Points, lines, planes and more

Make your own pictures!

1. The plane

We define $\mathbb{R}^2$ to be the set of all pairs of real numbers. A pair is understood to be ordered, so the pairs $(2, 7)$ and $(7, 2)$ are different. In set notation we write

$$\mathbb{R}^2 = \{\, (r, s) \;:\; r, s \in \mathbb{R}\,\}.$$

An element of $\mathbb{R}^2$ is also called a *vector*. This notion will be used in much greater generality soon. The two elements in a pair $v \in \mathbb{R}^2$ are called the *components* of v. Note that in the previous sentence the symbol v referred to a (whole) pair in $\mathbb{R}^2$; this is convenient when the separate components of v do not need to be referred to separately. It would have taken more notation to say that the elements r and s are the components of the pair (r, s).

There are many ways to think of $\mathbb{R}^2$. For example, the two components might indicate the amount of euros spent during a given year on food and clothing respectively. If person A spent 1500 on food and 1000 on clothing, then the pair $(1500, 1000)$ is associated to A. If person B spent 16000 on food and 200 on clothing, then the pair $(16000, 200)$ is associated to B. To their combined spending the pair $(17500, 1200)$ is associated.

We can also think of $\mathbb{R}^2$ as the (x, y)-plane. The pair (r, s) corresponds to the point whose x-coordinate equals r and whose y-coordinate equals s. Of course, none of the names of the variables matters here; we could also talk about a point in the (a, b)-plane, or the (food, clothing)-plane. We will often just talk about *the plane*, without specifying names for the two coordinates, and identify it with $\mathbb{R}^2$ by identifying a point in the plane with its pair of coordinates.

Yet another interpretation of elements of $\mathbb{R}^2$ comes from considering straight arrows from one point in the plane to another. The arrow from the point $P = (p_1, p_2)$ to the point $Q = (q_1, q_2)$ represents the vector $(q_1 - p_1, q_2 - p_2)$. Be aware that different arrows may represent the same vector.

Example 1.1. *The arrow from the point* $(1, 2)$ *to the point* $(2, -1)$ *represents the vector* $(1, -3)$*. So does the arrow from the point* $(-2, 0)$ *to* $(-1, -3)$*, as well as the arrow from the origin* O *to the point* $(1, -3)$*.*

Note that two arrows represent the same vector in $\mathbb{R}^2$ if and only if the arrows have the same direction and the same length. This interpretation is closely related to the identification of $\mathbb{R}^2$ with the plane as above: a vector or pair $v \in \mathbb{R}^2$ corresponds to a point P in the plane, if and only if the arrow from the origin O to P represents v, cf. example 1.1. In other words, it is easy to read off which vector an arrow starting at O represents: the arrow from O to a point P represents the vector that is the pair of coordinates of P. This is why the arrows starting at O play a special role in this context.

Although for us a pair $v \in \mathbb{R}^2$ needs no interpretation to be called a vector, people often think of arrows when talking about vectors. In physics, for example, a *speed vector* has a direction and a magnitude, just like a *force vector*; they are represented by any arrow with corresponding direction and appropriate length (which depends on the choice of a unit).

We define the sum of two vectors by taking the sum on each component. It is stated in the following definition, just as the scalar product of a vector, which is also taken componentwise.

Definition 1.2. *Given a scalar $\lambda \in \mathbb{R}$ and two elements $v, w \in \mathbb{R}^2$ with $v = (v_1, v_2)$ and $w = (w_1, w_2)$, we define the* sum $v + w$ *of v and w and the scalar product λv of λ and v by*

$$v + w = (v_1 + w_1, v_2 + w_2), \qquad \textit{and} \qquad \lambda v = (\lambda v_1, \lambda v_2).$$

We also abbreviate the vector $(0, 0)$ to 0.

Example 1.3. *Let us look back at the interpretation of $\mathbb{R}^2$ as spending on food and clothing. The pair $a = (1500, 1000)$ was associated to person A, while the pair $b = (16000, 200)$ was associated to person B, meaning that during the year in question, A and B spent 1500 and 16000 euros respectively on food and 1000 and 200 on clothing. Their combined spending corresponds to the sum $a+b = (17500, 1200)$.*

Note that the symbol 0 now refers to both the number $0 \in \mathbb{R}$ and the vector $(0, 0) \in \mathbb{R}^2$. The context will never leave doubt which element is meant. If v denotes a vector in $\mathbb{R}^2$, then in the expresssion $0v$, for instance, the symbol 0 denotes the real number zero, because we have not defined a product of two vectors; in the expression $0 + v$, the symbol 0 denotes $(0, 0)$, as we have not defined the sum of a scalar and a vector.

