Rational points on varieties, part II (surfaces)

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1. Introduction

- Introduction to questions about arithmetic geometry [5, 8, 15, 18].
- Counting points on varieties (Batyrev–Manin conjectures) [2, 12], [5, Section F5.4].
- Theorem of Segre–Manin [6, 13, 16, 17] and [10, Theorem 29.4 and 30.1].
- Conjecture: For every $t \in \mathbb{Q}$ there are $x, y, z, w \in \mathbb{Q}$ such that $t = \frac{x^4 - y^4}{z^4 - w^4}$, [9, Conjecture 2.5].

2. Picard group and canonical divisor

- Smoothness of points is defined by Jacobian criterion, or by regularity of the local ring, over the algebraic closure [4, Section I.5], [5, Section A1.4], or over the field of definition of the point [14, Proposition 3.5.22]. See also Exercise 3 below.
- Differentials [3, Chapter 16], [4, Section II.8], [5, Section A1.4], [7, Section XIX.3].
  - In particular: if K is a field extension of a field k, then
  $$\dim_K \Omega_{K/k} \geq \text{tr.d.}(K/k)$$
  with equality if and only if K is separably generated over k, i.e., there is a transcendence basis $\{x_\lambda\}$ for K/k, such that K is a separable algebraic extension of $k(\{x_\lambda\})$ [4, Theorem II.8.6A], [11, Theorem 59, p. 191]. For more about separably generatedness, see [3, Section A1.2].
  - If X is a smooth and irreducible variety over k, then the function field $k(X)$ is separably generated over k. (Proof: if X is smooth, then it is geometrically reduced, so the field extension $k(X)/k$ is separable [14, Proposition 2.2.20]. Since $k(X)/k$ is finitely generated, this implies that $k(X)/k$ is separably generated [3, Section A1.2].)
- Exterior product and exterior algebra [3, App A2.3], [7, Chapter XIX].
- Discrete valuation rings and regular local rings [1, Chapters 9 and 11, in particular Proposition 9.2 and Theorem 11.22], [3, Sections 10.3 and 11.1].
- Localization of a regular local ring at a prime ideal is regular [3, Corollary 19.14], [4, Theorem II.8.14A], [11, p. 139].
- Divisors and Picard group [4, Section II.6], [5, Section A2].
- Canonical divisor of complete intersection [4, Proposition II.8.20, Example II.8.20.3, Exercise II.8.4], [5, Exercise A.2.7].

Exercises

(1) Suppose P is a smooth point on a variety X over a field k. Set $n = \dim X$. Let $x_1, x_2, \ldots, x_n$ be local parameters at P, i.e., they generate the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Let $y_1, y_2, \ldots, y_n$ be local parameters as well. Show that there exists a function $f \in \mathcal{O}_{X,P}^*$ such that
$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = f \cdot dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n.$$

(2) For any $d \in \{2, 3, 4, 5\}$ and $t \in \{2, 3, 4, 5, 6\}$, and your choice of integer $M \geq 6$, count, for all $0 \leq m \leq M$, the number of rational points $[x : y : z : w] \in \mathbb{P}^3$ of height at most 2^m on the surface given by $t(z^d - w^d) = x^d - y^d$ that do not lie on the curves given by $z^d - w^d = x^d - y^d = 0$. (Give a table for each d.)

(3) Let $p > 2$ be a prime and set $k' = \mathbb{F}_p(s)$. Set $t = s^p$ and let $k \subset k'$ be the field $\mathbb{F}_p(t)$, so that k' is isomorphic to $k[u]/(u^p - t)$. Let $\bar{k}$ be an algebraic closure of k'. Let $C \subset \mathbb{A}^2_k(x, y)$ be the affine curve over k given by
$$y^2 = x^p - t.$$
Let $P \in C(k')$ be the point $P = (s, 0)$.

(a) Show that C is irreducible.
(b) Use the Jacobi criterion to show that C is not smooth at P, and C is smooth at all other points in $C(\overline{k})$.
(c) The local ring $\mathcal{O}_{C,P}$ (over k') is isomorphic to the localization of $k'[x,y]/(-y^2+x^p-t)$ at the maximal ideal $(y, x-s)$. Show that $\mathcal{O}_{C,P}$ is not regular.
(d) Show that the localization of $k[x,y]/(-y^2+x^p-t)$ at the maximal ideal (y, x^p-t) is regular.

(bonus) Suppose $X \subset \mathbb{P}^n$ is a smooth complete intersection over k of t hypersurfaces of degrees $e_1, \ldots, e_t$. Let $H \in \operatorname{Div} X$ be a hyperplane section of X, i.e., there is a hyperplane H' of $\mathbb{P}^n$ such that $H = H' \cap X$. You may assume that $H = H' \cap X$ is irreducible, given by $x_0 = 0$, and that for each i, the function $x_0/x_i \in k(X)$ is a generator of the maximal ideal of the local ring $\mathcal{O}_{X,H}$.
(a) Show that the canonical class of X is the class of $(-n-1+e_1)H$ if $t=1$.
(b) Show that the canonical class of X is the class of $(-n-1+\sum_i e_i)H$ in general.
See [5, Exercise A.2.7].

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