Linear algebra 2: exercises for Section 1

Ex. 1.1. Are the vectors $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ -1 \\ -4 \end{pmatrix}$ linearly independent?

Ex. 1.2. Are the vectors $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$ linearly independent?

Ex. 1.3. For which $x \in \mathbb{R}$ are the vectors $\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ x \end{pmatrix}$ linearly dependent?

Ex. 1.4. Compute $\det(M)$ for

$$M = \begin{pmatrix} -3 & -1 & 0 & -2 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Ex. 1.5. Give the kernel and the image of the map $\mathbb{R}^5 \to \mathbb{R}^3$ given by $x \mapsto Ax$ with

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 1 \\ 2 & -1 & 4 & 3 & 3 \\ -1 & 0 & -3 & -1 & 1 \end{pmatrix}.$$

Ex. 1.6. For any square matrix M show that $\mathrm{rk}(M^2) \leq \mathrm{rk}(M)$.

Ex. 1.7. Compute the characteristic polynomial, the complex eigenvalues and the complex eigenspaces of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ viewed as a matrix over $\mathbb{C}$.

Ex. 1.8. Find the eigenvalues and eigenspaces of the matrix $A = \begin{pmatrix} 11 & 9 \\ -12 & -10 \end{pmatrix}$. Is A diagonalizable?

Ex. 1.9. Same question for $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

Ex. 1.10. Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Ex. 1.11. Consider the map $f\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ given by $x \mapsto Ax$ where $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$. Show that $\mathbb{R}^2$ has a basis consisting of eigenvectors of f, and given the matrix of f with respect to this basis. For any positive integer n give a formula for the matrix representation of f^n, first with repect to the basis of eigenvectors, and then with repect to the standard basis.

Ex. 1.12. Suppose that M is a diagonalizable matrix. Show that $M^2 + M$ is diagonalizable.

Ex. 1.13. Is every 3×3 matrix whose characteristic polynomial is $X^3 - X$ diagonalizable? Is every 3×3 matrix whose characteristic polynomial is $X^3 - X^2$ diagonalizable?

Ex. 1.14. Let the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane $x + 2y + z = 0$. What are the eigenvalues and eigenspaces of f?

Ex. 1.15. What is the characteristic polynomial of the rotation map $\mathbb{R}^3 \to \mathbb{R}^3$ which rotates space around the line through the origin and the point $(1, 2, 3))$ by 180 degrees? Same question if we rotate by 90 degrees?