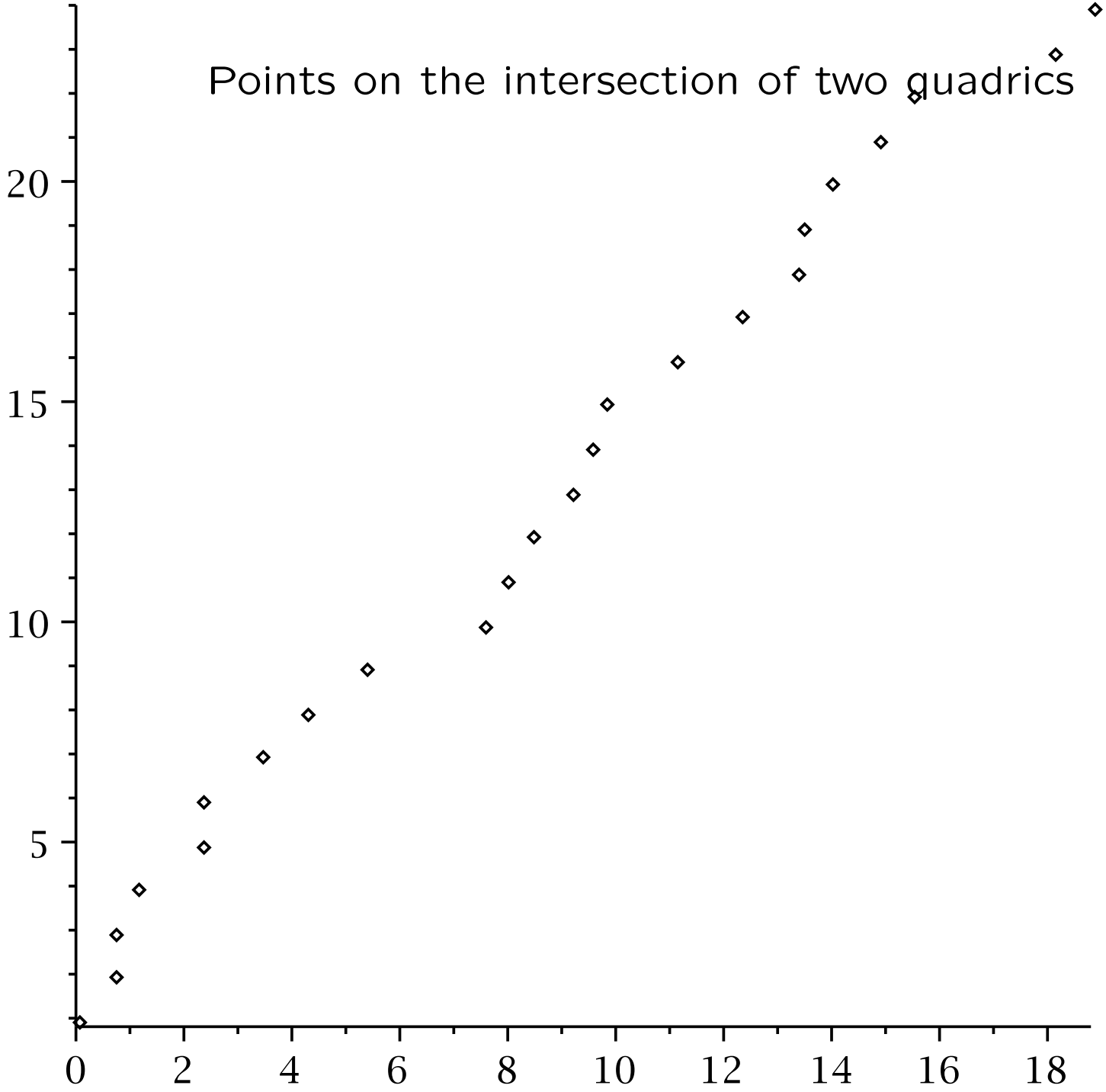
for all small enough $U = X_L - V$,

for Zariski open U,

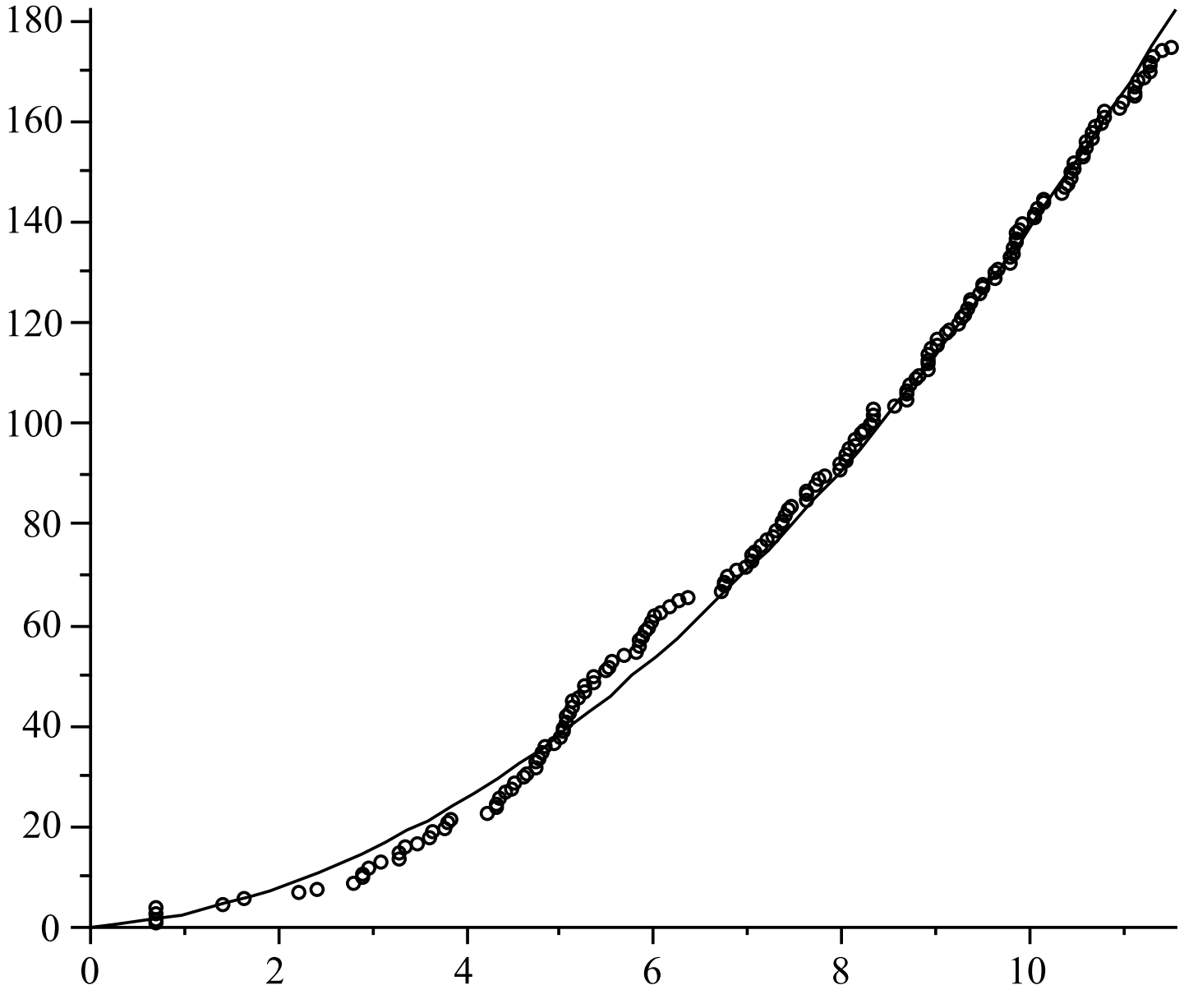
for higher Picard numbers.