In fact, not only the symbol 0 has a double meaning, but the symbol $+$ as well. If v and w are vectors, then the $+$ in the expression $v + w$ indicates the sum of two vectors, while the same symbol in the expression $3+2$ indicates the sum of two real numbers. Again the context will never leave doubt which sum is meant.

Because addition is just defined coordinatewise, the newly introduced sum behaves according to many of the same rules as the usual sum. Similarly, the vector 0 has the same properties as the number 0. Some of these rules and properties are summarized in the following proposition.

Proposition 1.4. *Suppose we have vectors $u, v, w \in \mathbb{R}^2$ and scalars $\lambda, \mu \in \mathbb{R}$. Then the following equalities hold.*

(1) $v + w = w + v$, *(commutivity)*
(2) $(u + v) + w = u + (v + w)$, *(associativity)*
(3) $\lambda(\mu v) = (\lambda\mu)v$, *(scalar product is associative)*
(4) $\lambda(v + w) = \lambda v + \lambda w$, *(distributivity I)*
(5) $(\lambda + \mu)v = \lambda v + \mu v$, *(distributivity II)*
(6) $0 + v = v$, *(vector zero behaves as zero)*
(7) $v + (-1)v = 0$, *($(-1)v$ is the negative of v)*
(8) $0v = 0$, *(multiplication by scalar zero gives vector zero)*
(9) $1v = v$. *(multiplication by 1 is the identity)*

Proof. Write u, v, and w as $u = (u_1, u_2)$, $v = (v_1, v_2)$, and $w = (w_1, w_2)$. Then we have

$$(u + v) + w = (u_1 + v_1, u_2 + v_2) + (w_1, w_2) = \big((u_1 + v_1) + w_1, (u_2 + v_2) + w_2\big)$$

and

$$u + (v + w) = (u_1, u_2) + (v_1 + w_1, v_2 + w_2) = \big((u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\big).$$

For each $i \in \{1,2\}$ we have $(u_i + v_i) + w_i = u_i + (v_i + w_i)$ because we know this rule holds for real numbers. This shows that the equality $(u+v)+w = u+(v+w)$ holds, which proves (2). The proofs of the other rules are left as an exercise for the reader. See Exercise 1.4. □

The rules in Proposition 1.4 allow us to manipulate expressions involving scalar products and sums without always having to write out everything in terms of the components. The rules themselves can all be proved trivially by writing out the vectors in terms of their components, reducing the rule to the corresponding rule for real numbers, which we know to hold.

It is clear that also if more vectors are involved, it does not matter which order we add vectors: for vectors $u, v, w, x \in \mathbb{R}^2$ we have for instance

$$((u+v)+w)+x = u+(v+(w+x)) = (v+x)+(w+u).$$

In general, the sum

$$\sum_{i=1}^{s} v_i = v_1 + v_2 + \ldots + v_s$$

of s vectors is just the vector consisting of the componentwise sums.

Before we give a general geometric interpretation of the newly defined sum, we look at a specific example.

Example 1.5. *Suppose we have vectors $v = (3,2)$ and $w = (1,4)$, corresponding to P and Q respectively. Then the sum $v+w$ corresponds to the point $S = (4,6)$, which is the unique point for which $OPSQ$ is a parallelogram in which OS is a diagonal. The point $R = (-1,-4)$ corresponds to $-w$. The difference $v - w$ corresponds to the point $T = (2,-2)$, which is the unique point for which $OTPQ$ is a parallelogram in which OP is a diagonal.*

Let $v, w \in \mathbb{R}^2$ be vectors and let P, Q, R, S, and T be points in the plane corresponding to v, w, $-w$, $v+w$, and $v-w$ respectively. Then the arrows from O to Q, from R to O, from P to S and from T to P all represent the vector w, which means that all these arrows have the same direction and the same length. We conclude that the quadrilaterals $OPSQ$, $OTPQ$ and $ORTP$ are all parallelograms. The point R, corresponding to $-w$, is the image of Q under reflection in the point O.

If P corresponds to a vector v and $c \in \mathbb{R}$ is a scalar, then cv corresponds to Q, where Q is obtained from P by scaling with respect to the origin O by a factor c. If c is negative, then the arrow from O to Q has the opposite direction of the arrow from O to P.

Example 1.6. *Suppose $w = (1,2)$, corresponding to the point P. Then $4w = (4,8)$, while $\frac{1}{2}w = (\frac{1}{2}, 1)$ and $-2w = (-2,-4)$.*

In physics, if s vectors $v_1, \ldots, v_s$ each represent a force on a given object, then the sum $v_1 + \ldots + v_s$ represents the combined force felt by the object. The object will accelerate in the direction of this sum. If the vectors sum to 0, then the object does not experience any acceleration.

