SQUARES FROM BLOCKS OF CONSECUTIVE INTEGERS : A PROBLEM OF ERDŐS AND GRAHAM

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Abstract. In this paper, we construct, given an integer $r \geq 5$, an infinite family of r non-overlapping blocks of five consecutive integers with the property that their product is always a perfect square. In this particular situation, this answers a question of Erdős and Graham in the negative.

1. Introduction

Let us define

$$f(n,k) = n(n+1)\cdots(n+k-1).$$

Then a beautiful result of Erdős and Selfridge [3] is that $f(n,k)$ is never a perfect power of an integer, provided $n \geq 1$ and $k \geq 2$. Likely motivated by this, Erdős and Graham [2] asked whether a similar statement might hold for $\prod_{j=1}^{r} f(n_j,k_j)$, specifically as to whether the Diophantine equation

$$\prod_{j=1}^{r} f(n_j,k_j) = x^2 \tag{1}$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_j \geq 4$ for $j = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(n_1, n_2, \ldots, n_r, x)$ with, say,

$$n_j + k_j \leq n_{j+1} \text{ for } 1 \leq j \leq r-1. \tag{2}$$

This latter condition enables us to avoid certain trivial cases where the blocks of integers are chosen to substantially overlap. Ulas [7] answered this question in the negative if

$$\{k_1, k_2, \ldots, k_r\} = \{4, 4, \ldots, 4\}$$

for $r = 4$ and $r \geq 6$ (subsequently extended to $r = 3$ and $r = 5$ by Bauer and the first author [1]; if $r = 2$ and $k_1 = k_2 = 4$, it is likely that equation (1) has at most finitely many solutions – see [5] for conditional results on this problem). Further, Ulas suggested that

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equation (1) should have infinitely many solutions in each case where the number of blocks is suitably large relative to the interval lengths (i.e. for r exceeding a constant that depends only on $\max_i\{k_i\}$). This assertion appears to be extremely difficult to prove (or, for that matter, disprove). As the values of the k_i increase, the techniques used to construct families of solutions to (1) in [1] and [7] seem likely to fail. In the present note, we will, however, provide an extension of the results of [1] and [7] to the case $k_i = 5$ $(1 \leq i \leq r)$. We prove the following result.

Theorem 1.1. *If $r \geq 5$ and $k_i = 5$ for $1 \leq i \leq r$, then there exist infinitely many $(r+1)$-tuples of positive integers $(n_1, n_2, \ldots, n_r, x)$ satisfying equation (1) and (2).*

2. Proof of Theorem 1.1

The argument in [1], in case $k_i = 4$ $(1 \leq i \leq r)$, proceeds as follows. From, for example, the polynomial identities

$$f(N,4)f(2N+3,4) = (2N^2+5N)(2N^2+5N+3)\left(2(N+2)(N+3)\right)^2$$

and

$$f(M,4) = (M^2+3M)(M^2+3M+2),$$

it follows, if N and M satisfy

$$2\left(2N^2+5N\right) = 3\left(M^2+3M\right), \tag{3}$$

that $f(N,4)f(M,4)f(2N+3,4)$ is a square. Writing $u = 4N+5$ and $v = 2M+3$, equation (3) is equivalent to the Pell-type equation

$$u^2 - 3v^2 = -2,$$

where we are interested in solutions with $u \equiv 1 \pmod 4$. Since it is readily shown that there are infinitely many such solutions, it is easy to conclude that there exist infinitely many solutions to (1) in the case $r = 3, k_1 = k_2 = k_3 = 4$ (and, since $\min\{n_i\}$ can be taken arbitrarily large, in fact, after a little work, for $r \geq 3$ and $k_i = 4$, $1 \leq i \leq r$).

It is unclear whether this argument can be extended to handle cases with $k_i \geq 5$. Instead, we search for four polynomials $p_i(t) \in \mathbb{Z}[t]$ $(1 \leq i \leq 4)$ with the property that $p_i(t) \neq p_j(t) + k$ for $1 \leq i, j, k \leq 4$ and $i \neq j$, and for which

$$f(p_1(t),5)f(p_2(t),5)f(p_3(t),5)f(p_4(t),5) = g(t)h^2(t), \tag{4}$$

where $g, h \in \mathbb{Z}[t]$ and g has degree at most two. If we can find such polynomials, we might optimistically hope that there exist infinitely many values of the parameter t for which $g(t)$ is a perfect square; the quadruples $(n_1, n_2, n_3, n_4) = (p_1(t), p_2(t), p_3(t), p_4(t))$ would provide a negative answer to the question of Erdős and Graham in the case $r = 4$ and $k_1 = k_2 = k_3 = k_4 = 5$. This hope turns out to be too optimistic, at least for the solutions to (4) that we find. However, an adaptation of this strategy will still allow us to answer the question negatively for $r \geq 5$.