The oscillating behaviour may or may not mean anything:

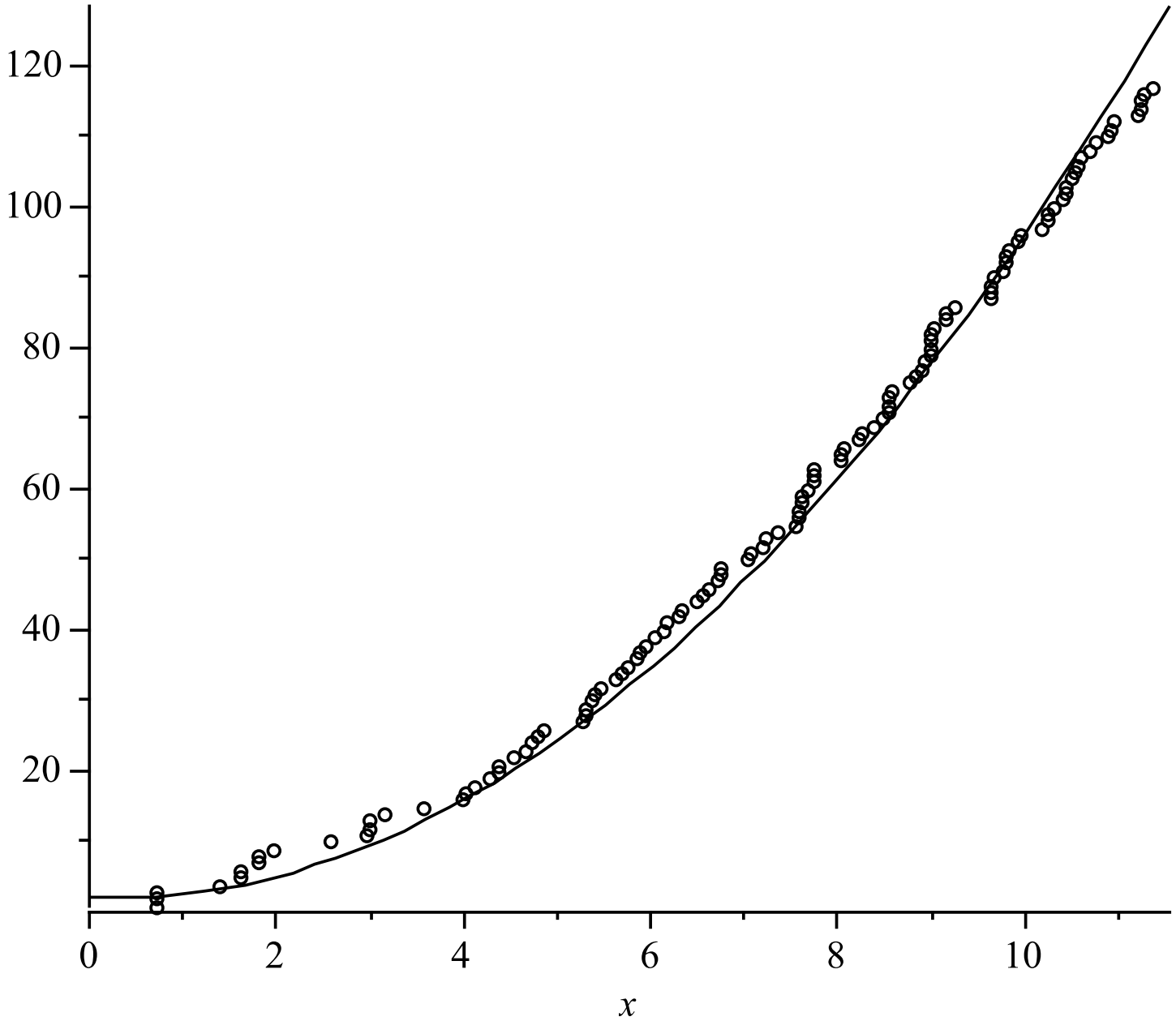

Randomized simulation
90
80
70
60
50
40
30
20
10
0
1
2
3
4


Points on the intersection of two quadrics
20
15
10
5
0
2
4
6
8
10
12
14
16
18

Graphs for (geometric) Picard number 2:



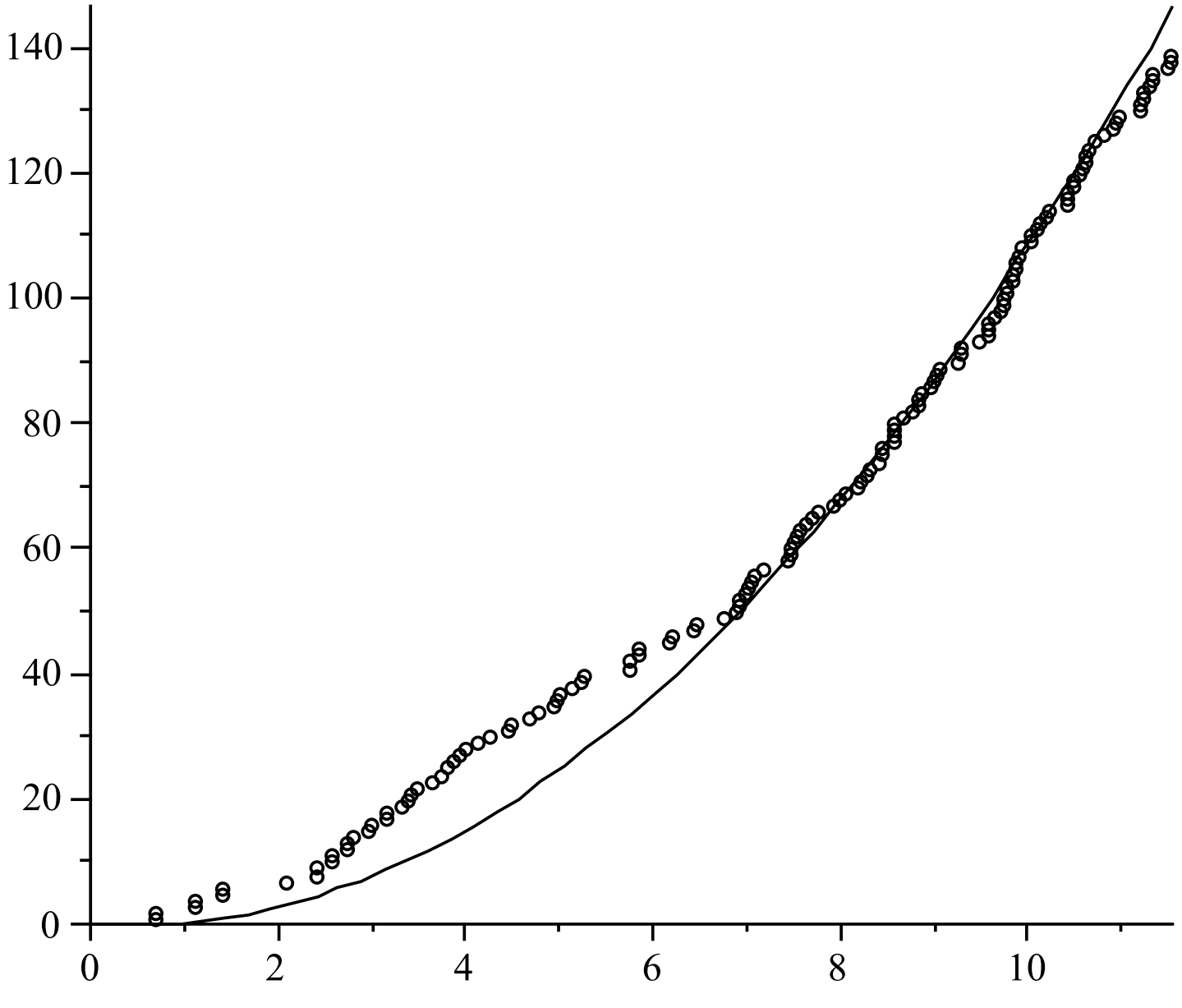
1/2*2.47*x^2+1.5*x
-2x^3w-x^2y^2-2x^2yz-5x^2yw-x^2z^2+xy^2z+3xy^2w+xyz^2-xyw^
-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



