Linear algebra 2: exercises for Section 5

Ex. 5.1. In each of the following cases indicate whether there exists a real 4×4-matrix A with the given properties. Here I denotes the 4×4 identity matrix.

1. $A^2 = 0$ and A has rank 1;
2. $A^2 = 0$ and A has rank 2;
3. $A^2 = 0$ and A has rank 3;
4. A has rank 2, and $A - I$ has rank 1;
5. A has rank 2, and $A - I$ has rank 2;
6. A has rank 2, and $A - I$ has rank 3.

Ex. 5.2. For the following matrices A, B give their Jordan normal forms, and decide if they are similar.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Ex. 5.3. Give the Jordan normal form of the matrix

$$\begin{pmatrix} 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 5 & 2 & -2 \\ 0 & -4 & 0 & 4 \end{pmatrix}$$

Ex. 5.4. Give the Jordan normal form of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Ex. 5.5. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute A^{100}.

Ex. 5.6. Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$.

1. Give the eigenvalues and eigenspaces of A.
2. Give a diagonal matrix D and a nilpotent matrix N for which $D+N = A$ and $DN = ND$.
3. Give a formula for A^n when $n = 1, 2, 3, \ldots$

Ex. 5.7. For the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

give a diagonalizable matix D and a nilpotent matrix N so that $A = D + N$ and $ND = DN$.

Ex. 5.8. For $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & -2 \\ 0 & 2 & 0 \end{pmatrix}$ compute the matrix e^A.

Ex. 5.9. Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\phi(x) = Ax$ where A is the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We proved in class that generalized eigenspaces for ϕ are ϕ-invariant. What are these spaces in this case? Give all other ϕ-invariant subspaces of $\mathbb{R}^3$.

Ex. 5.10. Compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Does A have a Jordan normal form as 4×4 matrix over $\mathbb{R}$? What is the Jordan normal form of A as a 4×4 matrix over $\mathbb{C}$?

Ex. 5.11. Suppose that for a 20×20 matrix A the rank of A^i for $i = 0, 1, \ldots 9$ is given by the sequence $20, 15, 11, 7, 5, 3, 1, 0, 0, 0$. What sizes are the Jordan-blocks in the Jordan normal form of A? Can you prove the formula you use for all matrices whose characteristic polynomial is a product of linear polynomials?