Cubic points on cubic curves and the Brauer-Manin obstruction on K3 surfaces

Ronald van Luijk
PIMS, Vancouver
University of British Columbia
Simon Fraser University

July 10, 2007
Bristol

Two problems:

(1) Are there cubic curves without cubic points?

(2) Is the Brauer-Manin obstruction the only one on K3 surfaces?

Goal:

(a) Explain the problems

(b) Relate them

Hasse Principle

Let X be a variety over $\mathbb{Q}$.
If X has no points over $\mathbb{R}$ then X has no points over $\mathbb{Q}$.
If X has no points over $\mathbb{Q}_p$ then X has no points over $\mathbb{Q}$.

Conics satisfy the **Hasse principle**:
If a conic C has a point over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every p,
then C has a point over $\mathbb{Q}$.

If a variety X over a number field k has points over every completion of k, then we say that X is **locally solvable everywhere** (**LSE**).

Cubic curves in general do **not** satisfy the Hasse principle.

The curve C given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ is **LSE**,
but has no points over $\mathbb{Q}$ (Selmer).

Cubic curves in general do **not** satisfy the Hasse principle.

The curve C given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ is **LSE**,
but has no points over $\mathbb{Q}$ (Selmer).

Question 1: Over what fields does C acquire points?

Cubic curves in general do **not** satisfy the Hasse principle.

The curve C given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ is **LSE**, but has no points over $\mathbb{Q}$ (Selmer).

Question 1: Over what fields does C acquire points?
Question 2: Over cubic fields that are galois?

Cubic curves in general do **not** satisfy the Hasse principle.

The curve C given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ is **LSE**,
but has no points over $\mathbb{Q}$ (Selmer).

Question 1: Over what fields does C acquire points?
Question 2: Over cubic fields that are galois?

Definition: A **cubic point** is a point over a cubic galois extension.

The line L: $711x + 172y + 785z = 0$ intersects C in three cubic points.

Cubic curves in general do **not** satisfy the Hasse principle.

The curve C given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ is **LSE**,
but has no points over $\mathbb{Q}$ (Selmer).

Question 1: Over what fields does C acquire points?
Question 2: Over cubic fields that are galois?

Definition: A **cubic point** is a point over a cubic galois extension.

The line $L\colon 711x + 172y + 785z = 0$ intersects C in three cubic points.

Question 3: Does every cubic curve that is **LSE** have cubic points?
(**unknown**)

Brauer-Manin obstruction.

Brauer-Manin obstruction.

Let K be a number field with ring of adèles

$$\mathbb{A}_K = \prod_{v \in M_K}{}' K_v \qquad \text{(almost all coordinates are integral).}$$

Brauer-Manin obstruction.

Let K be a number field with ring of adèles

$$\mathbb{A}_K = \prod_{v \in M_K}{}' K_v \qquad \text{(almost all coordinates are integral).}$$

Let X be a smooth, absolutely irreducible, projective variety over K.

Then the set of adèlic points is

$$X(\mathbb{A}_K) = \prod_{v \in M_K} X(K_v)$$

and this is nonempty if and only if X is **LSE**.

Brauer-Manin obstruction.

For any scheme Z we set $\operatorname{Br} Z = H^2_{\text{ét}}(Z, \mathbb{G}_m)$.
For any ring R we set $\operatorname{Br} R = \operatorname{Br} \operatorname{Spec} R$.

Brauer-Manin obstruction.

For any scheme Z we set $\operatorname{Br} Z = H^2_{\text{ét}}(Z, \mathbb{G}_m)$.
For any ring R we set $\operatorname{Br} R = \operatorname{Br} \operatorname{Spec} R$.

For any K-algebra S and any S-point $x\colon \operatorname{Spec} S \to X$, we get a homomorphism $x^*\colon \operatorname{Br} X \to \operatorname{Br} S$, yielding a map

$$\rho_S\colon X(S) \to \operatorname{Hom}(\operatorname{Br} X, \operatorname{Br} S).$$

Brauer-Manin obstruction.

For any scheme Z we set $\operatorname{Br} Z = H^2_{\text{ét}}(Z, \mathbb{G}_m)$.
For any ring R we set $\operatorname{Br} R = \operatorname{Br} \operatorname{Spec} R$.

For any K-algebra S and any S-point $x\colon \operatorname{Spec} S \to X$, we get a homomorphism $x^*\colon \operatorname{Br} X \to \operatorname{Br} S$, yielding a map

$$\rho_S\colon X(S) \to \operatorname{Hom}(\operatorname{Br} X, \operatorname{Br} S).$$

We will apply this to K and to the ring of adèles $\mathbb{A}_K$.

