Linear algebra 2: exercises for Section 7

Ex. 7.1. Let V and W be normed vector spaces over $\mathbb{R}$. For a linear map $f\colon V \to W$ let

$$||f|| = \sup_{x \in V,\ ||x||=1} ||f(x)||$$

1. Show that $B(V, W) = \{f \in \mathrm{Hom}(V, W)\colon ||f|| < \infty\}$ is a subspace of $\mathrm{Hom}(V, W)$, and that $||\cdot||$ is a norm on $B(V, W)$.
2. Show that $B(V, W) = \mathrm{Hom}(V, W)$ if V is finite dimensional.
3. Taking $V = W$ above, we obain a norm on $B(V, V)$. Show that $||f \circ g|| \leq ||f|| \cdot ||g||$ for all $f, g \in B(V, V)$.

Ex. 7.2. Consider the rotation map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates the plane by 45 degrees. For any norm on $\mathbb{R}^2$ the previous exercise defines a norm $||f||$. Show that $||f|| = 1$ when we take the standard euclidean norm $||\cdot||_2$ on $\mathbb{R}^2$. What is $||f||$ when we take the maximum norm $||\cdot||_\infty$ on $\mathbb{R}^2$?

Ex. 7.3. Consider $V = \mathbb{R}^n$ with the standard inner product and the norm $||\cdot||_2$. Suppose that $f\colon V \to V$ is a diagonalizable map whose eigenspaces are orthogonal (i.e., V has an orthogonal basis consisting of eigenvectors of f). Show that $||f||$ as defined in Ex. 7.1 above is equal to the largest absolute value of an eigenvalue of f.

Ex. 7.4. What is the sine of the matrix $\begin{pmatrix} \pi & \pi \\ 0 & \pi \end{pmatrix}$?

Ex. 7.5. Consider the vector space V of polynomial functions $[0, 1] \to \mathbb{R}$ with the sup-norm: $||f|| = \sup_{0 \leq x \leq 1} |f(x)|$. Consider the functional $\phi \in V^*$ defined by $\phi(f) = f'(0)$. Show that $\phi \notin B(V, \mathbb{R})$. [Hint: consider the polynomials $(1 - x)^n$ for $n = 1, 2, \ldots$.]