Exercise 1.1. *Given the vectors v and w in the following cases, find $v+w$, $v-w$, $2v-w$ and $-3w$, and draw the points corresponding to all six of these vectors.*

a) $v = (1,2)$ *and* $w = (-2,1)$,

b) $v = (-2, 3)$ *and* $w = (1, 1)$,
c) $v = (0, 0)$ *and* $w = (-1, -2)$.
d) $v = (2, -1)$ *and* $w = (4, -2)$.

Exercise 1.2. *In all rules of Proposition 1.4 containing the symbol* 0 *or* $+$*, what do these symbols stand for?*

Exercise 1.3. *True or false?*

a) *If* P*,* Q *and* R *are points in the plane and* $v, w \in \mathbb{R}^2$ *are vectors represented by the arrows from* P *to* Q *and from* Q *to* R *respectively, then* $v + w$ *is represented by the arrow from* P *to* R*.*
b) *If* P*,* Q *and* R *are points in the plane and* v *and* w *are vectors represented by the arrows from* P *to* R *and from* Q *to* R *respectively, then* $v - w$ *is represented by the arrow from* Q *to* P*.*

Exercise 1.4. *Prove some of the remaining rules of Proposition 1.4.*

2. Lines in the plane

This section is not complete yet (but covers everything from the first lecture).

Definition 2.1. *The inner product of two vectors* $v = (v_1, v_2)$ *and* $w = (w_1, w_2)$ *is given by*

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2.$$

Note that the inner product takes two vectors and makes a **number**.

Example 2.2. *We have* $\langle (7, 2), (-1, 3) \rangle = 7 \cdot (-1) + 2 \cdot 3 = -7 + 6 = -1$.

Proposition 2.3. *Given three vectors* $u, v, w \in \mathbb{R}^2$ *and a scalar* $\lambda \in \mathbb{R}$*, we have*

(1) $\langle v, w \rangle = \langle w, v \rangle$,
(2) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,
(3) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,
(4) $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle = \langle u, \lambda v \rangle$.

Proof. See Exercise 2.2. □

We now look at the line L in the (x, y)-plane given by $3y = 2x + 1$ and the line M given by $3y = 2x$ (make picture). The lines can also be given by

$$L : \langle (2, -3), (x, y) \rangle = -1,$$
$$M : \langle (2, -3), (x, y) \rangle = 0.$$

In fact, if we identify the plane with $\mathbb{R}^2$, which we will do for the rest of the section, then we do not need to specify the components x and y separately and we can write the lines as

$$L = \{ u \in \mathbb{R}^2 \ : \ \langle (2, -3), u \rangle = -1 \}$$
$$M = \{ u \in \mathbb{R}^2 \ : \ \langle (2, -3), u \rangle = 0 \}.$$

For every point (x, y) on M we have $y = \frac{2}{3}x$, so if we set $s = \frac{1}{3}x$, then we have $x = 3s$ and $y = 2s$; in other words, there exists a number s such that $(x, y) = (3s, 2s) = s(3, 2)$. We conclude that the line M consists of all points corresponding to a real multiple of the vector $(3,2)$. The important thing to notice is that M goes through the origin. In set notation, we could write

$$\begin{aligned} M &= \{\, (x,y) \in \mathbb{R}^2 \;:\; 3y = 2x \,\} \\ &= \{\, (3s,2s) \;:\; s \in \mathbb{R} \,\} = \{\, s(3,2) \;:\; s \in \mathbb{R} \,\}. \end{aligned}$$

For every point (x,y) on L we have $y = \frac{1}{3}(2x+1)$, so if we set $s = \frac{1}{3}(x-1)$, then we have $x = 3s+1$ and $y = 2s+1$; in other words, there exists a number s such that $(x,y) = (3s+1, 2s+1) = s(3,2) + (1,1)$. We conclude that the line L consists of all points corresponding to a vector that is the sum of the fixed vector $(1,1)$ and a real multiple of the vector $(3,2)$. We need the fixed vector $(1,1)$ because L does not go through the origin. In set notation, we could write

$$\begin{aligned} L &= \{\, (x,y) \in \mathbb{R}^2 \;:\; 3y = 2x+1 \,\} \\ &= \{\, (3s+1, 2s+1) \;:\; s \in \mathbb{R} \,\} \\ &= \{\, (1,1) + s(3,2) \;:\; s \in \mathbb{R} \,\}. \end{aligned}$$