We define $d_i = \deg(p_i)$ and narrow our search to the case where $(d_1, d_2, d_3, d_4) = (1, 1, 2, 2)$. Under this restriction, the irreducible factors of $f(p_1(t), 5)f(p_2(t), 5)$ are all linear, so if (4) holds, then the product $f(p_3(t), 5)\,f(p_4(t), 5)$ has at most one single irreducible quadratic factor, up to squares of polynomials. In the following table, we list quadratic polynomials $p_3(t)$ and $p_4(t)$ with this property; while we do not prove that this list is exhaustive, we know of no other examples (up to scaling and translation).

N	$p_3(t)$	$p_4(t)$	N	$p_3(t)$	$p_4(t)$
$4t^2 + 3t - 4$	N	$2N$	$24t^2 + 13t$	$N - 2$	$2N$
$8t^2 + t - 3$	N	$2N$	$24t^2 + 19t + 2$	$N - 2$	$2N$
$12t^2 + t - 4$	N	$2N$	$24t^2 + 23t + 1$	$N - 2$	$2N$
$12t^2 + 5t - 6$	N	$2N$	$32t^2 + 14t + 1$	$N - 2$	$2N$
$12t^2 + 7t - 3$	N	$2N$	$36t^2 + 23t + 2$	$N - 2$	$2N$
$12t^2 + 11t - 4$	N	$2N$	$36t^2 + 31t + 5$	$N - 2$	$2N$
$2t^2 + 2t - 3$	$N - 1$	$2N$	$18t^2 + 6t - 3$	$N - 1$	$2N$
$4t^2 + t - 2$	$N - 1$	$2N$	$24t^2 + 4t - 3$	$N - 1$	$2N$
$6t^2 + 2t - 3$	$N - 1$	$2N$	$24t^2 + 20t + 1$	$N - 1$	$2N$
$8t^2 + t - 3$	$N - 1$	$2N$	$36t^2 + 3t - 2$	$N - 1$	$2N$
$12t^2 + t - 4$	$N - 1$	$2N$	$36t^2 + 21t + 1$	$N - 1$	$2N$
$12t^2 + 5t - 6$	$N - 1$	$2N$	$36t^2 + 27t + 3$	$N - 1$	$2N$
$12t^2 + 5t - 1$	$N - 1$	$2N$	$24t^2 + t - 7$	N	$2N$
$12t^2 + 7t - 3$	$N - 1$	$2N$	$24t^2 + 5t - 4$	N	$2N$
$12t^2 + 11t - 4$	$N - 1$	$2N$	$24t^2 + 11t - 3$	N	$2N$
$12t^2 + 11t + 1$	$N - 1$	$2N$	$24t^2 + 17t - 4$	N	$2N$
$4t^2 + t - 2$	$N - 2$	$2N$	$36t^2 + 5t - 4$	N	$2N$
$8t^2 + 7t + 1$	$N - 2$	$2N$	$36t^2 + 13t - 3$	N	$2N$
$12t^2 + t - 4$	$N - 2$	$2N$	$36t^2 + 19t + 1$	$2N - 1$	$3N$
$12t^2 + 5t - 1$	$N - 2$	$2N$	$36t^2 + 35t + 7$	$2N - 1$	$3N$
$12t^2 + 7t - 3$	$N - 2$	$2N$	$36t^2 + 19t + 1$	$2N - 2$	$3N$
$12t^2 + 11t + 1$	$N - 2$	$2N$	$36t^2 + 35t + 7$	$2N - 2$	$3N$
$4t^2 + t - 2$	$N - 3$	$2N$	$24t^2 + 13t + 1$	$N - 3$	$4N$
$24t^2 + 7t - 4$	$N - 2$	$2N$	$24t^2 + 19t + 3$	$N - 3$	$4N$

From this list, it is relatively easy to check that the only case where there exist linear polynomials $p_1(t)$ and $p_2(t)$ and polynomials f, g for which (4) holds is when $N = 4t^2+t-2$, $p_3(t) = N-3$ and $p_4(t) = 2N$. Indeed, one easily checks that the polynomials

$$\begin{aligned}
p_1(t) &= 4t-4,\\
p_2(t) &= 4t+1,\\
p_3(t) &= 4t^2+t-5,\\
p_4(t) &= 8t^2+2t-4\\
g(t) &= 2\,(4t^2+t-4)\\
h(t) &= 2^{-3}(4t^2+t-2)(4t^2+t-1)\prod_{j=-4}^{5}(4t+j)
\end{aligned}$$

satisfy (4). Unfortunately, $g(t)$ is not a perfect square for any integer t, as it is never a square modulo 25.

