ON CHARACTER VARIETIES OF TWO-BRIDGE KNOT GROUPS

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Abstract. We find explicit models for the $\mathrm{PSL}_2(\mathbb{C})$- and $\mathrm{SL}_2(\mathbb{C})$-character varieties of the fundamental groups of complements in $\mathbb{S}^3$ of an infinite family of two-bridge knots that contains the twist knots. We compute the genus of the components of these character varieties, and deduce upper bounds on the degree of the associated trace fields. We also show that these knot complements are fibered if and only if they are commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, resolving a conjecture of Hoste and Shanahan.

1. Introduction

Given a finitely generated group Γ, the set of all representations $\Gamma \to \mathrm{SL}_2(\mathbb{C})$ naturally carries the structure of an algebraic set. So does the set of characters of these representations. Often the components of this last set that contain only characters of abelian representations are well understood. The union of the other components is called the $\mathrm{SL}_2(\mathbb{C})$-character variety of Γ. Over the last few decades, the $\mathrm{SL}_2(\mathbb{C})$-character variety of the fundamental groups of hyperbolic 3-manifolds has proven to be an effective tool in understanding their topology (see [5], [6], [7]). The same can be said for their $\mathrm{PSL}_2(\mathbb{C})$-character variety, defined in §2.1.2, but in general it is difficult to find even the simplest invariants of these varieties, such as the number of irreducible components.

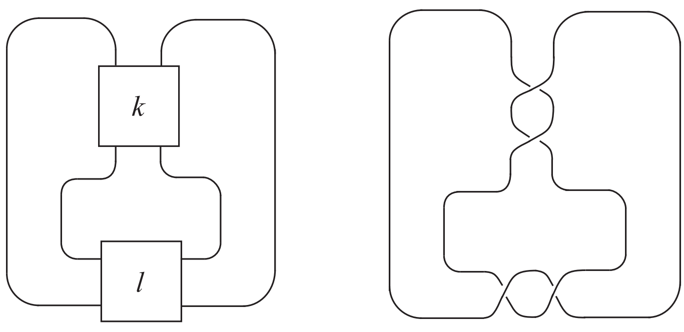

Figure 1. The knot $J(k,l)$ and the figure-eight knot $J(2,-2)$.

In this paper we consider the case that Γ is a knot group, i.e., the fundamental group of the complement in $\mathbb{S}^3$ of a knot. We look at the knots $J(k,l)$ as described in Figure 1, where k and l are integers denoting the number of half twists in the labeled boxes; positive numbers correspond to right-handed twists and negative numbers correspond to left-handed twists. Note that $J(k,l)$ is a knot if and only if kl is even; otherwise it is a two-component link. The subfamilies of knots $J(\pm 2, l)$, with $l \in \mathbb{Z}$, consist of all twist knots, containing the figure-eight knot $J(2,-2)$ and the trefoil $J(2,2)$. The complement of the knot $J(k,l)$ is hyperbolic if and only if $|k|, |l| \geq 2$ and $J(k,l)$ is not the trefoil.

We compute the genus of every component of the character varieties associated to these knots. This is the first time such results have been found for an infinite family of knots. In particular it shows that the genus of both character varieties of a knot complement can be arbitrarily large, which was not known before.

More precisely, for any nonzero integers k and l with kl even, we let $M(k,l)$ denote the complement $\mathbb{S}^3 \setminus J(k,l)$ and let $X(k,l)$ and $Y(k,l)$ denote the $\mathrm{SL}_2(\mathbb{C})$- and $\mathrm{PSL}_2(\mathbb{C})$-character variety of the fundamental group $\pi_1(M(k,l))$. Both varieties are curves and $X(k,l)$ is a double cover of $Y(k,l)$. Our first main result is a non-recursively defined model for $Y(k,l)$. Secondly, we construct a projective birational model for $Y(k,l)$ that we prove to be smooth and irreducible when $J(k,l)$ is hyperbolic and $k \neq l$. For $k = l > 2$ the curve $Y(k,l)$ has two smooth components and we identify which of the two is the canonical component $Y_0(k,l)$, defined in §2.1.2. The results, and those for $X(k,l)$ and its canonical component $X_0(k,l)$, defined in §2.1.1, are summarized in the following theorems.

Theorem 1.1. *Let k, l be any nonzero integers with l even, $|k| \geq 2$, and $k \neq l$.*

(1) *The curve $Y(k,l)$ is irreducible. It has geometric genus*
$$(\lfloor |k|/2 \rfloor - 1)(|l|/2 - 1)$$
and is hyperelliptic if and only if $|k| \leq 5$ or $|l| \leq 5$.

