Banff 5 dec. 2008

During this conference, many examples of K3's, e.g.,

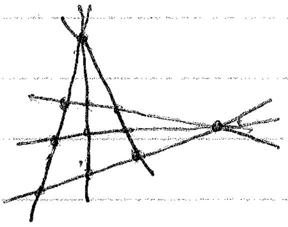
- smooth (diagonal) quartics
  - A. Sarti's talk, (with simplectic / non-simplectic autom's)
  - M. Schütt's talk: Dwork pencil, Fermat
  - Martin Bright & Evis Ieronymou: diagonal
  - Ronald v. Luijk: with $\rho = 1$, counting pts ---
- intersect. (1,1) & (2,2) in $\mathbb{P}^2 \times \mathbb{P}^2$ (with involutions), Joe Silverma
- K3 elliptic surfaces with section
  - T. Shioda ⟷ Inose pencil,
  - Noriko Yui ⟷ certain quotients of Fermat + non-simplectic autom
  - Thomas Dedieu: as counterexamples to his self-rat. map conjecture
- (2,2,2) in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
  - Arthur Baragat upto finite index all auto's
  - Serge Cantat dynamics of $\sigma_1 \sigma_2 \sigma_3$, pictures!
- Kummer's of $E_1 \times E_2$
  - Argentin's example in Olivier Wittenberg's talk

  [& more general Kummer of Jac $(g=2)$. Michael Stoll

[so: few lectures without examples,
Ekat. Amerik, ~~Yui~~ Arnaud Beauville, Chad
~~Thomas~~ ⊳]

Kummer of $E_1 \times E_2$: $y^2 = f_1(x_1) \cdot f_2(x_2)$

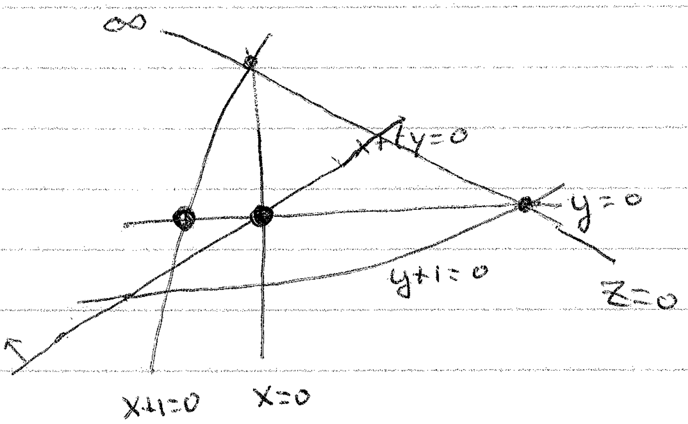
double cover of $\mathbb{P}^2$ ram. over



autom of order 25 example
of Noriko: also like this

This talk: more examples, of the same kind.

work with Bert van Geemen



$\underbrace{\underset{\text{lines}}{6} + \underset{E_p}{6} + \underset{E_{\underline{q}}^{l}}{6} + \text{posch}}_{19}$

parameter t ;

double cover : $\underset{\searrow \text{desingularize}}{X_t}$: $s^2 = x(x+1)y(y+1)(x+ty)$

blow up double pts : except. curve over P yields
rational -2 curve on double cover

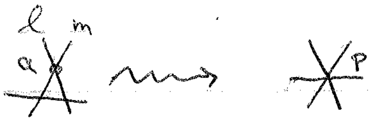
P triple pt : - - - exc. curve meets 3 lines l
blow up in such pt gives E_P^l

X_t is K3, $t \neq 0, \infty$

<u>Lemma</u>: 19-dim is saturated in $H_2(X, \mathbb{Z})$,

and $(\ )^{\perp} = \mathbb{Z}(2) \overset{\perp}{\oplus} \mathbb{Z}(2) \overset{\perp}{\oplus} \mathbb{Z}(-$

(by intersect. matrix + degeneration to $t=-1$

, then $E_a \longmapsto E_p + E_p^{\ell} + E_p^{m}$,

now orthog. $\boxtimes$

It follows that (at least generically), X_t is not Kummer

(since Km has different transc. lattice)

In X_t: two disjoint $-E_8$'s:

one:

$$\begin{tikzcd}[column sep=small] (x) \arrow[r, dash] & E_{001}^{x} \arrow[r, dash] & E_{001} \arrow[r, dash] \arrow[d, dash] & E_{001}^{y} \arrow[r, dash] & (y) \arrow[r, dash] & E_{-101} \arrow[r, dash] & (x+1) \\ & & E_{001}^{x+ty} & & & & \end{tikzcd}$$

Then $\exists$ Shioda-Inose structure,

i.e, simplectic invol. ι s.t

$$\begin{tikzcd} X_t \arrow[loop left, "\iota"] \arrow[dr, "\pi", "2:1"'] & & A = \text{ab. surface} \arrow[loop above, "-1"] \arrow[dl, "2:1"'] \\ & V_t = \text{Kummer!} & \end{tikzcd}$$

$$T_{X_t}(2) \xrightarrow[\sim]{\pi_*} T_{V_t}$$

Explicit?

methods to find ι :

① look at the two E_8's. ι^* has to interchange ....

moreover, ι has isolated fixpts (8 of them).