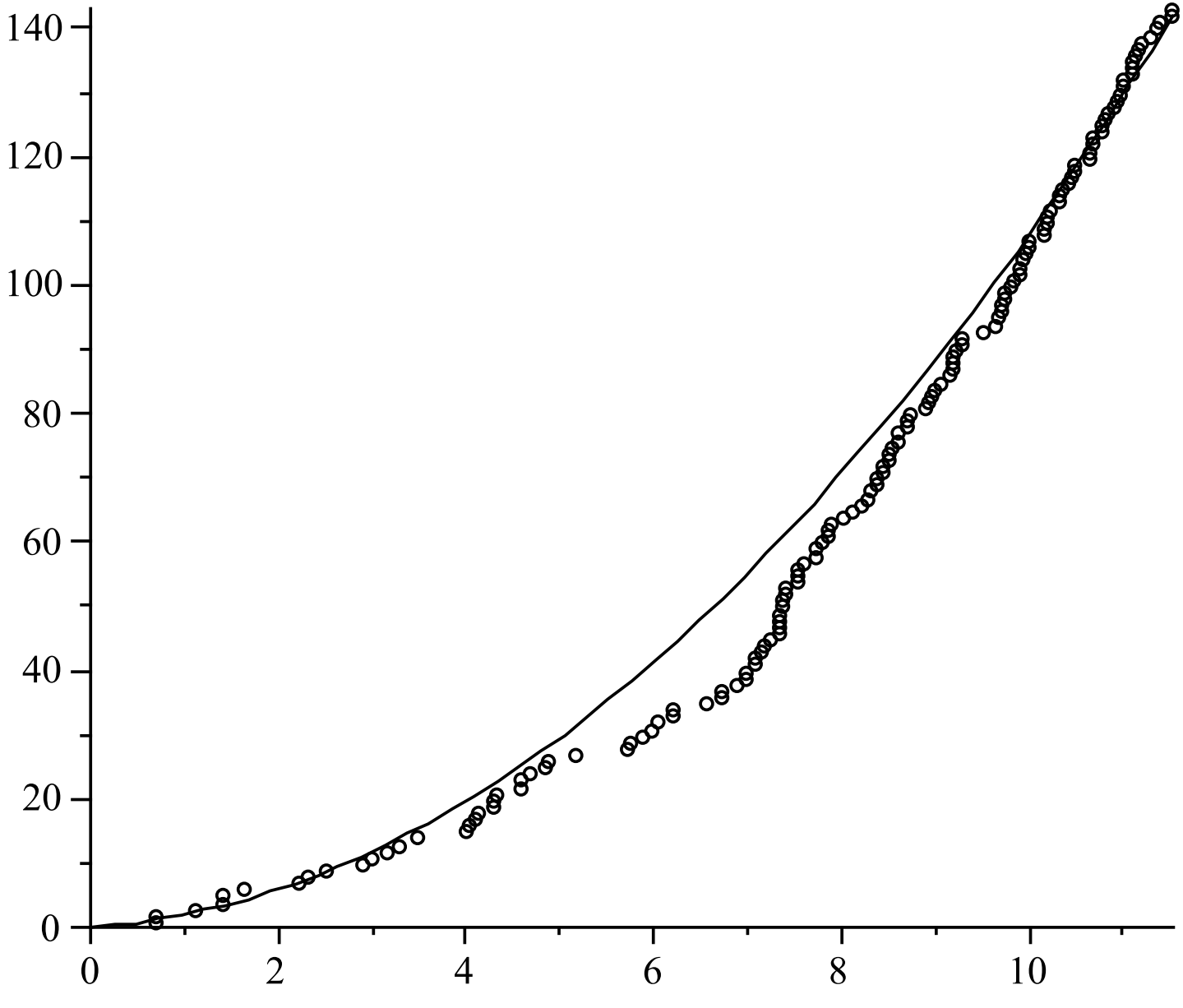
1/2*1.98*x^2-x/2+2

-2x^3w-x^2y^2-2x^2yz-x^2z^2-5x^2zw+xy^2z-2xy^2w+xyz^2-6xyw

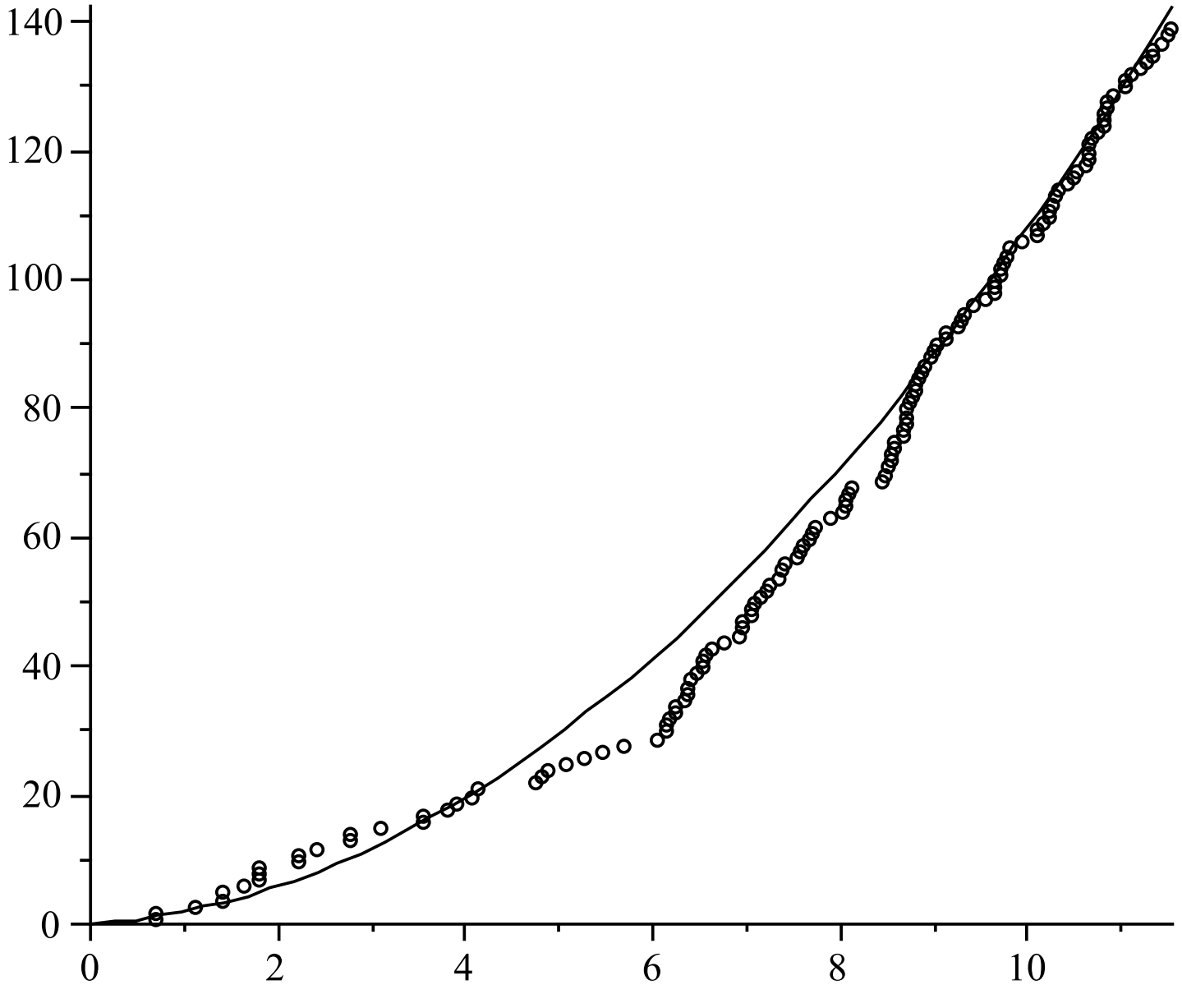
-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