(2) *If $|l| > 2$, then the curve $Y(l,l)$ has two components. The component $Y_0(l,l)$ has genus 0. The other component has genus $(|l|/2-2)^2$ and is hyperelliptic if and only if $|l| \leq 6$.*

Theorem 1.2. *Suppose l is a nonzero even integer, say $l = 2n$. If $k \neq l$ is an integer satisfying $|k| \geq 2$, then $X(k,l)$ is irreducible and its genus equals*
$$3|mn| - |m| - a|n| + b,$$
with $m = \lfloor k/2 \rfloor$ and
$$a = \begin{cases} 4 & \text{if } k \text{ is odd and } k < 0, \\ 1 & \text{otherwise.} \end{cases} \qquad b = \begin{cases} 2 & \text{if } k \text{ is odd and } k < 0 < l, \\ 1 & \text{if } k \text{ is odd and } l < 0, \\ -1 & \text{if } k \text{ is even and } kl > 0, \\ 0 & \text{otherwise.} \end{cases}$$
If $|l| > 2$, then $X(l,l)$ has two components, namely $X_0(l,l)$ of genus $|n| - 1$ and an other component of genus $3n^2 - 7|n| + 5$.

Precisely two knots in this family have canonical components of their $\mathrm{SL}_2(\mathbb{C})$-character varieties that have genus 1, namely the figure-eight knot $J(2,-2)$ and the 7_4 knot $J(4,4)$.

Recent results have shown that arithmetic properties of the $\mathrm{SL}_2(\mathbb{C})$- and $\mathrm{PSL}_2(\mathbb{C})$-character varieties can give information about topological invariants such as the commensurability classes of knot complements ([4], [15], [16]). Our irreducibility results allow us to use a criterion of Calegari and Dunfield [4] to prove a conjecture of Hoste and Shanahan [15, Conj. 1] about commensurability classes of the knots $J(k,l)$. Note that fibered means fibered over $\mathbb{S}^1$. The result is the following.

Theorem 1.3. *The manifold $M(k,l)$ is fibered if and only if $M(k,l)$ is commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.*

If K is a hyperbolic knot, let $[F(K):\mathbb{Q}]$ denote the degree of the trace field $F(K)$ of K over $\mathbb{Q}$, i.e., the field generated by all traces of elements in the image of a lift $\pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{SL}_2(\mathbb{C})$ of the discrete faithful representation (see [13] and §2.1.1). From the non-recursively defined model for $Y(k,l)$ we can deduce an upper bound for the degree of the trace field of $J(k,l)$. The following theorem says that for all hyperbolic $J(k,l)$ this bound is of the same order of magnitude as the genus of $X_0(k,l)$.

Theorem 1.4. *Let k and l be integers for which $J(k,l)$ is a hyperbolic knot. Then the degree $[F\big(J(k,l)\big):\mathbb{Q}]$ of the trace field of $J(k,l)$ is bounded by $\frac{1}{2}|kl|$. It is bounded by $\frac{1}{2}kl-1$ if $kl>0$ and by $|l|-1$ if $k=l$.*

In §2.2.1 we define the family of two-bridge knots $K(p,q)$, parametrized by pairs (p,q) of coprime odd integers satisfying $-p<q\le p$. For all nonzero integers k,l with kl even, the knot $J(k,l)$ is ambient isotopic with $K(p,q)$ for the unique such p,q for which the image of q/p in $\mathbb{Q}/\mathbb{Z}$ equals that of $l/(1-kl)$; we find from the roughest bound in Theorem 1.4 that $(p-1)/2$ is an upper bound for the degree of the trace field of $K(p,q)$. This also follows for general two-bridge knots from a result of Riley [25, §3].

For any nonzero integers k and l with kl even, let $c(k,l)$ denote the crossing number of the knot $J(k,l)$, i.e., the minimum number of crossings in any projection of the knot. For the hyperbolic twist knots $J(2,l)$ the smallest bounds of Theorem 1.4 are in fact equalities and directly related to the crossing number $c(2,l)$ by [13, Thm. 1, Cor. 1]. We immediately obtain an interesting corollary.

Corollary 1.5. *For any integer $l\neq -1,0,1,2$ the genus of the $\mathrm{SL}_2(\mathbb{C})$-character variety $X(2,l)=X_0(2,l)$ of $J(2,l)$ equals*

$$c(2,l)-3=[F\big(J(2,l)\big):\mathbb{Q}]-1.$$

It is easy to check the degree of the trace field of $J(k,l)$ for small values of $|k|$ and $|l|$, where the smallest upper bounds given in Theorem 1.4 are in fact equalities. We therefore wonder the following.