From class field theory we have

$$0 \to \quad \operatorname{Br} K \to \quad \operatorname{Br} \mathbb{A}_K \to \quad \mathbb{Q}/\mathbb{Z}$$

Applying $\operatorname{Hom}(\operatorname{Br} X, \_)$ we find ...

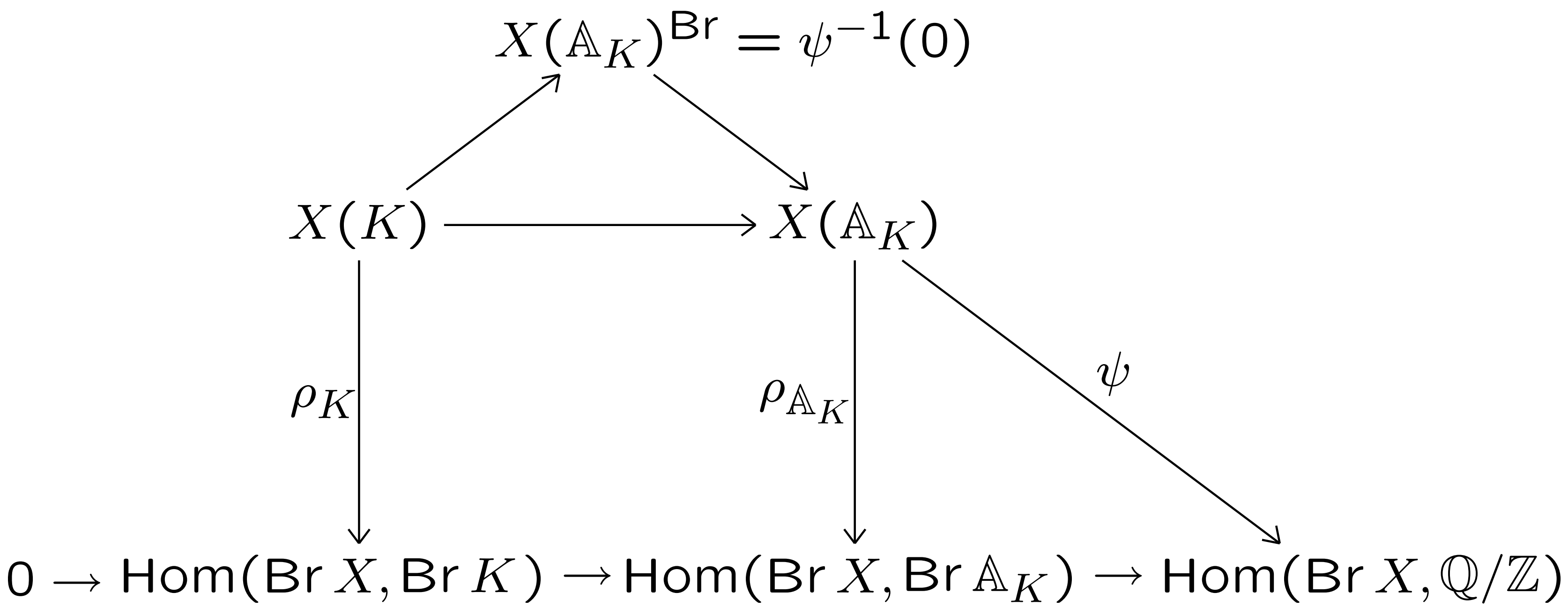
$$0 \to \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, K) \to \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, \mathbb{A}_K) \to \mathrm{Hom}(\mathrm{Br}\, X, \mathbb{Q}/\mathbb{Z})$$

$$\begin{array}{ccccccc} & & X(K) & & X(\mathbb{A}_K) & & \\ & & \downarrow{\scriptstyle \rho_K} & & \downarrow{\scriptstyle \rho_{\mathbb{A}_K}} & & \\ 0 & \to & \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, K) & \to & \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, \mathbb{A}_K) & \to & \mathrm{Hom}(\mathrm{Br}\, X, \mathbb{Q}/\mathbb{Z}) \end{array}$$

