Linear algebra 2: exercises for Section 8

Ex. 8.1. Let V_1, V_2, U, W be vector spaces over a field F, and let $b\colon V_1 \times V_2 \to U$ be a bilinear map. Show that for each linear map $f\colon U \to W$ the composition $f \circ b$ is bilinear.

Ex. 8.2. Let V, W be vector spaces over a field F. If $b\colon V \times V \to W$ is both bilinear and linear, show that b is the zero map.

Ex. 8.3. Give an example of two vector spaces V, W over a field F and a bilinear map $b\colon V \times V \to W$ for which the image of b is not a subspace of W.

Ex. 8.4. Let V, W be two 2-dimensional subspaces of the standard $\mathbb{R}$-vector space $\mathbb{R}^3$. The restriction of the standard inner product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ to $\mathbb{R}^3 \times W$ is a bilinear map $b\colon \mathbb{R}^3 \times W \to \mathbb{R}$.

1. What is the left kernel of b? And the right kernel?
2. Let $b'\colon V \times W \to \mathbb{R}$ be the restriction of b to $V \times W$. Show that b' is degenerate if and only if the angle between V and W is $90°$.

Ex. 8.5. Let V, W be finite-dimensional vector spaces over a field F and $b : V \times W \to F$ a bilinear form with left kernel V_0 and right kernel W_0. Show that b induces the *non-degenerate* bilinear form

$$b' : V/V_0 \times W/W_0 \to F, \quad (v + V_0, w + W_0) \longmapsto b(v, w)\,.$$

and conclude that $\dim(V/V_0) = \dim(W/W_0)$.

Ex. 8.6. Let V be a vector space over $\mathbb{R}$, and let $b\colon V \times V \to \mathbb{R}$ be a symmetric bilinear map. Let the "quadratic form" associated to b be the map $q\colon V \to \mathbb{R}$ that sends $x \in V$ to $b(x, x)$. Show that b is uniquely determined by q.

Ex. 8.7. Let V be a vector space over $\mathbb{R}$, and let $b\colon V \times V \to \mathbb{R}$ be a bilinear map. Show that b can be uniquely written as a sum of a symmetric and a skew-symmetric bilinear form.