Question 1.6. *Let k and l be integers for which $J(k,l)$ is a hyperbolic knot. Is the degree $[F\big(J(k,l)\big):\mathbb{Q}]$ of the trace field of $J(k,l)$ equal to $-\frac{1}{2}kl$ if $kl<0$? Is it equal to $\frac{1}{2}kl-1$ if $kl>0$ and $k\neq l$ and equal to $|l|-1$ if $k=l$?*

In fact, for all p,q as above with $p<100$ and $K(p,q)$ hyperbolic, we checked that when the character variety of the two-bridge knot $K(p,q)$ is irreducible, then the degree of the trace field $F\big(K(p,q)\big)$ of $K(p,q)$ equals the upper bound $(p-1)/2$ proven by Riley. We therefore also wonder the following.

Question 1.7. *Let p and q be coprime odd integers with $-p<q<p$ for which the knot $K(p,q)$ is hyperbolic. Assume that the $\mathrm{PSL}_2(\mathbb{C})$-character variety of the fundamental group of the complement of $K(p,q)$ is irreducible. Is the degree of the trace field of $K(p,q)$ equal to $(p-1)/2$?*

The paper is set up as follows. In the next section we describe character varieties in general and in particular for two-bridge knots, a family of knots that contains our family. This includes the definition of the canonical component. In §2.3 we describe the family $J(k,l)$ as a subfamily of the two-bridge knots and find the fundamental groups of their complements. In §2.4 we give a brief summary of the theory of Newton polygons and algebraic curves.

The two models for $Y(k,l)$ are defined in §3 and §4. More precisely, the standard model $C(k,l)$ is given non-recursively in Proposition 3.8 and the smooth model $D(k,l)$ is given in (12). The birationality is proven in Proposition 4.4. Proposition 4.6 identifies which component of the new model of $Y_0(l,l)$ corresponds with the canonical component, after which we can prove Theorem 1.4.

We find the number of components of $Y(k,l)$ for all integers k and l and prove that all components are smooth in §5. In §6 we use this to prove Theorems 1.1 and 1.2. Theorem 1.3 is proved in the final section, §7.

2. Preliminaries

2.1. **Representation and character varieties.** We will begin with some background material concerning the representation and character varieties of finitely generated groups, and knot groups in particular. Standard references for this material are [6] and [7].

Let Γ be any finitely generated group with generating set $\{\gamma_1,\dots,\gamma_N\}$. The set $\mathrm{R}(\Gamma)=\mathrm{Hom}(\Gamma,\mathrm{SL}_2(\mathbb{C}))$ can be given the structure of an affine algebraic set defined over $\mathbb{Q}$ by using the entries of the images of the γ_i under $\rho\in\mathrm{R}(\Gamma)$ as coordinates for ρ. We therefore will refer to $\mathrm{R}(\Gamma)$ as the $\mathrm{SL}_2(\mathbb{C})$-*representation variety* of Γ. The isomorphism class of this variety does not depend on the choice of generators. In general, $\mathrm{R}(\Gamma)$ need not be irreducible.

2.1.1. $\mathrm{SL}_2(\mathbb{C})$-*character varieties.* The character of a representation ρ is the function $\chi_\rho\colon\Gamma\to\mathbb{C}$ defined by $\chi_\rho(\gamma)=\mathrm{tr}(\rho(\gamma))$. Define the set of characters $\tilde{X}(\Gamma)=\{\chi_\rho:\rho\in\mathrm{R}(\Gamma)\}$, which is often denoted by $X(\Gamma)$ elsewhere in the literature, but we will reserve that notation for a particular subset of $\tilde{X}(\Gamma)$. For all $\gamma\in\Gamma$ we define the function $t_\gamma\colon\mathrm{R}(\Gamma)\to\mathbb{C}$ by $t_\gamma(\rho)=\chi_\rho(\gamma)$. Let T be the subring of the ring of all functions from $\mathrm{R}(\Gamma)$ to $\mathbb{C}$ that is generated by 1 and the functions t_γ for $\gamma\in\Gamma$. The ring T is finitely generated, for instance by the elements