It is possible, however, to use this construction to answer the question of Erdős and Graham in the negative, for values of $r \geq 5$ (with $k_i = 5$ for all i).

Lemma 2.1. *Suppose $D < 100$ is a positive squarefree integer. Then the equation*

$$g(t) = Ds^2 \tag{5}$$

has infinitely many solutions in positive integers s and t if and only if

$$D \in \{5, 7, 13, 37, 47, 58, 67, 73, 83, 97\}.$$

Proof. In each case, there either exists a local obstruction to solvability or infinitely many solutions. A useful online resource for such questions is `http://www.alpertron.com.ar/QUAD.HTM`. The solutions one finds are readily described in terms of binary recurrence sequences. □

Proof of Theorem 1.1. We will use induction on r. First suppose $r \in S = \{5, 6, 7, 9, 10, 11, 12\}$. We take an integer D and $r-4$ integers $n_1, \dots, n_{r-4}$ satisfying

$$\prod_{j=1}^{r-4} f(n_j, 5) = Dy^2$$

for some y, as in the following table.

r	D	$n_1, \ldots, n_{r-4}$
5	5	2
6	7	$1, 6$
7	13	$1, 6, 24$
9	7	$1, 8, 13, 24, 32$
10	47	$1, 7, 14, 20, 32, 44$
11	13	$1, 6, 24, 30, 64, 132, 152$
12	13	$1, 8, 13, 32, 57, 118, 272, 548$

By Lemma 2.1, there are infinitely many pairs (s, t) of integers satisfying $g(t) = Ds^2$, with $p_1(t) > n_{r-4} + 4$, so that (1) and (2) are satisfied with $n_{r-4+i} = p_i(t)$ $(1 \le i \le 4)$ and $x = Dysh(t)$. This proves the statement of the theorem for $r \in S$.

Now suppose $r \ge 5$ and $r \notin S$ and set $q = r - 8$, so that $q = 0$, or $q \ge 5$. In both cases (in the former trivially and in the latter by the induction hypothesis), there are positive integers $n_1, \ldots, n_q, w$ satisfying (1) and (2) with q instead of r and w instead of x.

By Lemma 2.1, there are infinitely many quadruples (s_1, t_1, s_2, t_2) of integers, with $p_1(t_1) > n_q + 4$ and $p_1(t_2) > p_4(t_1) + 4$ satisfying $g(t_i) = 5s_i^2$ $(i = 1, 2)$; for each quadruple, (1) and (2) hold with $n_{q+j} = p_j(t_1)$ and $n_{q+4+j} = p_j(t_2)$ $(1 \le j \le 4)$ and $x = 5ws_1s_2h(t_1)h(t_2)$. This proves the theorem for all $r \ge 5$. $\square$

Remark Geometrically speaking, equation (1) determines a double cover of the affine space $\mathbb{A}^r$ with coordinates $n_1, \ldots, n_r$. We are interested in studying integral points on this variety. The inequalities in (2) imply that we exclude certain subvarieties given by, for instance, $n_j + l = n_{j+1}$ for some $0 \le l < k_j$. One might wonder for which r-tuples $k_1, k_2, \ldots, k_r$ the double cover is of so-called general type. In such cases, Lang's conjecture implies that even the set of rational points is not dense on the variety, so we might expect very few integral points. For $k_1 = \ldots = k_r = k$, however, the variety is of general type if and only if k is at least 5 and we just proved that for $k = 5$ there are infinitely many integral points when $r \ge 5$. Since Lang's conjecture leaves the possibility that certain subvarieties contain infinitely many integral points, it does not actually provide information about our problem. Indeed, the points we have found all lie on the subvariety given by $n_r = 2n_{r-1} + 6$.

3. Concluding remarks

We know of no solutions whatsoever to equation (1) with $k_i = 5$ for $1 \le i \le r$ and $r \le 3$. For $r = 4$, there are many solutions (in all likelihood, infinitely many), including, by way of example, ones with $(n_1, n_2, n_3, n_4) = (1, 14, 24, 48)$ and $(n_1, n_2, n_3, n_4) = (17, 24, 33, 74)$. If we consider products of blocks of consecutive integers of length six or more, the techniques discussed here appear unlikely to yield analogous results (though such results may well still be true).

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