Linear algebra 2: exercises for Chapter 7 and first part of Chapter 8

Let V and W be normed vector spaces over $\mathbb{R}$. For a linear map $f\colon V \to W$ let

$$||f|| = \sup_{x\in V,\ ||x||=1} ||f(x)||$$

Ex. 7.1. Consider $V = \mathbb{R}^n$ with the standard inner product and the norm $||\cdot||_2$. Suppose that $f\colon V \to V$ is a diagonalizable map whose eigenspaces are orthogonal (i.e., V has an orthogonal basis consisting of eigenvectors of f). Show that $||f||$ as defined above is equal to the largest absolute value of an eigenvalue of f.

Ex. 7.2.

1. Show that $B(V,W) = \{f \in \mathrm{Hom}(V,W)\colon ||f|| < \infty\}$ is a subspace of $\mathrm{Hom}(V,W)$, and that $||\cdot||$ is a norm on $B(V,W)$.
2. Show that $B(V,W) = \mathrm{Hom}(V,W)$ if V is finite-dimensional.
3. Taking $V = W$ above, we obtain a norm on $B(V,V)$. Show that $||f \circ g|| \leq ||f|| \cdot ||g||$ for all $f, g \in B(V,V)$.

Ex. 7.3. Consider the rotation map $f\colon\mathbb{R}^2 \to \mathbb{R}^2$ which rotates the plane by 45 degrees. For any norm on $\mathbb{R}^2$ the previous exercise defines a norm $||f||$. Show that $||f|| = 1$ when we take the standard euclidean norm $||\cdot||_2$ on $\mathbb{R}^2$. What is $||f||$ when we take the maximum norm $||\cdot||_\infty$ on $\mathbb{R}^2$?

Ex. 7.4. Consider the vector space V of polynomial functions $[0,1] \to \mathbb{R}$ with the sup-norm: $||f|| = \sup_{0\leq x\leq 1} |f(x)|$. Consider the functional $\phi \in V^*$ defined by $\phi(f) = f'(0)$. Show that $\phi \notin B(V,\mathbb{R})$. [Hint: consider the polynomials $(1-x)^n$ for $n = 1, 2, \ldots$.]

Ex. 7.5. What is the sine of the matrix $\begin{pmatrix} \pi & \pi \\ 0 & \pi \end{pmatrix}$?

Ex. 8.1. Let V_1, V_2, U, W be vector spaces over a field F, and let $b\colon V_1 \times V_2 \to U$ be a bilinear map. Show that for each linear map $f\colon U \to W$ the composition $f \circ b$ is bilinear.

Ex. 8.2. Let V, W be vector spaces over a field F. If $b\colon V \times V \to W$ is both bilinear and linear, show that b is the zero map.

Ex. 8.3. Give an example of two vector spaces V, W over a field F and a bilinear map $b\colon V \times V \to W$ for which the image of b is not a subspace of W.

Ex. 8.4. Let V, W be two 2-dimensional subspaces of the standard $\mathbb{R}$-vector space $\mathbb{R}^3$. The restriction of the standard inner product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ to $\mathbb{R}^3 \times W$ is a bilinear map $b\colon \mathbb{R}^3 \times W \to \mathbb{R}$.

1. What is the left kernel of b? And the right kernel?
2. Let $b'\colon V \times W \to \mathbb{R}$ be the restriction of b to $V \times W$. Show that b' is degenerate if and only if the angle between V and W is $90°$.

Ex. 8.5. Let V be a vector space over $\mathbb{R}$, and let $b\colon V \times V \to \mathbb{R}$ be a symmetric bilinear map. Let the "quadratic form" associated to b be the map $q\colon V \to \mathbb{R}$ that sends $x \in V$ to $b(x, x)$. Show that b is uniquely determined by q.

Ex. 8.6. Let V be a vector space over $\mathbb{R}$, and let $b\colon V \times V \to \mathbb{R}$ be a bilinear map. Show that b can be uniquely written as a sum of a symmetric and a skew-symmetric bilinear form.