Linear representations of finite groups and random walks

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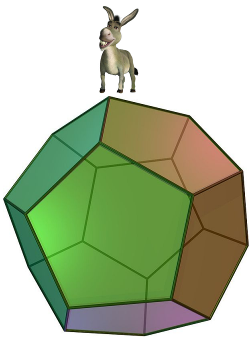
Linear representations

Let G be a finite group. A linear representation of G (over $\mathbb{C}$) is a group morphism $\rho : G \longrightarrow \mathrm{GL}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space. It is said to be irreducible if there is no nontrivial subvector space which is stable under G. Irreducible representations play more or less the same role in the theory as prime numbers in $\mathbb{Z}$. The first basic result is that every representation is the direct sum of irreducible representations. Moreover we can prove that there are (up to isomorphism) only a finite number of distinct irreducible representations of a given group G.

If ρ is a representation, its *character* is by definition the function $G \longrightarrow \mathbb{C}$ which maps an element $g \in G$ to $\mathrm{Tr}(\rho(g))$. Characters are a very powerful tool for studying representations. Among other facts: two representations are isomorphic if and only if they have the same character, we can see on the character if a given representation is irreducible or not, and if it is not irreducible, we can compute very easily its decomposition into irreducibles.

Application to random walks

Imagine that a donkey is walking randomly on the vertices of a dodecahedron. At the step 0, he is at the vertex number one. At each step, he moves to an adjacent vertex which is chosen randomly.



Can you estimate how close to random is his position after n steps ? In other words, if p_n^i is the probability for beeing at the vertex number i after n steps, how fast does $\sum_{i=1}^{20} |p_n^i - \frac{1}{20}|$ go to zero ? An answer can be given with standard linear algebra and heavy computations, but the language of representations allows for a much better understanding and easier computations.

What can be done in a bachelor seminar ?

The theory of linear representations of finite groups and characters has to be learned. For this purpose, the first chapter of the book of Serre, *Linear representations of finite groups*, may serve as a reference. As examples, one can compute all the irreducible representations for the isometry groups of some Platon solids. After that, the above application to random walks could be treated. Or, we can imagine some excursion into discrete Fourier analysis. An other possibility is to study some applications to spectroscopy.

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