Clearly, the lines L and M are closely related. If we add $(1,1)$ to a point on M, we get a point on L. We say that L is the translation of M over the vector $(1,1)$. We could write

$$L = \{\, (1,1) + P \;:\; P \in M \}.$$

In fact, suppose T is any point on the line L. If P is a point on M, then we have $\langle (2,-3), P \rangle = 0$ and $\langle (2,-3), T \rangle = -1$, so by rule (2) of Proposition 2.3 we find

$$\langle (2,-3), P+T \rangle = \langle (2,-3), P \rangle + \langle (2,-3), T \rangle = 0 + -1 = -1.$$

We conclude that $P+T$ is also a point on L. We find that L is the translation of M over any point on L. In other words, we have

$$L = \{\, P + T \;:\; P \in M \}.$$

Example 2.4. *Suppose we want to parametrise the line N given by $2y = 7x - 1$. We rewrite it as $7x - 2y = 1$, or*

$$\langle (7,-2), (x,y) \rangle = 1.$$

The points on the associated line H through the origin, given by $2y = 7x$, or

$$\langle (7,-2), (x,y) \rangle = 0,$$

are parametrised by $(x,y) = (2s, 7s) = s(2,7)$. In set notation we have

$$H = \{\, s(2,7) \;:\; s \in \mathbb{R} \,\}$$

Now all we need is one point on N, say $T = (1,3)$. Then N is the translate of H over T and we have

$$N = \{(1,3) + s(2,7) \;:\; s \in \mathbb{R}\} = \{(2s+1, 7s+3) \;:\; s \in \mathbb{R}\}.$$

Indeed, it is easy to check that $(x,y) = (2s+1, 7s+3)$ is a parametrisation of the line N.

Conversely, given a parametrisation of a line, eliminating the parameter yields an equation for the line.

Example 2.5. *Suppose the line L is parametrised by $(x,y) = (3s+1, 4s-2) = (1,-2) + s(3,4)$. To find an equation for L, we eliminate the variable s from the set of eqations*

$$\begin{aligned} x &= 3s+1, \\ y &= 4s-2. \end{aligned}$$

Multiply the first equation by 4, the second by 3, and subtract the results to get $4x - 3y = 10$.

Exercise 2.1. *Draw arrows representing the given vectors v and w, and compute their inner product.*

- $v = (-2,5)$ *and* $w = (7,1)$,
- $v = 2(-3,2)$ *and* $w = (1,3) + (-2,4)$,
- $v = (-3,4)$ *and* $w = (4,3)$,
- $v = (-3,4)$ *and* $w = (8,6)$,
- $v = (2,-7)$ *and* $w = (x,y)$,
- $v = w = (a,b)$.

Exercise 2.2. *Prove some of the rules of Proposition 2.3.*

Exercise 2.3. *We have seen that the equation $3y = 2x+1$ can also be written as $\langle (2,-3), (x,y) \rangle = -1$. Therefore, in set notation the line L given by $3y = 2x+1$ can be written as*

$$\begin{aligned} L &= \{\, (x,y) \in \mathbb{R}^2 \;:\; 3y = 2x+1 \,\} \\ &= \{\, (x,y) \in \mathbb{R}^2 \;:\; \langle (2,-3), (x,y) \rangle = -1 \,\} \\ &= \{ v \in \mathbb{R}^2 \;:\; \langle (2,-3), v \rangle = -1 \,\} \\ &= \{\, (3s+1, 2s+1) \;:\; s \in \mathbb{R} \,\} \\ &= \{\, (1,1) + s(3,2) \;:\; s \in \mathbb{R} \,\}. \end{aligned}$$

Write each of the following lines in the (x,y)-plane in these five manners.

a) *The line given by $-2y = 3x+2$.*
b) *The line given by $3x = 2-y$.*
c) *The line given by $\langle r, (x,y) \rangle = a$ with $r = (-3,4)$ and $a = 7$.*
d) *The line given by $\langle r, (x,y) \rangle = a$ with $r = (2,1)$ and $a = -3$.*
e) *The line parametrised by $x = 3s+1$ and $y = -2s-5$.*
f) *The line parametrised by $(x,y) = (-2,3) + t(1,-2)$.*

Exercise 2.4. *True or false?*

a) *If $p \in \mathbb{R}^2$ is a nonzero vector and $a, b \in \mathbb{R}$ are real numbers, then the lines L and M given by*

$$\begin{aligned} L &= \{ u \in \mathbb{R}^2 \;:\; \langle p, u \rangle = a \} \\ M &= \{ u \in \mathbb{R}^2 \;:\; \langle p, u \rangle = b \} \end{aligned}$$

are parallel.

b) *If $p, q \in \mathbb{R}^2$ are nonzero vectors and $a, b \in \mathbb{R}$ are real numbers, and the lines L and M given by*

$$\begin{aligned} L &= \{ u \in \mathbb{R}^2 \;:\; \langle p, u \rangle = a \} \\ M &= \{ u \in \mathbb{R}^2 \;:\; \langle q, u \rangle = b \} \end{aligned}$$

are parallel, then $p = q$.

