Linear algebra 2: exercises for Chapter 2 (Direct sums)

You may use Theorem 3.1 (Cayley-Hamilton) for these exercises. This theorem states that for every square matrix A, we have $P_A(A) = 0$, where P_A is the characteristic polynomial of A.

Ex. 2.1. Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation around the line through the origin and the point $(1, 1, 1)$ by 120 degrees. Decompose $\mathbb{R}^3$ as a direct sum of two subspaces that are each stable under ϕ.

Ex. 2.2. Consider the vector space $V = \mathbb{R}^3$ with the linear map $\phi\colon V \to V$ given by the matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ -2 & -1 & 1 \\ -3 & -1 & 2 \end{pmatrix}$$

Decompose $\mathbb{R}^3$ as a direct sum of two subspaces that are each stable under ϕ.

Ex. 2.3. Same question for

$$\begin{pmatrix} 0 & 1 & 1 \\ 5 & -4 & -3 \\ -6 & 6 & 5 \end{pmatrix}$$

Ex. 2.4. Consider the vector space $V = \mathbb{R}^4$ with the linear map $\phi\colon V \to V$ that permutes the standard basis vectors in a cycle of length 4. What is the characteristic polynomial of ϕ? Decompose $\mathbb{R}^4$ into a direct sum of 3 subspaces that are all stable under ϕ.

Ex. 2.5. A nonzero endomorphism f of a vector space V is said to be a *projection* if $f^2 = f$. Suppose f is such a projection.

1. Show that the image of f is equal to the kernel of $f - \mathrm{id}_V$, i.e., the eigenspace E_1 at eigenvalue 1.
2. Show that V is the direct sum of the kernel E_0 of f and E_1.
3. Show that $f = f_0 \oplus f_1$ where f_0 is the zero-map on E_0 and f_1 is the identity map on E_1.

Ex. 2.6. An endomorphism f of a vector space V is said to be a *reflection* if f^2 is the identity on V. Suppose f is such a reflection. Show that V is the direct sum of two subspaces U and W for which $f = \mathrm{id}_U \oplus (-\mathrm{id}_W)$.