$$t_{\gamma_{i_1}\cdots\gamma_{i_r}},\ 1\le i_1<\ldots<i_r\le N$$

(see [7], Proposition 1.4.1). This implies that a character $\chi\in\tilde{X}(\Gamma)$ is determined by its values on finitely many elements of Γ. If $h_1,\dots,h_m$ are generators of T, then the map $\mathrm{R}(\Gamma)\to\mathbb{C}^m$ given by $\rho\mapsto(h_1(\rho),\dots,h_m(\rho))$ induces an injection $\tilde{X}(\Gamma)\to\mathbb{C}^m$. This gives $\tilde{X}(\Gamma)$ the structure of a closed algebraic subset of $\mathbb{C}^m$, but the fact that it is closed is quite nontrivial (see [7, Proposition 1.4.4]). It follows that $\tilde{X}(\Gamma)$ has the structure of an abstract affine algebraic variety with coordinate ring $T_{\mathbb{C}}=T\otimes\mathbb{C}$. Different sets of generators of T give different models for $\tilde{X}(\Gamma)$, all isomorphic over $\mathbb{Z}$. We refer to $\tilde{X}(\Gamma)$ as the $\mathrm{SL}_2(\mathbb{C})$-*character variety* of Γ.

A representation $\rho\in\mathrm{R}(\Gamma)$ is *reducible* if all the $\rho(\gamma)$ with $\gamma\in\Gamma$ have a common one-dimensional eigenspace, otherwise it is called *irreducible*. A representation ρ is *abelian* if its image is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$, and *nonabelian* otherwise. Note that every irreducible representation is necessarily nonabelian, although there do exist nonabelian reducible representations. For fundamental groups of knot complements in $\mathbb{S}^3$ these are all metabelian (see [12, Section 1]).

The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathrm{R}(\Gamma)$ by conjugation. Let $\hat{\mathrm{R}}(\Gamma)$ denote the set of orbits. Two representations $\rho, \rho' \in \mathrm{R}(\Gamma)$ are *conjugate* if they lie in the same orbit. Since two conjugate representations give the same character, the trace map $\mathrm{R}(\Gamma) \to \tilde{X}(\Gamma)$ induces a well-defined map $\hat{\mathrm{R}}(\Gamma) \to \tilde{X}(\Gamma)$. Note that if Γ is finite, then this map is a bijection, but in general it need not be injective. It is injective when restricted to irreducible representations; if $\rho, \rho' \in \mathrm{R}(\Gamma)$ have equal characters $\chi_\rho = \chi_{\rho'}$, and ρ is irreducible, then ρ and ρ' are conjugate (see [7, Proposition 1.5.2]).

Let $\tilde{X}_{\mathrm{a}}(\Gamma)$, and $\tilde{X}_{\mathrm{na}}(\Gamma)$ denote the set of characters of abelian and nonabelian representations $\rho \in \mathrm{R}(\Gamma)$ respectively. The set $\tilde{X}_{\mathrm{a}}(\Gamma)$ is a Zariski closed subset of $\tilde{X}(\Gamma)$ (see [12, Propositions 1.3(ii) and 1.7(1)]).

We can say more when Γ is the fundamental group of a knot complement in $\mathbb{S}^3$. We will **assume** this to be case from now on, say $\Gamma = \pi_1(M)$ is the fundamental group of the 3-manifold $M = \mathbb{S}^3 \setminus K$ for the knot K in $\mathbb{S}^3$. Then $\tilde{X}_{\mathrm{na}}(\Gamma)$ is also a Zariski closed subset of $\tilde{X}(\Gamma)$ and $\tilde{X}_{\mathrm{a}}(\Gamma)$ is isomorphic to $\mathbb{A}^1$ (see [12, Proposition 1.7(2) and Corollary 1.10]). As the characters of abelian representations are well understood, we will focus only on $\tilde{X}_{\mathrm{na}}(\Gamma)$, which we will also denote by $X(\Gamma)$. By abuse of language, we will refer to $X(\Gamma)$ as the $\mathrm{SL}_2(\mathbb{C})$-*character variety* of Γ as well.

If M is a hyperbolic knot complement, then M is isomorphic to a quotient of hyperbolic 3-space $\mathbb{H}^3$ by a discrete group. By Mostow-Prasad rigidity there is then a discrete faithful representation $\overline{\rho}_0 \colon \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ that is unique up to conjugation, defining an action of Γ on $\mathbb{H}^3$ whose quotient $\mathbb{H}^3/\Gamma$ is isomorphic with M. Moreover, the representation $\overline{\rho}_0$ can be lifted to a discrete faithful representation $\Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. Fix such a lift and call it ρ_0. By work of Thurston [33], the character of ρ_0 is contained in a unique component of $X(\Gamma)$, which has dimension 1 and which will be denoted by $X_0(\Gamma)$. In all cases presented in this paper, we will see that $X_0(\Gamma)$ does not depend on the choice of lift ρ_0.

