Rational points on varieties, part II (surfaces)

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1. Intersection Theory and Blowing up

- Extended moving lemma and intersection numbers being constant within divisor classes [6, A2.3.1].
- Intersection pairing on $\operatorname{Pic} X$ when X is normal and projective surface [5, Theorem V.1.1], [6, Section A2.3], [7, Appendix B].
- Self intersection: $C \cdot D = \deg_C \mathcal{L}(D) \otimes \mathcal{O}_C$ restricted to $C = D$ [5, Lemma V.1.3].
- $X \subset \mathbb{P}^n$ a surface, $H \in \operatorname{Div} X$ a hyperplane section, $C \subset X$ a curve. Then $H^2 = H \cdot H = \deg X$ [6, A2.3], and $H \cdot C = \deg C$. [5, Exercise V.1.2].
- Adjunction formula $2g(C) - 2 = C \cdot (C + K_X)$ for smooth curve C on smooth projective surface X [5, Proposition V.1.5], [6, Theorem A4.6.2]. If C is not smooth, then you should use the *arithmetic genus* instead.
- Riemann-Roch for surfaces [5, Theorem V.1.6], [6, Theorem A4.6.3].
- Kodaira Vanishing [5, Remark II.7.15, Exercise V.4.12], [6, Remark A4.6.3.2].
- Let $f\colon S \to S'$ be a surjective morphism of smooth, irreducible, projective surfaces that is generically finite of degree d. Then for any $D, D' \in \operatorname{Div} S'$, we have $(f^*D) \cdot (f^*D') = d(D \cdot D')$ [3, Proposition I.8] for characteristic zero, [6, A2.3.2] for f finite, combine [11, Propositions 5.2.32 and 9.2.11] for the general case.
- Blow-up [3, Section II.1], [5, Section I.4], [6, A1.2.6.(f)].
  - effect on Pic.
  - effect on canonical divisor.
  - self intersection is -1.

2. Exercises

(1) Let X be a nice surface over a field k, and $P \in X(k)$ a point. Let $\pi\colon \tilde{X} \to X$ be the blow-up of X at P. Suppose $C \subset X$ is an irreducible curve with multiplicity m at P. Let $\tilde{C}$ be the strict transform of C on $\tilde{X}$.
  (a) Show that we have $\tilde{C}^2 = C^2 - m^2$.
  (b) Show that the arithmetic genera of C and $\tilde{C}$ are related by $p_a(\tilde{C}) = p_a(C) - \frac{1}{2}m(m-1)$.
  (c) Consider $X = \mathbb{P}^2$. Show that an irreducible curve of degree d in $\mathbb{P}^2$ has at most $\frac{1}{2}(d-1)(d-2)$ singular points, and that if equality holds, then all singular points are double points. You may use that the arithmetic genus of a nice curve is nonnegative.
  (d) Suppose $k = \mathbb{Q}$. Let $C \subset \mathbb{P}^2(x, y, z)$ be given by $x^2z^2 = x^4 + y^4$. Then C is smooth outside the point $P = [0 : 0 : 1]$, which is a double point on C. (This type of singularity is called a *tacnode*.) Show that the strict transform $\tilde{C}$ on the blow-up $\tilde{X}$ of X at P has one singular point, say R, and that R is a node on $\tilde{C}$.
  (e) Let $\tilde{\tilde{X}}$ be the blow-up of $\tilde{X}$ at R, and let $\tilde{\tilde{C}}$ be the strict transform of $\tilde{C}$ on $\tilde{\tilde{X}}$. Show that $\tilde{\tilde{C}}$ is smooth and has genus 1.

(2) It is a fact that if Y is a nice surface and $E \subset Y$ is a nice curve that is isomorphic to $\mathbb{P}^1$ and has self intersection $E^2 = -1$, then E is an *exceptional curve* in the sense that there exists a nice surface X and a point P on X and a morphism $\pi\colon Y \to X$ that is (isomorphic to) the blow-up of X at P, with E corresponding to the exceptional curve above P.
  (a) Let Y be a nice surface with canonical divisor K_Y, and $E \subset Y$ a nice curve. Show that E is an exceptional curve if and only if $E^2 = E \cdot K_Y = -1$.
  (b) Let Y be a nice surface on which the anticanonical divisor $-K_Y$ is ample. Let $E \subset Y$ be a nice curve. Show that E is an exceptional curve if and only if its self intersection E^2 is negative.

(3) Let $C \subset \mathbb{P}^3$ be a nice curve that is the complete intersection of two surfaces of degrees d and e. Using intersection theory on one of these two surfaces, show that the genus of C equals $1 + \frac{1}{2}de(d + e - 4)$.

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