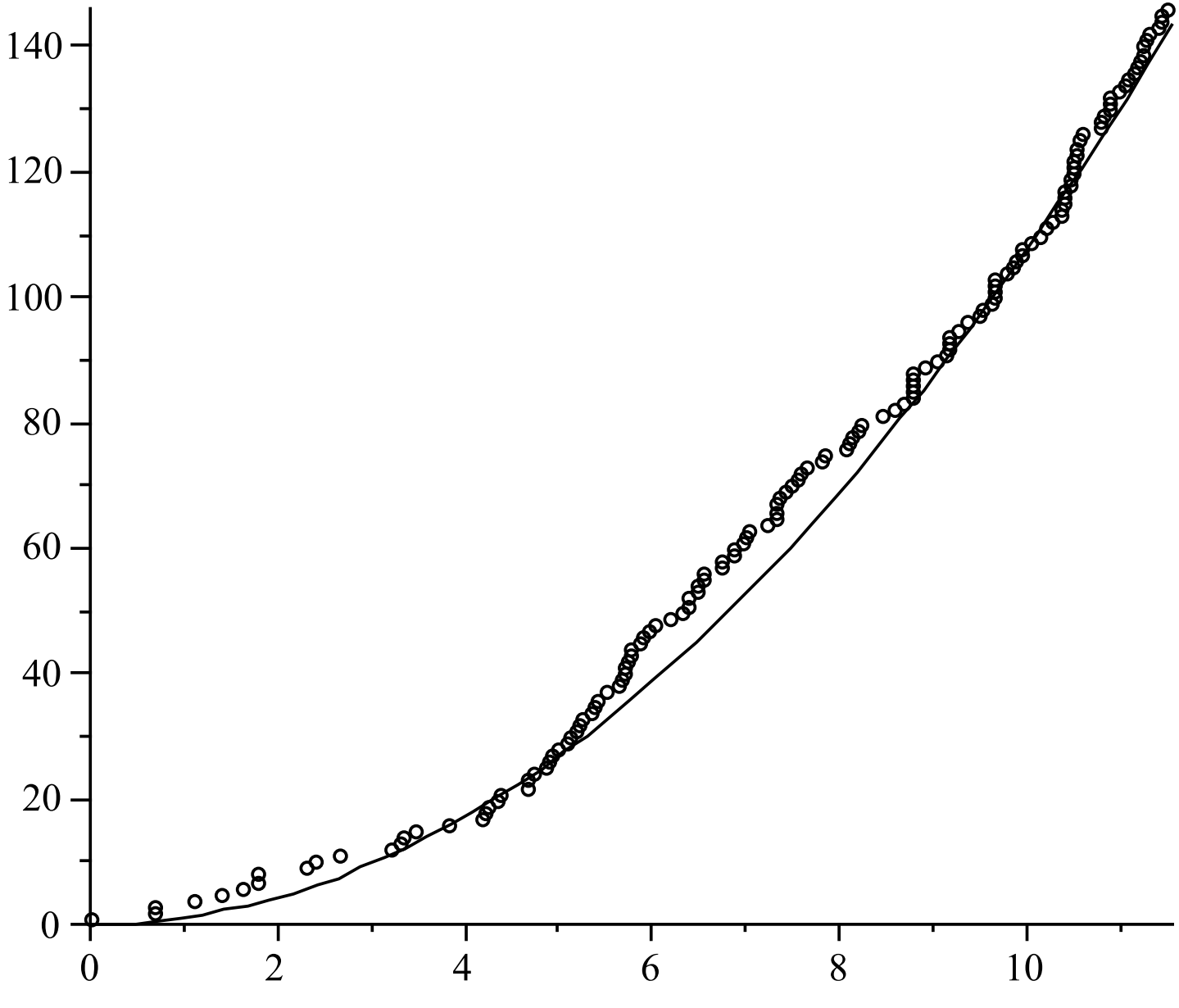
1/2*2.37*x^2-x
-2x^3w-x^2y^2-2x^2yz-x^2z^2-5x^2zw+xy^2z-2xy^2w+xyz^2-xyw^
-2xzw^2+4xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



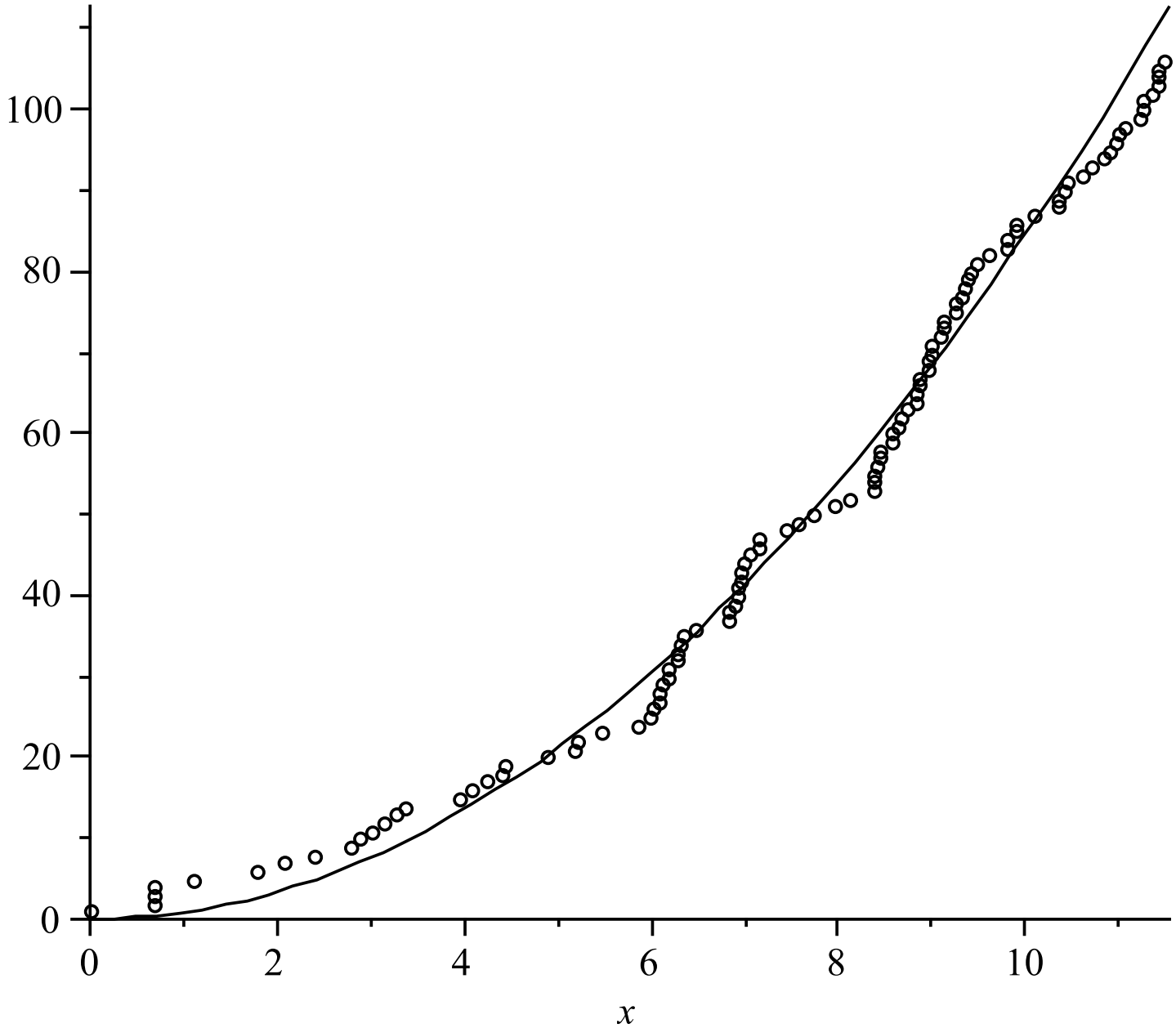
1/2*1.96*x^2+x
-2x^3w-x^2y^2-2x^2yz-x^2z^2-5x^2zw+xy^2z+3xy^2w+xyz^2-xyw^
-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



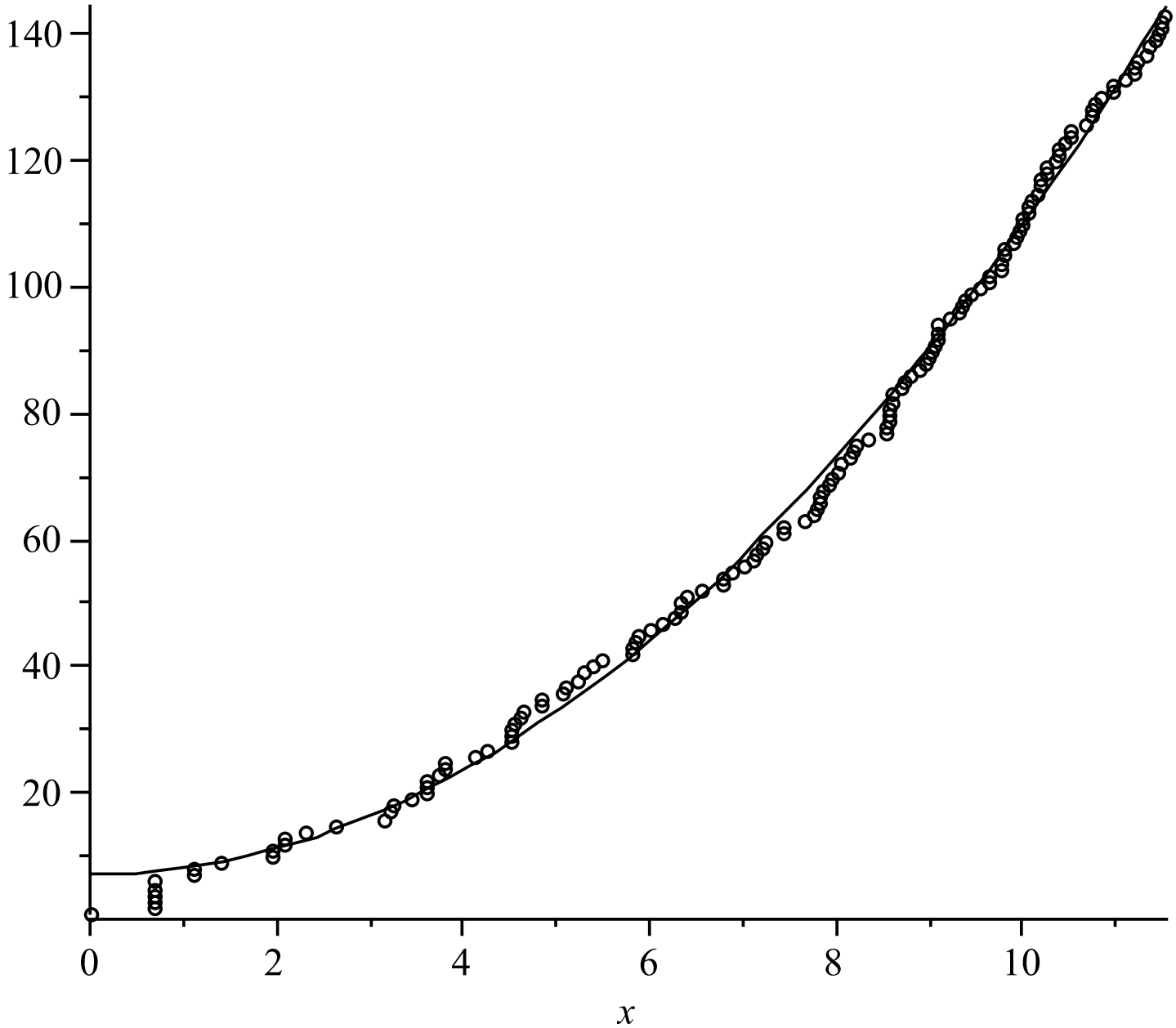
1/2*1.962*x^2+x
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-5xyzw-
xyw^2+5xz^2w-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