2.1.2. $\mathrm{PSL}_2(\mathbb{C})$-*character varieties.* There are various constructions for the $\mathrm{PSL}_2(\mathbb{C})$-representation and character varieties of Γ, none of which are quite as standard. We refer the reader to the general definition in [16, §2.1] and to [1, §3], and [8]. Since in our case Γ is the fundamental group of a knot complement in $\mathbb{S}^3$, the definitions simplify dramatically. Note that $\mu_2 \cong \{\pm 1\}$ is isomorphic to the kernel of the homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$.

The first simplification comes from the fact that we have $H^2(\Gamma, \mu_2) = 0$ (see [1, page 756], or [8, remark after Lemma 2.1]. Under this condition, the $\mathrm{PSL}_2(\mathbb{C})$-*character variety* $\tilde{Y}(\Gamma)$ is isomorphic to the quotient $\tilde{X}(\Gamma)/\mathrm{Hom}(\Gamma, \mu_2)$, where $\sigma \in \mathrm{Hom}(\Gamma, \mu_2)$ acts on $\chi_\rho \in \tilde{X}(\Gamma)$ by $(\sigma\chi_\rho)(\gamma) = \sigma(\gamma)\chi_\rho(\gamma)$ for all $\gamma \in \Gamma$.

The second simplification comes from a better understanding of $\mathrm{Hom}(\Gamma, \mu_2)$ in our specific case. Since Γ is a knot group, there are presentations for Γ where the generators γ_i are all meridians of K. For such a presentation, the γ_i are all conjugate and we have $t_{\gamma_i} = t_{\gamma_j}$ for $1 \leq i, j \leq N$. In fact, the Wirtinger presentation (see [29, Section 3.D]) is such a presentation where the relations are of length 4 in the generators and their inverses, with one relation for each crossing. Therefore, there is a well-defined notion of *parity* of an element $\gamma \in \Gamma$, based on the parity of the length of γ as a word in terms of meridians. Let $\Gamma_{\mathrm{e}} \subset \Gamma$ denote the subgroup of index 2 consisting of all even $\gamma \in \Gamma$. Any $\sigma \in \mathrm{Hom}(\Gamma, \mu_2)$ sends all the (conjugate) meridians to the same element, so σ is trivial on Γ_{e}, and we find $\mathrm{Hom}(\Gamma, \mu_2) \cong$

$\mathrm{Hom}(\Gamma/\Gamma_{\mathrm{e}}, \mu_2) \cong \mathrm{Hom}(\mu_2, \mu_2) \cong \mu_2$. The induced action of μ_2 on $R(\Gamma)$ is given by $(-\rho)(\gamma) = -\rho(\gamma)$ for $\gamma \notin \Gamma_{\mathrm{e}}$ and $(-\rho)(\gamma) = \rho(\gamma)$ for $\gamma \in \Gamma_{\mathrm{e}}$. The induced action on $\tilde{X}(\Gamma)$ is given by $-\chi_\rho = \chi_{-\rho}$, and the corresponding action on T by negating t_γ for all $\gamma \notin \Gamma_{\mathrm{e}}$. We conclude that the $\mathrm{PSL}_2(\mathbb{C})$-character variety $\tilde{Y}(\Gamma)$ is isomorphic to $\tilde{X}(\Gamma)/\mu_2$ and its coordinate ring is $T_{\mathrm{e}} \otimes \mathbb{C}$, where $T_{\mathrm{e}} = T^{\mu_2}$ is the subring of T of all elements invariant under μ_2.

We let $Y(\Gamma)$ denote the image of $X(\Gamma) = \tilde{X}_{\mathrm{na}}(\Gamma)$ under the quotient map $\tilde{X}(\Gamma) \to \tilde{Y}(\Gamma)$. As for $X(\Gamma)$, by abuse of language, we will refer to $Y(\Gamma)$ as the $\mathrm{PSL}_2(\mathbb{C})$-*character variety* of Γ. If M is hyperbolic, then we denote the component of $Y(\Gamma)$ that contains the character of the discrete faithful representation of Γ by $Y_0(\Gamma)$, obtaining a map $X_0(\Gamma) \to Y_0(\Gamma)$.

2.2. **Character varieties of two-bridge knot complements.** The knots $J(k,l)$ that we are interested in are part of a larger family, the so-called two-bridge knots. As we will use some results on two-bridge knots, we now describe these knots and their character varieties.

