Representations of the symmetric group

The theory of linear representations is pervasive in many areas of mathematics and science; its goal is essentially to understand all the ways in which a given group can act on a vector space. More precisely, a *(complex) representation* of a finite group G on a finite-dimensional complex vector space V is a map $G \times V \to V$, which we write $(g, v) \mapsto g \cdot v$, such that

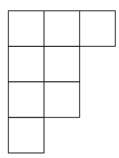
$$g \cdot (\alpha v + \beta w) = \alpha(g \cdot v) + \beta(g \cdot w), \quad g \cdot (h \cdot v) = (g \cdot h) \cdot v, \quad \text{and} \quad 1_G \cdot v = v$$

for all group elements $g, h \in G$, scalars $\alpha, \beta \in \mathbf{C}$ and vectors $v, w \in V$. If we choose a basis for V, such an action gives rise to a group homomorphism of G into the group of $\mathrm{GL}_n(\mathbf{C})$ of $n \times n$ invertible matrices, where $n = \dim V$ is called the *degree* of the representation.

A representation is said to be *irreducible* if it cannot be written as a direct sum of two representations of smaller degree. The irreducible representations are the basic building blocks of all representations, for any representation can be decomposed as a direct sum of irreducible ones. Moreover, there is only a finite number of these irreducible representations (up to isomorphism), and the *theory of characters* allows one to easily decompose any given representation into its irreducible constituents.

The problem of understanding all the representations of G is thus reduced to that of understanding the irreducibles. In general, we know that their number is equal to the number of conjugacy classes in G, but do not really have a comprehensive description of them. For the symmetric groups S_n, however, we can parametrize explicitly the irreducible representations by the conjugacy classes in a way which is uniform for all n.

The conjugacy class of a permutation can be described by the sizes of the cycles in its disjoint cycle decomposition, which is a *partition* of n into positive integers. For example, the permutation $(153)(26)(48) \in S_8$ gives rise to the partition $8 = 3 + 2 + 2 + 1$. Partitions are often represented graphically by the means of *Young diagrams*, in which the number of boxes in each row represent a different part, e.g. the following for $(3, 2, 2, 1)$.



One can construct an irreducible representation V_λ associated to such a diagram λ by considering the action of S_n on certain spaces of polynomials, and obtain this way a complete set of representatives for the irreducible representations of S_n.

A nice project would be to understand this construction, as well as certain results relating the combinatorics of Young diagrams (and of certain embellished versions of them called *Young tableaux*) to the representation theory of S_n. The amount of general representation theory which is needed can be adapted to the taste and background of the student (a previous acquaintance with representations is not requisite).

Possible reference: B. Sagan, *The symmetric group: Representations, combinatorial algorithms and symmetric functions*, GTM 203, Springer, 2001.

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