3. Other dimensions

We now generalize the definition from the previous section to higher dimension. Let n be a positive integer. We define $\mathbb{R}^n$ to be the set of all n-tuples of real numbers. In set notation we write

$$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \ : \ x_1, \ldots, x_n \in \mathbb{R} \}.$$

Again, an element of $\mathbb{R}^n$ is also called a *vector* and its entries are called *components*. We abbreviate the vector $(0, 0, \ldots, 0)$ by 0.

Definition 3.1. *Given a scalar $\lambda \in \mathbb{R}$ and two elements $v, w \in \mathbb{R}^n$ with $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, we define the* sum *$v + w$ of v and w and the* scalar product *λv of λ and v by*

$$v + w = (v_1 + w_1, \ldots, v_n + w_n), \qquad \textit{and} \qquad \lambda v = (\lambda v_1, \ldots, \lambda v_n).$$

The same rules hold as before.

Proposition 3.2. *Suppose we have vectors $u, v, w \in \mathbb{R}^n$ and scalars $\lambda, \mu \in \mathbb{R}$. Then the following equalities hold.*

(1) $v + w = w + v$, *(commutivity)*
(2) $(u + v) + w = u + (v + w)$, *(associativity)*
(3) $\lambda(\mu v) = (\lambda\mu)v$, *(scalar product is associative)*
(4) $\lambda(v + w) = \lambda v + \lambda w$, *(distributivity I)*
(5) $(\lambda + \mu)v = \lambda v + \mu v$, *(distributivity II)*
(6) $0 + v = v$, *(vector zero behaves as zero)*
(7) $v + (-1)v = 0$, *($(-1)v$ is the negative of v)*
(8) $0v = 0$, *(multiplication by scalar zero gives vector zero)*
(9) $1v = v$. *(multiplication by 1 is the identity)*

Proof. The proofs are left as an exercise for the reader. See Exercise 3.2. □

Again, the sum

$$\sum_{i=1}^{s} v_i = v_1 + v_2 + \ldots + v_s$$

of s vectors in $\mathbb{R}^n$ is just the vector consisting of the componentwise sums.

For $n = 1$ we just get the real line, while for $n = 2$ we retrieve the story of the previous section.

For $n = 3$, the situation is similar to that of $n = 2$. The triples in $\mathbb{R}^3$ correspond to points in space: the components of a vector $v \in \mathbb{R}^3$ correspond to the coordinates with respect to an unnamed coordinate system. Each vector in $\mathbb{R}^3$ is again represented by an arrow from one point in space to another. Two arrows represent the same vector if and only if they have the same direction and the same length. If two vectors $v, w \in \mathbb{R}^n$ correspond to points P and Q and their sum $v + w$ corresponds to the point S, then $OPSQ$ is a parallelogram in space, in which OS is a diagonal.

In physics, a force in space corresponds to a vector in $\mathbb{R}^3$ that is represented by an arrow with the same direction as the force and appropriate length, according to the magnitude of the force and the choice of unit. As in the plane, if s vectors $v_1, \ldots, v_s$ each correspond to a force on a given object, then the sum $v_1 + \ldots + v_s$ corresponds to the combined force felt by the object.

Also in physics, sometimes $\mathbb{R}^4$ is interpreted as space with an extra coordinate to indicate time.

Exercise 3.1. *Given the vectors v and w in the following cases, find $v+w$, $v-w$, $v-3w$ and $-2v$, and draw the points corresponding to all six of these vectors if they lvie in $\mathbb{R}^n$ for $n \leq 3$.*

a) $v = (-3)$ *and* $w = (1)$,
b) $v = (0, 1, 2)$ *and* $w = (-1, 0, 1)$,
c) $v = (\pi, \sqrt{2}, 2, -7)$ *and* $w = (\sqrt{3}, e, 11, \frac{7}{4})$.
d) $v = (3, 6, 3, -6)$ *and* $w = (1, 2, 1, -2)$.

Exercise 3.2. *Prove some of the rules of Proposition 3.2.*

4. Inner product

Definition 4.1. *Given two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, we define their inner product to be*

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n.$$