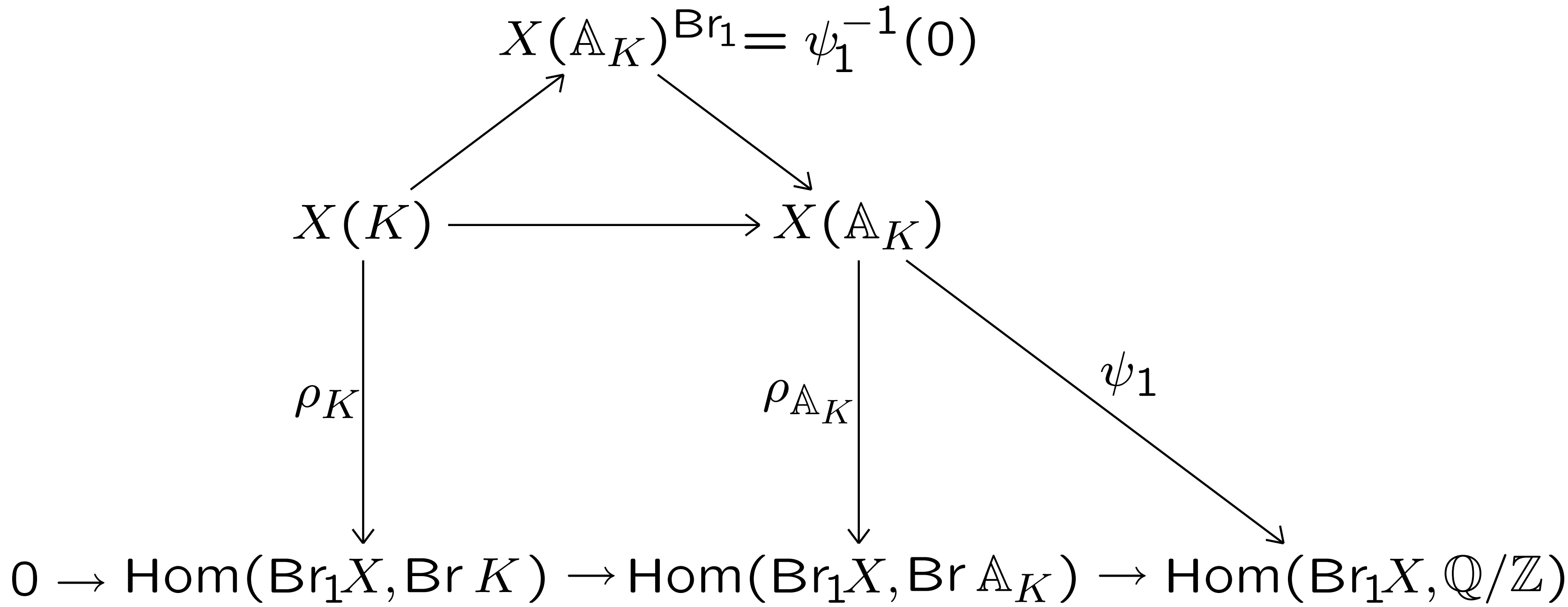
$$
\begin{array}{ccccc}
 & X(K) & \longrightarrow & X(\mathbb{A}_K) & \\
 & \Big\downarrow{\scriptstyle \rho_K} & & \Big\downarrow{\scriptstyle \rho_{\mathbb{A}_K}} & \searrow{\scriptstyle \psi} \\
0 \to & \mathrm{Hom}(\mathrm{Br}\,X, \mathrm{Br}\,K) & \to & \mathrm{Hom}(\mathrm{Br}\,X, \mathrm{Br}\,\mathbb{A}_K) & \to \mathrm{Hom}(\mathrm{Br}\,X, \mathbb{Q}/\mathbb{Z})
\end{array}
$$

$$
\begin{array}{ccccc}
 & & X(\mathbb{A}_K)^{\mathsf{Br}} = \psi^{-1}(0) & & \\
 & \nearrow & & \searrow & \\
X(K) & & \longrightarrow & & X(\mathbb{A}_K) \\
\downarrow \rho_K & & & & \downarrow \rho_{\mathbb{A}_K} \quad \searrow \psi \\
0 \to \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, K) & & \longrightarrow & & \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, \mathbb{A}_K) \to \mathrm{Hom}(\mathrm{Br}\, X, \mathbb{Q}/\mathbb{Z})
\end{array}
$$

$$X(\mathbb{A}_K)^{\mathsf{Br}} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset$$



$$X(\mathbb{A}_K)^{\mathrm{Br}_1} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset$$



$$\mathrm{Br}_1\, X = \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$$

$$X(\mathbb{A}_K)^{\mathsf{Br}(1)} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset.$$

There is a **Brauer-Manin obstruction** to the Hasse principle if

$$X(\mathbb{A}_K) \neq \emptyset \qquad \text{and} \qquad X(\mathbb{A}_K)^{\mathsf{Br}} = \emptyset.$$

$$X(\mathbb{A}_K)^{\mathrm{Br}(1)} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset.$$

There is a **Brauer-Manin obstruction** to the Hasse principle if

$$X(\mathbb{A}_K) \neq \emptyset \qquad \text{and} \qquad X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset.$$

For a class $\mathcal{S}$ of varieties over K the Brauer-Manin obstruction is the **only obstruction** to the Hasse principle if for every $X \in \mathcal{S}$ we have

$$X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset \quad \Leftrightarrow \quad X(K) = \emptyset.$$

$$X(\mathbb{A}_K)^{\mathrm{Br}(1)} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset.$$

There is a **Brauer-Manin obstruction** to the Hasse principle if

$$X(\mathbb{A}_K) \neq \emptyset \qquad \text{and} \qquad X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset.$$

For a class $\mathcal{S}$ of varieties over K the Brauer-Manin obstruction is the **only obstruction** to the Hasse principle if for every $X \in \mathcal{S}$ we have

$$X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset \quad \Leftrightarrow \quad X(K) = \emptyset.$$

Conjecture: The Brauer-Manin obstruction is the only obstruction to the Hasse principle for **rationally connected varieties**.