1/2*2.15*x^2
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-xyw^2
-2xzw^2+4xw^3+y^3w+2y^2z^2-5y^2zw-y^2w^2+2z^3w+w^4



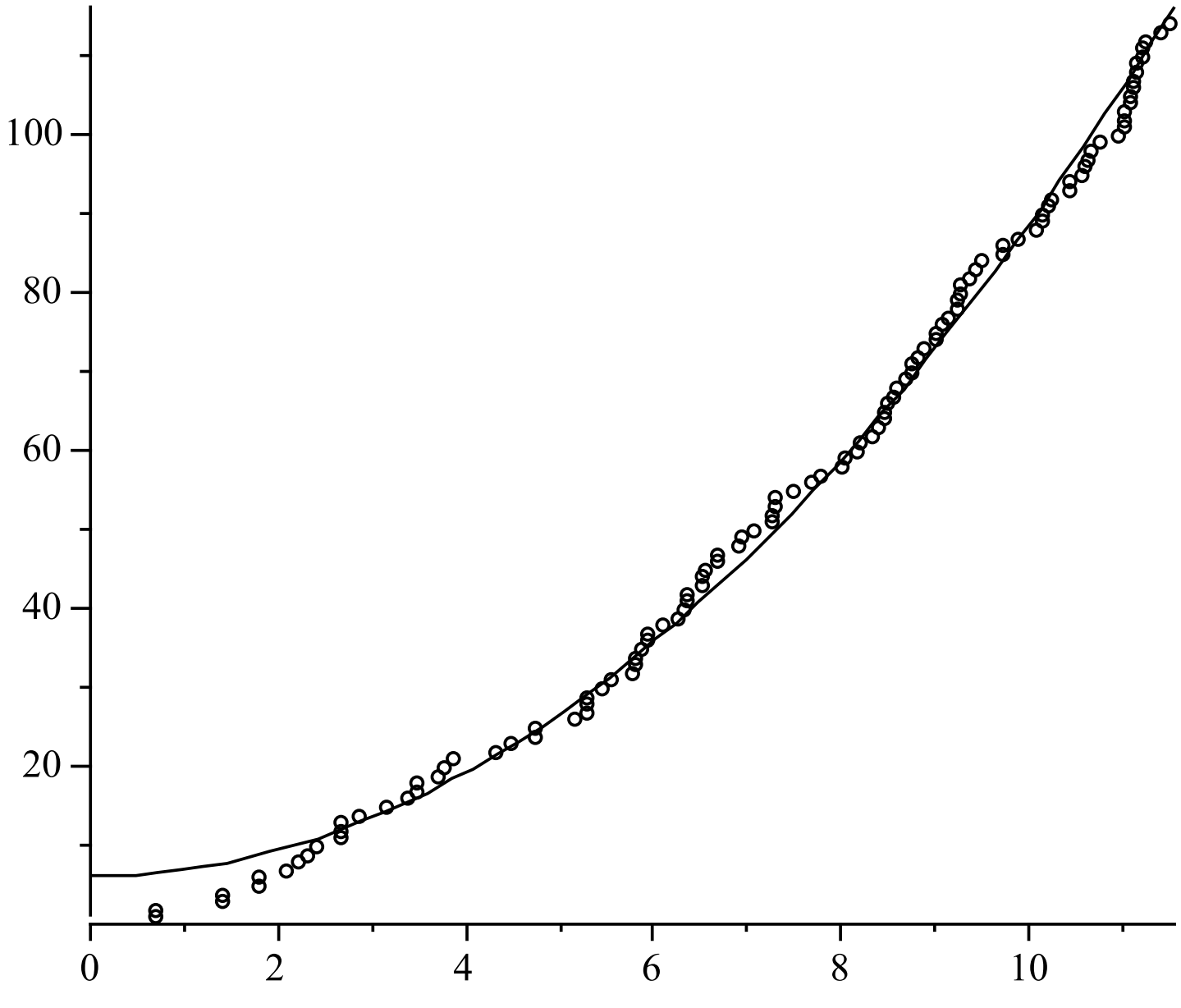
1/2*1.69*x^2
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-xyw^2
-2xzw^2+4xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w-5zw^3+w^4



