Linear algebra 2: exercises for Chapter 6

Ex. 6.1. Define $\phi_i\colon \mathbb{R}^n \to \mathbb{R}$ by $\phi_i(x_1,\ldots,x_n) = x_1+x_2+\cdots+x_i$ for $i = 1,2,\ldots n$. Show that $\phi_1,\ldots,\phi_n$ is a basis of $(\mathbb{R}^n)^*$, and compute its dual basis of $\mathbb{R}^n$.

Ex. 6.2. Let V be an n-dimensional vector space, let $v_1,\ldots,v_n \in V$ and let $\phi_1,\ldots,\phi_n \in V^*$. Show that $\det((\phi_i(v_j))_{i,j})$ is non-zero if and only if $v_1,\ldots,v_n$ is a basis of V and $\phi_1,\ldots,\phi_n$ is a basis of V^*.

Ex. 6.3. Let V be the 3-dimensional vector space of polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree at most 2. In each of the following cases, we define $\phi_i \in V^*$ for $i = 0,1,2$. In each case, indicate whether ϕ_0,ϕ_1,ϕ_2 is a basis of V^*, and if so, give the dual basis of V.

1. $\phi_i(f) = f(i)$
2. $\phi_i(f) = f^{(i)}(0)$, i.e., the ith derivative of f evaluated at 0.
3. $\phi_i(f) = f^{(i)}(1)$
4. $\phi_i(f) = \int_{-1}^{i} f(x)dx$

Ex. 6.4. For each positive integer n show that there are constants $a_1,a_2,\ldots,a_n$ so that

$$\int_0^1 f(x)e^x dx = \sum_{i=1}^n a_i f(i)$$

for all polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$ of degree less than n.

Ex. 6.5. Suppose V is a finite dimensional vector space and W is a subspace. Let $f\colon V \to V$ be a linear map so that $f(w) = w$ for $w \in W$. Show that $f^T(v^*) - v^* \in W^o$ for all $v^* \in V^*$.

Conversely, if you assume that $f^T(v^*) - v^* \in W^o$ for all $v^* \in V^*$, can you show that $f(w) = w$ for $w \in W$?

*** Ex. 6.6.** Let V be a finite-dimensional vector space and let $U \subset V$ and $W \subset V^*$ be subspaces. We identify V and V^{**} via α_V (so $W^\circ \subset V$). Show that

$$\dim(U^\circ \cap W) + \dim U = \dim(U \cap W^\circ) + \dim W\,.$$

Ex. 6.7. Let $\phi_1,\ldots,\phi_n \in (\mathbb{R}^n)^*$. Prove that the solution set C of the linear inequalities $\phi_1(x) \geq 0,\ldots,\phi_n(x) \geq 0$ has the following properties:

1. $\alpha, \beta \in C \implies \alpha + \beta \in C$.
2. $\alpha \in C,\ t \in \mathbb{R}_{\geq 0} \implies t\alpha \in C$.
3. If $\phi_1, \ldots, \phi_n$ form a basis of $(\mathbb{R}^n)^*$, then

$$C = \{t_1\alpha_1 + \ldots + t_n\alpha_n : t_i \in \mathbb{R}_{\geq 0}, \forall i \in \{1, \ldots, n\}\},$$

where $\alpha_1, \ldots, \alpha_n$ is the basis of $\mathbb{R}^n$ dual to $\phi_1, \ldots, \phi_n$.