2.2.1. *Two-bridge knots.* two-bridge knots are those knots admitting a projection with only two maxima and two minima. To every two-bridge knot we can associate a pair (p,q) of coprime odd integers with $-p < q \le p$, such that the two-bridge knot is ambient isotopic to the knot $K(p,q)$ we now define. As described in [3, Chapter 12], to a pair (p,q) as above, we associate the sequence $[a_1, \ldots, a_s]$ of entries in the continued fraction

$$\frac{q}{p} + \epsilon = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_s}}}}}$$

where $\epsilon \in \{0,1\}$ is such that $0 < \frac{q}{p} + \epsilon \le 1$ and where these entries satisfy $a_i \ge 1$ and they are chosen such that s is odd, which is possible by replacing the last entry a of the usual continued fraction by the two elements $a-1$ and 1 if necessary. Then $K(p,q)$ is the knot presented by the so-called 4-plat in Figure 2, where the j-th block between the two middle strands consists of a_{2j-1} left-handed half twists, and the j-th block between the two left-most strands consists of a_{2j} right-handed half twists. The knots $K(p,q)$ and $K(p',q')$ (with (p,q) and (p',q') as above) are ambient isotopic if and only if $p = p'$ and either $q = q'$ or $qq' \equiv 1 \pmod{p}$ (see [3, Theorem 12.6]); if $p = p'$ and $qq' \equiv 1 \pmod{p}$, then the 4-plat presentation of $K(p',q')$ is obtained from turning the 4-plat presentation of $K(p,q)$ upside down, i.e., reversing the sequence $[a_1, \ldots, a_s]$, which comes down to rotating about a "horizontal" line in $\mathbb{S}^3$. Indeed, it is well known that the fractions q/p and q'/p' of the continued fractions associated to any sequence of numbers of odd length and its reverse respectively, satisfy $p = p'$ and $qq' \equiv 1 \pmod{p}$.

If $p = p'$ and $qq' \equiv 1 \pmod{p}$, then turning the 4-plat $K(p,q)$ upside down induces isomorphisms between the fundamental groups and character varieties of $K(p,q)$ and $K(p',q')$. Now assume $q = q'$, so $q^2 \equiv 1 \pmod{p}$, and let $Y(p,q)$ denote the $\mathrm{PSL}_2(\mathbb{C})$-character variety associated to $K(p,q)$. Then turning the 4-plat presentation upside down induces an automorphism of $Y(p,q)$. If furthermore

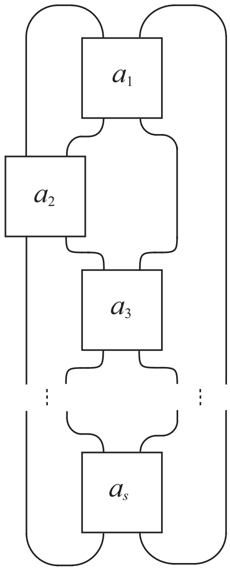

FIGURE 2. The 4-plat corresponding to $[a_1, a_2, \ldots, a_s]$ for s odd.

$K(p, q)$ is hyperbolic, then Ohtsuki [22] proves that $Y(p, q)$ is reducible by showing that the canonical component $Y_0(p, q)$ is fixed by this involution, while other components are not. This fact will be used in §4 to determine $Y_0(2n, 2n)$.

The fundamental group $\pi_1(\mathbb{S}^3 \setminus K(p, q))$ of the knot complement $\mathbb{S}^3 \setminus K(p, q)$ has a presentation

$$\Gamma = \langle\, a, b \mid wa = bw \,\rangle, \tag{1}$$

where

$$w = a^{e_1} b^{e_2} \cdots a^{e_{p-2}} b^{e_{p-1}} \tag{2}$$

with $e_i = (-1)^{\lfloor \frac{iq}{p} \rfloor}$. This presentation follows from the canonical Schubert normal form [30] of the two-bridge diagram of $K(p, q)$ (see [25, Prop. 1], [20, (2.1)], [19, Prop. 1]).

2.2.2. *Character Varieties.* As in the previous section, for any $\gamma \in \Gamma$, let t_γ be the function $t_\gamma \colon \mathrm{R}(\Gamma) \to \mathbb{C}, \rho \mapsto \mathrm{tr}(\rho(\gamma))$, and let T be the subring of the ring of all functions from $\mathrm{R}(\Gamma)$ to $\mathbb{C}$ that is generated by 1 and these functions. Since a and b are conjugate in Γ, we have $t_a = t_b$. Therefore, the ring T is generated by t_a and t_{ab} (see §2.1.1), which are the most common traces used as coordinates to define the $\mathrm{SL}_2(\mathbb{C})$-character variety of $K(p, q)$. We will use slightly different coordinates, which define a nicer model. For any $\rho \in \mathrm{R}(\Gamma)$, the matrices $\rho(b)$ and $\rho(b^{-1})$ have the same traces, so we have $t_{b^{-1}} = t_b = t_a$. Using a and b^{-1} as generators of Γ, we may also use t_a and $t_{ab^{-1}}$ as coordinates. Therefore, the $\mathrm{SL}_2(\mathbb{C})$-character variety $X(\Gamma)$ may be identified with the image of $\mathrm{R}_{\mathrm{na}}(\Gamma)$ under the map $(t_{ab^{-1}}, t_a) \colon \mathrm{R}(\Gamma) \to \mathbb{A}^2$, where $\mathrm{R}_{\mathrm{na}}(\Gamma)$ is the set of nonabelian representations. For any $\lambda_0 \in \mathbb{C}^*$ and $r_0 \in \mathbb{C}$, we set