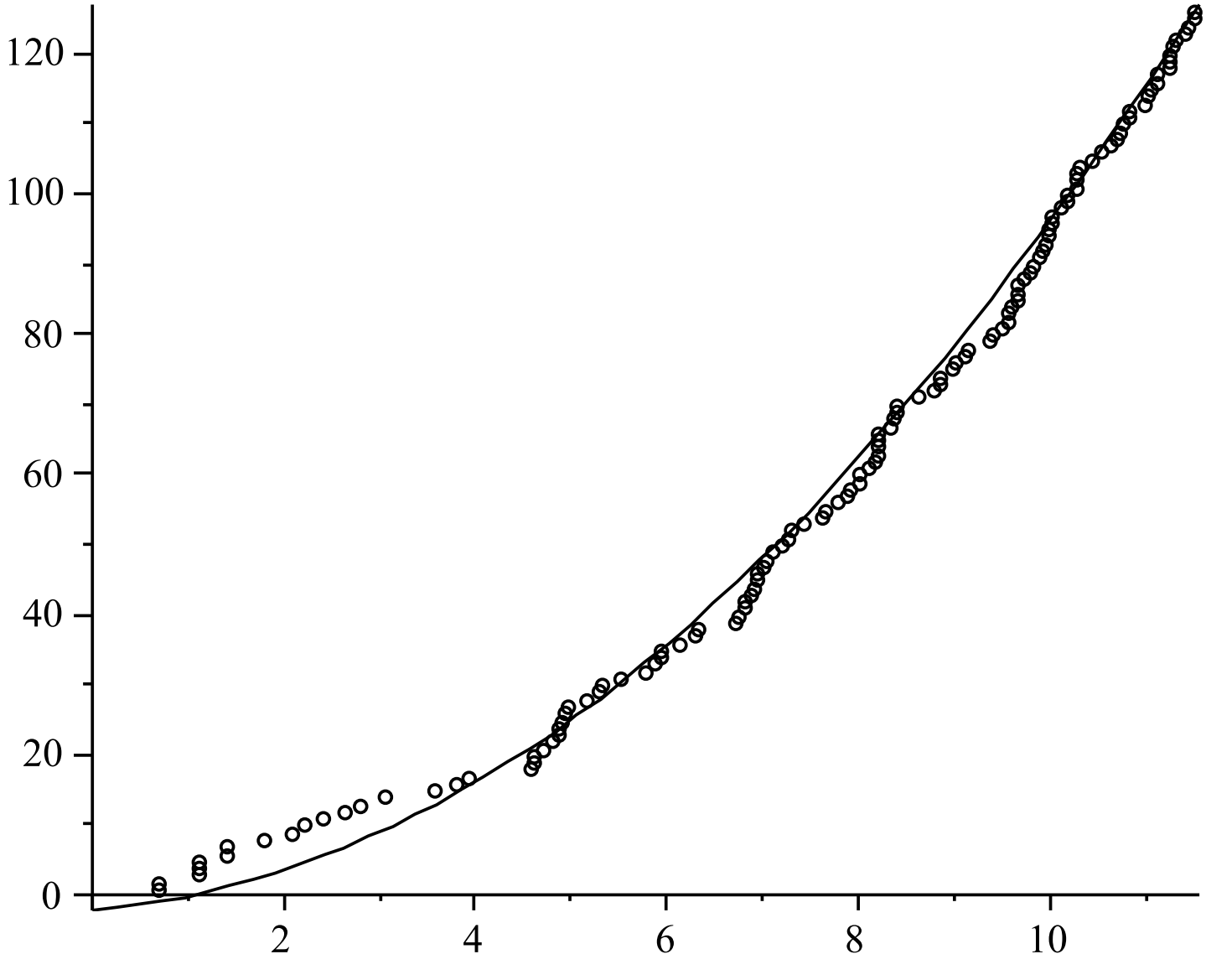
1/2*2.06*x^2+7

-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-xyw^2

-2xzw^2+4xw^3+y^3w+2y^2z^2-y^2w^2-5yz^2w+2z^3w+w^4

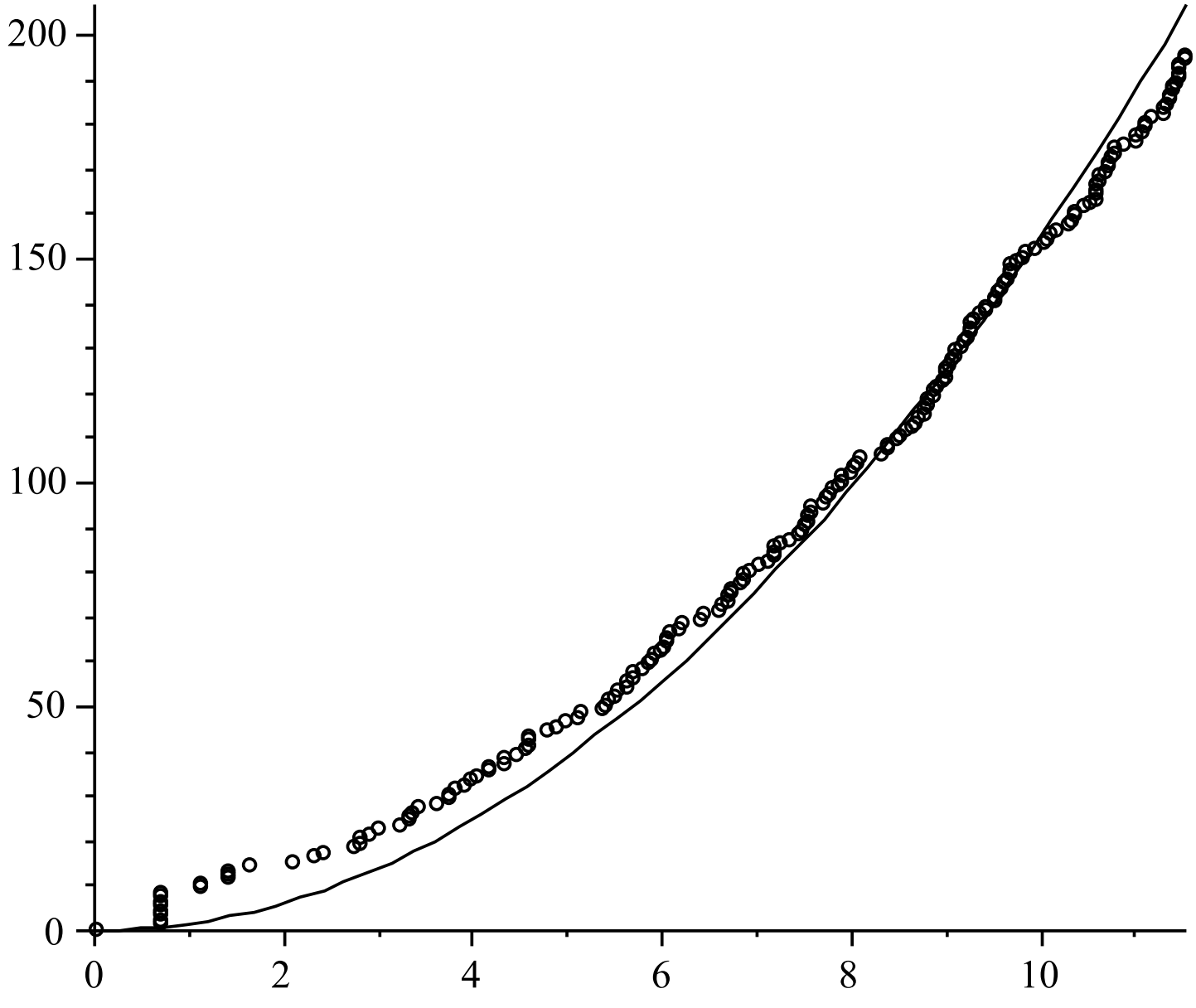


1/2*1.65*x^2+6
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-
xyw^2+3xzw^2+4xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



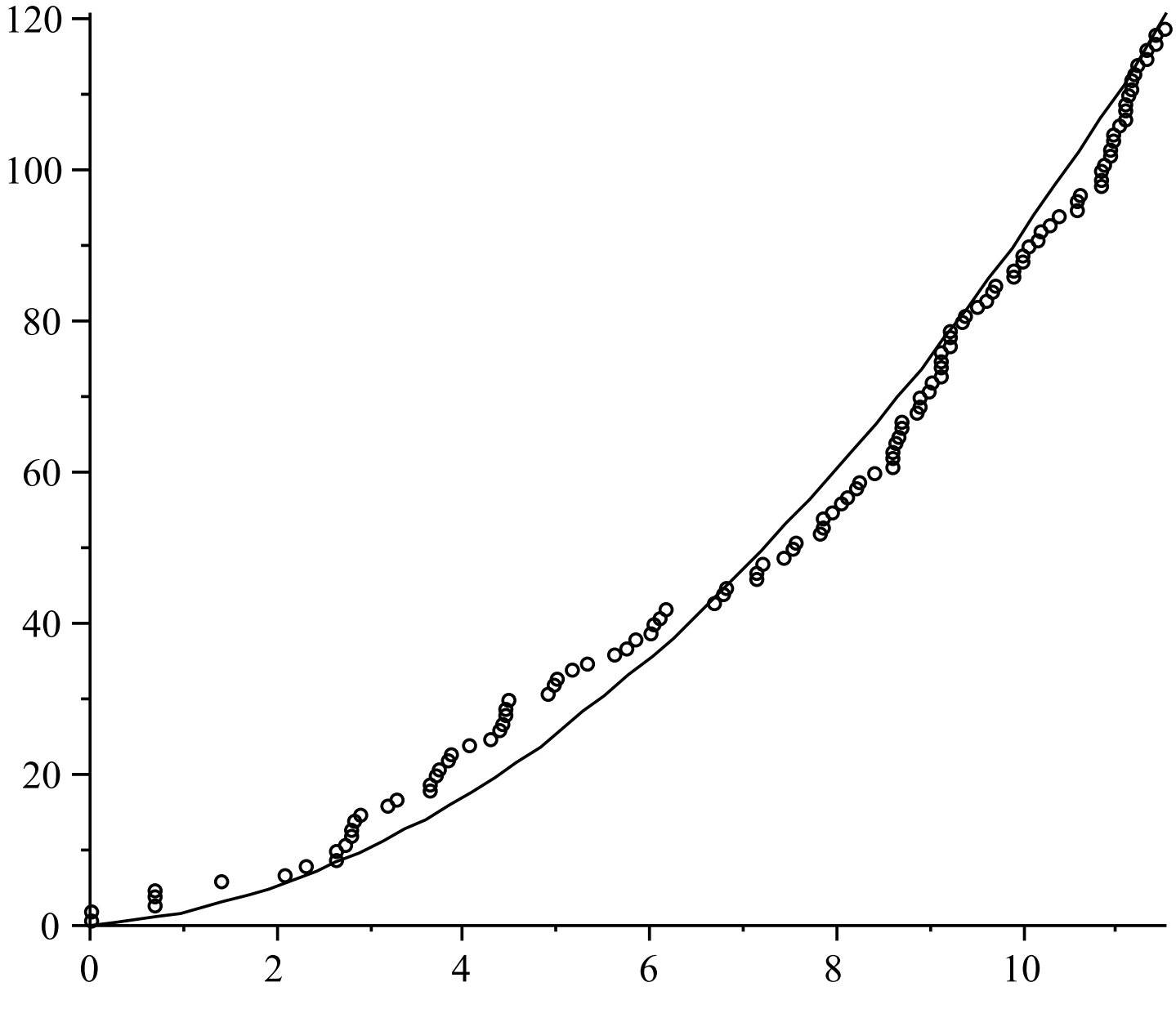
1/2*1.76*x^2+x-2

-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-xyw^2+5xz^2w-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+5yzw^2+2z^3w^4