Definition: A **K3 surface** is a smooth, absolutely irreducible, projective surface X with trivial canonical sheaf and $H^1(X, \mathcal{O}_X) = 0$.

Examples of K3's:
smooth surfaces of degree 4 in $\mathbb{P}^3$,
Kummer surfaces.

Question 4: Is the Brauer-Manin obstruction the only obstruction to the Hasse principle for **K3 surfaces**?
(**unknown**)

Relating the two problems

Relating the two problems

Let C be a smooth cubic curve over K in $\mathbb{P}^2$ and ρ the automorphism

$$\rho\colon\ C \times C \to C \times C, \quad (P, Q) \mapsto (Q, R),$$

with R the third intersection point of C with the line through P and Q.

Relating the two problems

Let C be a smooth cubic curve over K in $\mathbb{P}^2$ and ρ the automorphism

$$\rho\colon\ C \times C \to C \times C, \quad (P, Q) \mapsto (Q, R),$$

with R the third intersection point of C with the line through P and Q.

Let X_C be the minimal desingularization of the quotient $(C \times C)/\rho$.

Then X_C is a K3 surface.

Theorem (vL)
Let C be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3+by^3+cz^3=0$ and suppose
(i) C is **LSE**,
(ii) $abc \in K^*$ is not a cube,
(iii) C has no cubic points (with K as ground field).
Then

$$X_C(\mathbb{A}_K)^{\mathrm{Br}_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

(algebraic Brauer-Manin obstruction is not the only one).

Theorem (vL)
Let C be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3+by^3+cz^3=0$ and suppose
(i) C is **LSE**,
(ii) $abc \in K^*$ is not a cube,
(iii) C has no cubic points (with K as ground field).
Then

$$X_C(\mathbb{A}_K)^{\mathrm{Br}_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

sketch of proof:
(iii) implies $X_C(K) = \emptyset$.
Indeed, $T \in X_C(K)$ corresponds to a galois-invariant orbit $\{(P,Q),(Q,R),(R,P)\}$ of ρ on $C \times C$, so galois acts by even permutations only and P,Q,R are defined over some cubic extension that is galois.

Theorem (vL)
Let C be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3+by^3+cz^3=0$ and suppose
(i) C is **LSE**,
(ii) $abc \in K^*$ is not a cube,
(iii) C has no cubic points (with K as ground field).
Then

$$X_C(\mathbb{A}_K)^{\mathrm{Br}_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

sketch of proof:
(iii) implies $X_C(K) = \emptyset$.
(ii) implies $\mathrm{Br}_1 X_C = \mathrm{Br}\, K$.
Indeed, $\mathrm{Br}_1 X_C / \mathrm{Br}\, K \cong H^1(K, \mathrm{Pic}\, \overline{X}_C)$, and $\mathrm{Pic}\, \overline{X}_C$ is defined over $K(\zeta_3, \sqrt[3]{a/c}, \sqrt[3]{b/c})$, with galois group contained in $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$.
The only subgroups with nontrivial $H^1(K, \mathrm{Pic}\, \overline{X}_C)$ all fix $\sqrt[3]{abc}$.

Theorem (vL)

Let C be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3+by^3+cz^3=0$ and suppose

(i) C is **LSE**,

(ii) $abc \in K^*$ is not a cube,

(iii) C has no cubic points (with K as ground field).

Then

$$X_C(\mathbb{A}_K)^{\mathrm{Br}_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

sketch of proof:

(iii) implies $X_C(K) = \emptyset$.

(ii) implies $\mathrm{Br}_1 X_C = \mathrm{Br}\, K$.

(i) implies that X_C is **LSE**, so $X_C(\mathbb{A}_K) \neq \emptyset$.

Theorem (vL)

Let C be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3 + by^3 + cz^3 = 0$ and suppose

(i) C is **LSE**,

(ii) $abc \in K^*$ is not a cube,

(iii) C has no cubic points (with K as ground field).

Then

$$X_C(\mathbb{A}_K)^{\mathrm{Br}_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

sketch of proof:

(iii) implies $X_C(K) = \emptyset$.

(ii) implies $\mathrm{Br}_1 X_C = \mathrm{Br}\, K$.

(i) implies that X_C is **LSE**, so $X_C(\mathbb{A}_K) \neq \emptyset$.

Done!