$\rightsquigarrow$ find

$$\iota : (s ; x : y : z) \longmapsto \left(\frac{-s}{x^2y^2z^2} , \frac{1}{y} : \frac{1}{x} : \frac{1}{z} \right)$$

② look for ell. filration :

one that works is (affine, $z=1$)

$$(x, y, s) \longmapsto \frac{s}{y(x+ty)} =: \alpha$$

~~filrat~~ Can be written as

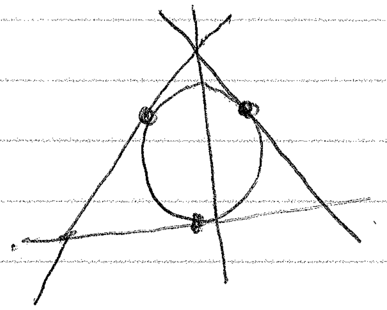
$$X_t : \eta^2 + (1-\alpha^2)\xi\eta + t\alpha^2\eta = \xi^3 + t\alpha^2\xi^2$$

Have $\iota : P \longmapsto (0,0) - P$

preserves filration.

So can calculate the quotient;

$$V_t \longleftrightarrow \eta^2 = (v+u+1)(u+t)(v^2-2u)u$$



4 lines + smooth conic

Kummer ??

Aside   Kummer of $\mathrm{Jac}(g=2)$.

Curve C: $y^2 = \prod(x-\alpha_i) = f(x)$

$\mathrm{Jac}\, C \overset{\sim}{\dashrightarrow} S^2C = \frac{C\times C}{\leftrightarrow}$,

Km : mod out by -1 = induced by hyperell invol ι on C,
so

$$\mathrm{Km} \overset{\sim}{\dashrightarrow} \frac{C\times C}{\langle \leftrightarrow, \iota\times\iota\rangle}$$

functions gen by x_i, y_i $(i=1,2,\ y_i^2 = f(x_i))$

$\leftrightarrow$: $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$

$\iota\times\iota$: $(y_1, y_2) \mapsto (-y_1, -y_2)$

Invariants : $\eta = y_1 y_2$

$\xi = x_1 x_2$

$\zeta = x_1 + x_2$

relation: $\eta^2 = \prod(x_1-\alpha_i)(x_2-\alpha_i)$

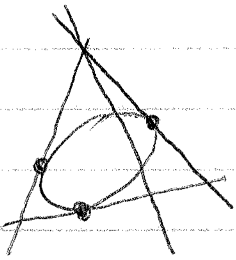
$= \prod(\xi - \alpha_i\zeta + \alpha_i^2)$

∴ double cover ramified over 6 lines

(& C can be reconstructed:
the 6 lines are tangent to conic
$\zeta^2 = 4\xi$,
the tangency pts are under $C\times C \to \mathrm{Km} \xrightarrow{2:1} \mathbb{P}^2$
$(\alpha_i, 0), (\alpha_i, 0) \longmapsto$ pt.

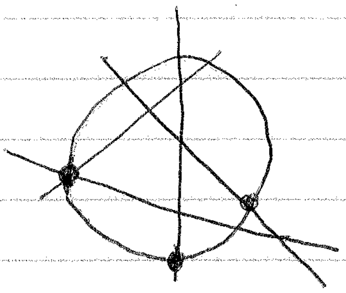
Nice, but my ram. locus $\neq$ 6 lines.

Remarkable solution, Bert van Geemen



apply Cremona with center the 3 tangent pts

result: line not cont. center pt
yields conic,
other comp. yield lines



Do again: center = triple pt + 2 pts

result: 6 lines, $\longleftrightarrow$ cover W_t

+ config. over $\mathbb{Q}(t)$

(but map not.)

so $\quad \varphi : V_t \xrightarrow{\cong} W_t$,

def. over quadr. ext of $\mathbb{Q}(t)$.

See : $\varphi^* \omega + (\varphi^\sigma)^* \omega \neq 0$

thus $\varphi + \varphi^\sigma$ is corresp over $\mathbb{Q}(t)$, preserving transcend. lattice.

Now W_t.

Indeed, the 6 lines are tangent to a conic

However, over $\mathbb{Q}(t)$, no rat. pts

$\Big\downarrow$ as before

$$C : \quad y^2 = x\,(x^2 - 4x + 4 + 4t)\,(x^2 + 4x + 4 + 4t)$$

C comes with invol $\quad x \mapsto \dfrac{4+4t}{x}$

① over quadr ext. $\quad W_t \cong \mathrm{Km}(\mathrm{Jac}\, C)^{(-t-1)}$

same trick : $\varphi + \varphi^\tau$ is / $\mathbb{Q}(t)$ + regular 2-form not $\mapsto 0$.

fun

finally,

quotient C by invol.

yields

$$E_t : \quad y^2 = (x-1)\left(x^2 - \frac{1}{t+1}\right),$$

one gets

$$\mathrm{Km}\left(\mathrm{Jac}(C)^{(-t-1)}\right. \xrightarrow{\cong} \mathrm{Km}\left(E_t \times E_t^{(t+1)}\right)$$

↑

3^{rd} time: over ext,

same trick works.

Conclusion: $\forall t$ s.t E_t ellipt, non CM:

3 dim transc. part is irred. HS / Galois rep $/\mathbb{Q}(t)$.

so same for X_t, with $\cong$ transc. part.

Coroll.

<u>Remark</u>: Ahlgren, Ono, Penniston 2002, Amer. J.:

$t \neq 0, 1, \in \mathbb{F}_q$, char $\neq 2$ $\quad \chi : \mathbb{F}_q^* \longrightarrow \pm 1$ quadr., $\chi(0) = 0$

$$(t+1)\sum_{x,y} \chi\big(xy(x+1)(y+1)(x+ty)\big) + \underline{q} = \left(\sum_{x \in \mathbb{F}_q} \chi\left((x-1)\left(x^2 - \tfrac{1}{t+1}\right)\right)\right)^2$$