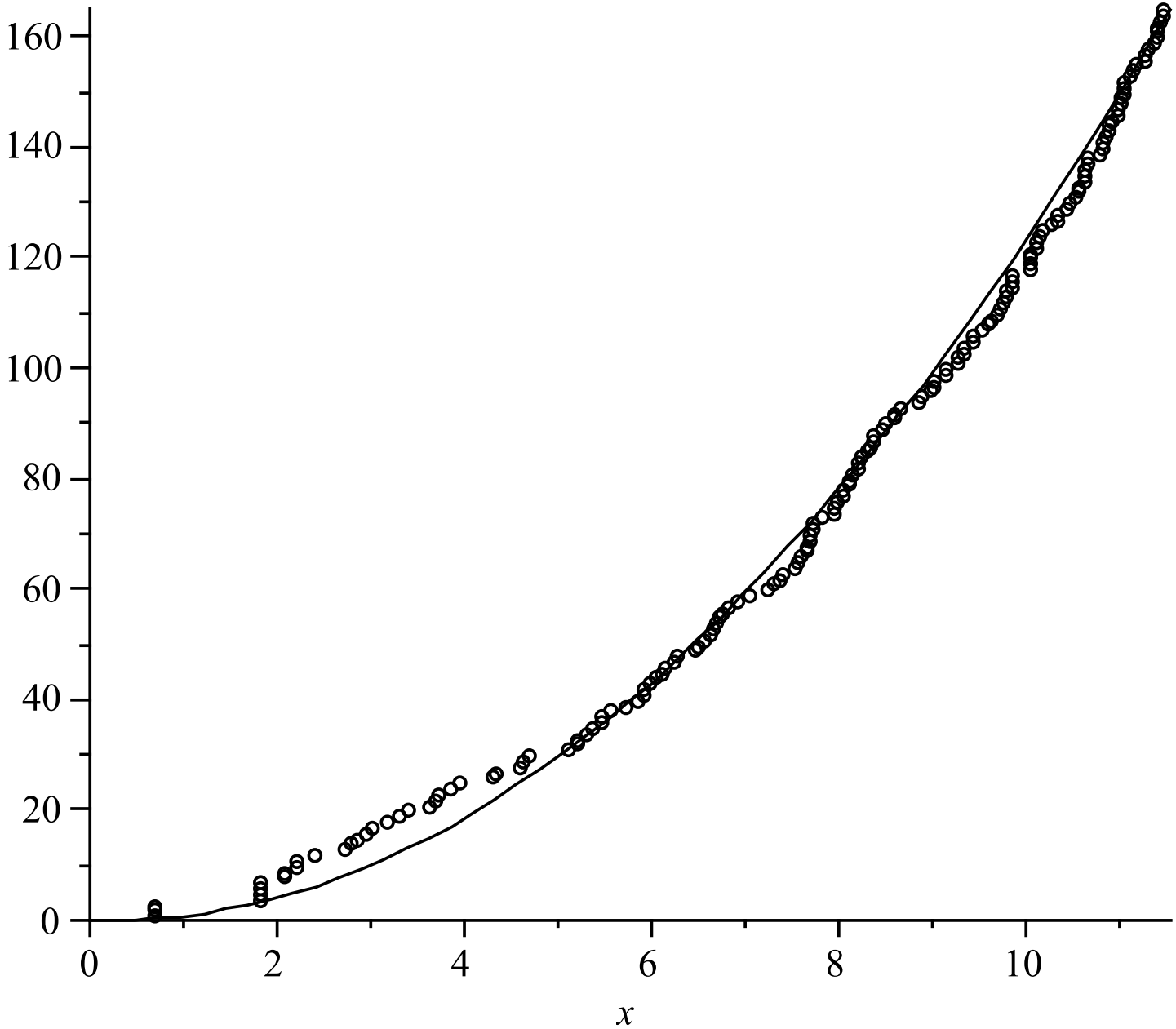


88

1/2*3.10*x^2
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z+3xy^2w+xyz^2-xyw^2-2xzw^
xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w-5z^2w^2+w^4

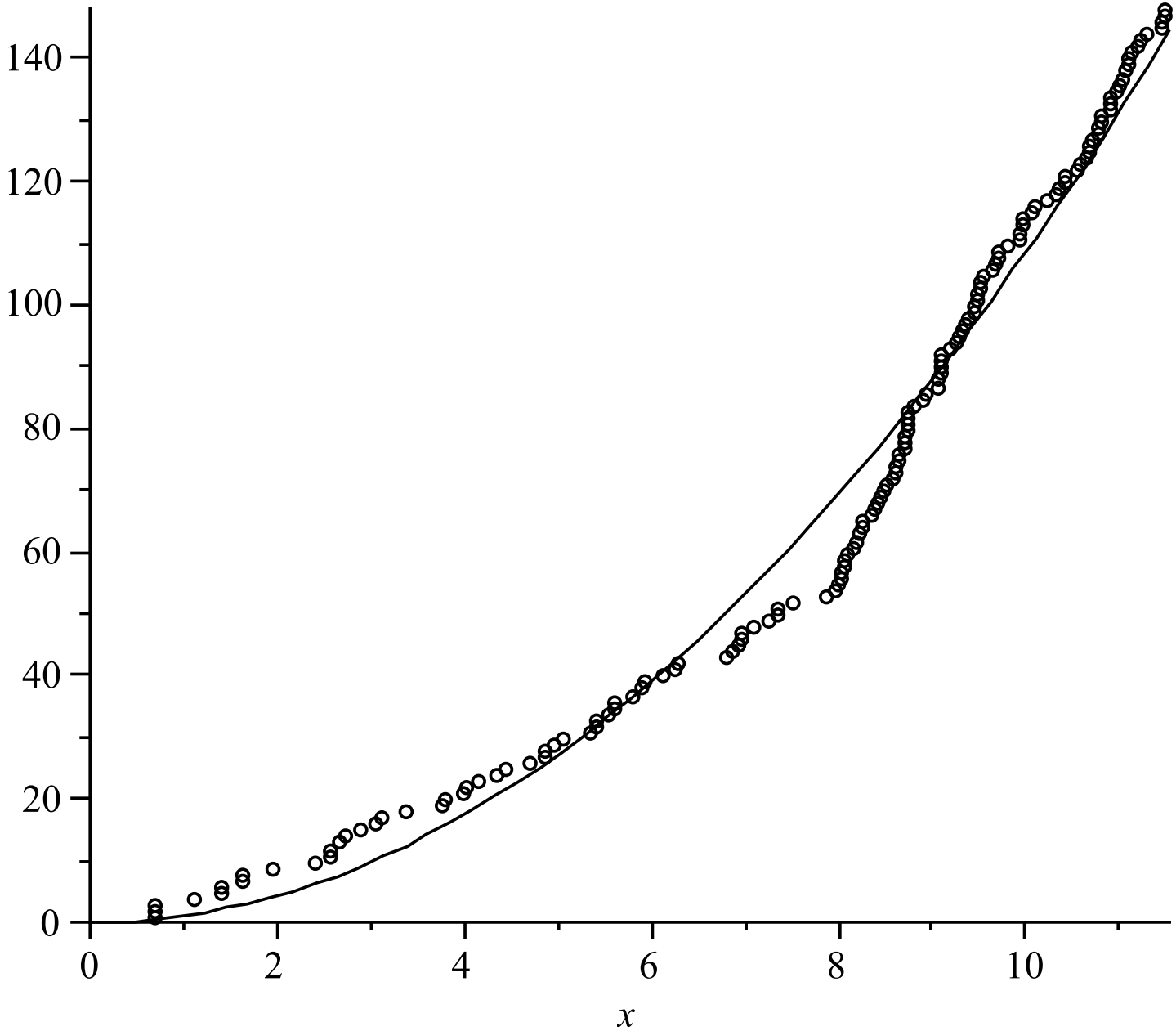


1/2*1.64*x^2+x
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z+3xy^2w+xyz^2-xyw^2-2xzw^
xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