$$A(\lambda_0) = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0^{-1} \end{pmatrix}, \quad B(\lambda_0, r_0) = \begin{pmatrix} \lambda_0 & 0 \\ 2 - r_0 & \lambda_0^{-1} \end{pmatrix}.$$

The entry $2 - r_0$ in $B(\lambda_0, r_0)$ is chosen so that $A(\lambda_0)B(\lambda_0, r_0)^{-1}$ has trace r_0.

Proposition 2.1. *Let $\rho \in \mathrm{R}(\Gamma)$ be a nonabelian representation. Then there are $\lambda_0 \in \mathbb{C}^*$ and $r_0 \in \mathbb{C}$ such that ρ is conjugate to the representation ρ' determined by $\rho'(a) = A(\lambda_0)$ and $\rho'(b) = B(\lambda_0, r_0)$. Conversely, any representation ρ' of this form is nonabelian and $(t_{ab^{-1}}, t_a)(\rho') = (r_0, \lambda_0 + \lambda_0^{-1})$.*

Proof. Since a and b are conjugate in Γ, they have the same trace. This and the fact that they do not commute is enough to conclude the first statement by [27, Lemma 7]. For the second statement, suppose that ρ' satisfies the given conditions. Then we have $t_{ab^{-1}}(\rho') = \mathrm{tr}(\rho'(ab^{-1})) = r_0$, and $t_a(\rho') = \mathrm{tr}(\rho'(a)) = \lambda_0 + \lambda_0^{-1}$. If ρ' were abelian, then from $\rho'(w)\rho'(a) = \rho'(b)\rho'(w)$ we would find $\rho'(a) = \rho'(b)$, which is a contradiction. This finishes the proof. □

A nonabelian representation ρ is irreducible if and only if the r_0 in Proposition 2.1 satisfies $r_0 \neq 2$. Consider a point $P = (r_0, x_0) \in \mathbb{A}^2$. By Proposition 2.1, the point P is contained in $X(\Gamma)$ if and only if there is a $\lambda_0 \in \mathbb{C}^*$ with $x_0 = \lambda_0 + \lambda_0^{-1}$ such that the assignments $a \mapsto A(\lambda_0)$ and $b \mapsto B(\lambda_0, r_0)$ can be extended to a representation $\rho \in \mathrm{R}(\Gamma)$. Choose either λ_0 for which we have $x_0 = \lambda_0 + \lambda_0^{-1}$, and let $W(\lambda_0, r_0)$ denote the right-hand side of (2) with $A(\lambda_0)$ and $B(\lambda_0, r_0)$ substituted for a and b respectively. Then the assignment extends to a representation if and only if we have $W(\lambda_0, r_0)A(\lambda_0) = B(\lambda_0, r_0)W(\lambda_0, r_0)$, which results in four equations in λ_0 and r_0. The following proposition states that these equations reduce to a single equation in r_0 and x_0, which is therefore independent of the choice of λ_0. (Note that $x_0 = \lambda_0 + \lambda_0^{-1}$ and so $x_0^2 - 2 = \lambda_0^2 + \lambda_0^{-2}$.)

Proposition 2.2. *Consider the ring $\mathbb{Q}[r, \lambda, \lambda^{-1}]$ and let I denote the ideal generated by the four entries of the matrix $W(\lambda, r)A(\lambda) - B(\lambda, r)W(\lambda, r)$. Then I is generated by*

$$F = W_{11} + (\lambda^{-1} - \lambda)W_{12}, \tag{3}$$

where W_{ij} denotes the (i, j)-entry of $W(\lambda, r)$. Moreover, if we set $y = \lambda^2 + \lambda^{-2}$, then F is contained in the subring $\mathbb{Q}[r, y]$ of $\mathbb{Q}[r, \lambda, \lambda^{-1}]$.