90

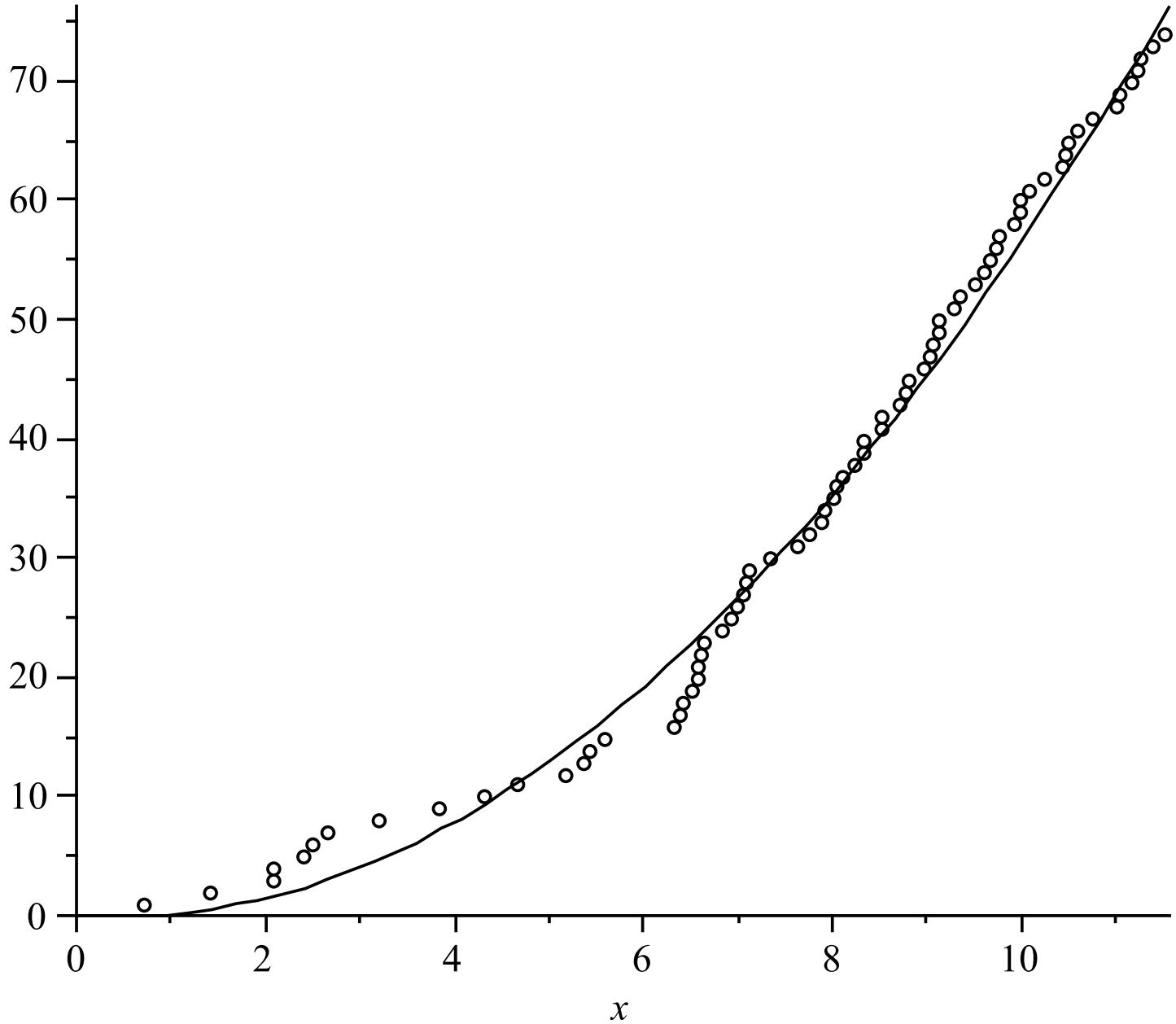
1/2*2.56*x^2-x/2
-2x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z+3xy^2w+xyz^2-
xyw^2+5xz^2w-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



1/2*2.17*x^2

3x^3w-x^2y^2-2x^2yz-5x^2yw-x^2z^2+xy^2z-2xy^2w+xyz^2-xyw^2

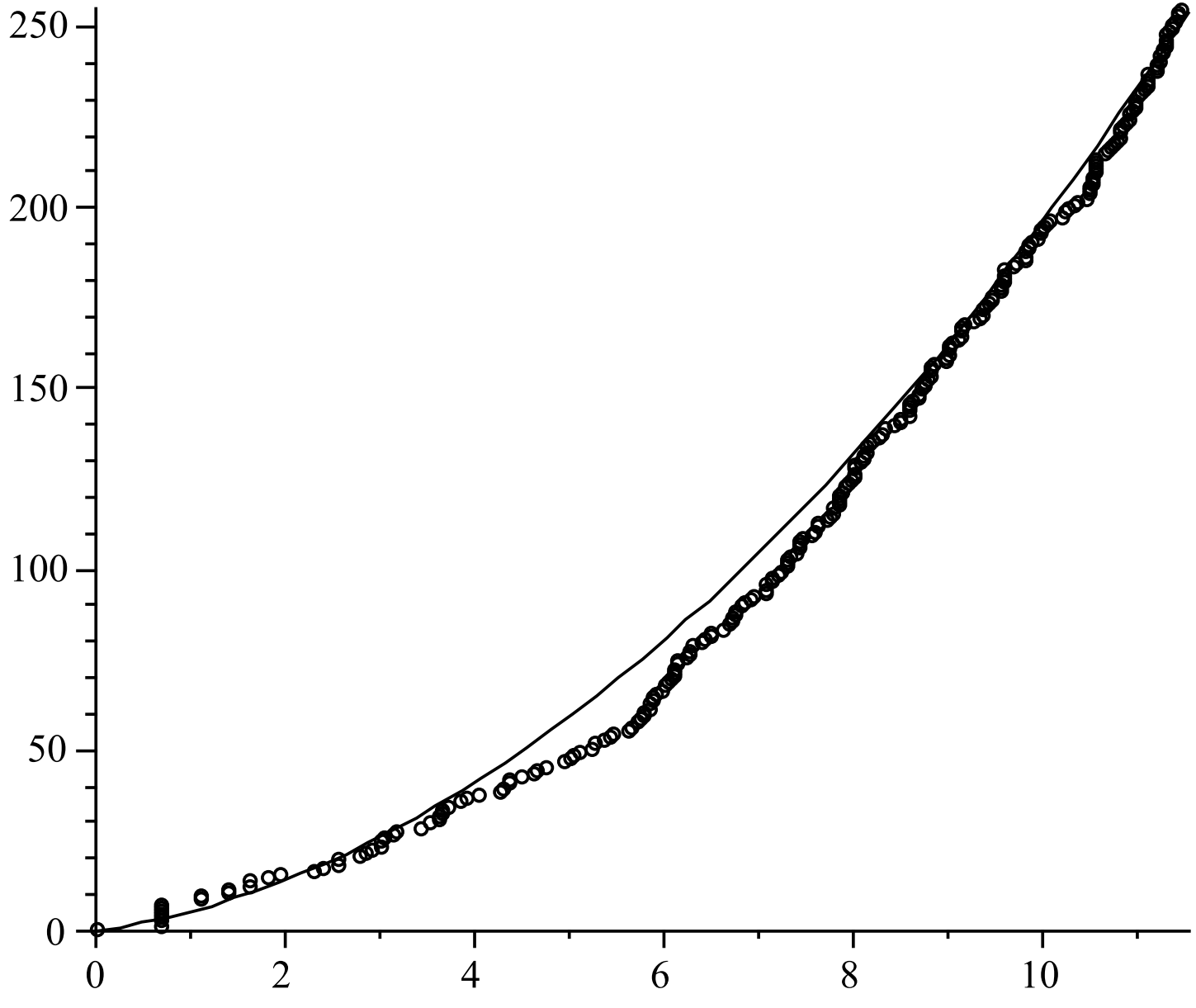
-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



1/2*1.23*x^2-x/2

3x^3w-x^2y^2-2x^2yz-x^2z^2-5x^2zw+xy^2z-2xy^2w+xyz^2-xyw^2

-2xzw^2-xw^3+y^3w+2y^2z^2-y^2w^2+2z^3w+w^4



93

1/2*3.12*x^2+4*x

3x^3w-x^2y^2-2x^2yz-x^2z^2+xy^2z-2xy^2w+xyz^2-xyw^2-2xzw^2-xy

y^3w+2y^2z^2-y^2w^2+2z^3w-5z^2w^2+w^4