Proof. See [26, Theorem 1]. □

We conclude that $X(\Gamma)$ is given in $\mathbb{A}^2(r, x)$ by $F = 0$, where F is viewed as a polynomial in $x = \lambda + \lambda^{-1}$. In particular, if $K(p, q)$ is hyperbolic, then the canonical component $X_0(\Gamma)$ will be an irreducible component of this algebraic set.

The coordinate ring of $X(\Gamma)$ is $\mathbb{C}[r, x]/(F)$ with F as in Proposition 2.2. Note again that r and x correspond to $t_{ab^{-1}}$ and t_a. The involution $\chi \to -\chi$ from §2.1.2 fixes r and sends x to $-x$. This implies that the coordinate ring of $Y(\Gamma)$ is isomorphic to the subring $\mathbb{C}[r, x^2]/(F) \cong \mathbb{C}[r, y]/(F)$, with $y = x^2 - 2$ corresponding to t_{a^2}. That is, $Y(\Gamma)$ is given in $\mathbb{A}^2(r, y)$ by $F = 0$ with F viewed as a polynomial in $y = \lambda^2 + \lambda^{-2}$. Therefore the double cover $X(\Gamma) \to Y(\Gamma)$ is given by $(r, x) \mapsto (r, x^2 - 2)$.

The projective closure of this model of $Y(\Gamma)$ has bad singularities at infinity. We will see that in the case of the subfamily of two-bridge knots of the form $J(k, l)$, discussed in the next section, there is an other model of $Y(\Gamma)$, whose coordinates are $t_{ab^{-1}}$ and the trace of another element, that has a smooth projective closure in $\mathbb{P}^1 \times \mathbb{P}^1$. This will allow us, for instance, to compute the geometric genus of the irreducible components of $Y(\Gamma)$ and $X(\Gamma)$ for that family.

Remark 2.3. *The trace map* $\hat{\mathrm{R}}_{\mathrm{na}}(\Gamma) \to X(\Gamma)$ *from the set of conjugacy classes of nonabelian representations in* $\mathrm{R}(\Gamma)$ *to the set of their characters is injective when restricted to irreducible representations, as discussed in §2.1. The reader be warned, however, that for reducible representations this is not the case. As stated correctly in* [2], *a representation of the form mentioned in Proposition 2.1 with* λ_0 *and* r_0 *is conjugate to the representation of the same form with* λ_0^{-1} *and* r_0*, but in general only in a group larger than* $\mathrm{SL}_2(\mathbb{C})$*. For* $r_0 = 0$ *these representations are not conjugate in* $\mathrm{SL}_2(\mathbb{C})$*, while they do have the same characters.*

2.3. **A family of two-bridge knots.** We are interested in the family of knots of the form $J(k,l)$ as described in the introduction (see Figure 1). Note that $J(k,l)$ is a knot precisely when kl is even, which we will almost always assume to be the case. Note also that $J(k,l)$ is symmetric in k and l. We will often make use of this symmetry and assume that l is even. Note furthermore that there is an obvious rotation of $\mathbb{S}^3$ taking $J(k,l)$ to its reverse when l is even, and that $J(-k,-l)$ is the mirror image of $J(k,l)$. Sometimes we will use this to assume without loss of generality that k or l is nonnegative. These are not the only equivalences among the knots, as for any integer l the knots $J(2,l)$ and $J(-2,l-1)$ are equivalent.

If kl is even, then the knot $J(k,l)$ is ambient isotopic to the two-bridge knot $K(p,q)$ for the unique odd and coprime integers p and q with $-p < q \leq p$ for which the image of $\frac{q}{p}$ in $\mathbb{Q}/\mathbb{Z}$ equals that of $\frac{l}{1-kl}$. Note that for any integer l the knots $J(2,l)$ and $J(-2,l-1)$ give the same p and q. When $|k|$ and $|l|$ are large enough, the following table shows to which sequence of numbers the corresponding 4-plat is associated.

$[1, k-2, 1, l-2, 1]$	for $k, l > 2$,
$[1, k-1, -l]$	for $k > 1$ and $l < 0$,
$[-k, l-1, 1]$	for $k < 0$ and $l > 1$,
$[-k-1, 1, -l-1]$	for $k, l < -1$.

Indeed, these 4-plats are easily checked to be ambient isotopic with $J(k,l)$ (see Figure 3 for the case $k, l > 2$). The remaining cases have small $|k|$ or $|l|$ and are also easily checked. Note that $J(l,k)$ is ambient isotopic with $K(p',q')$ for p', q' coprime odd integers such that $-p' < q' \leq p'$ and $\frac{q'}{p'} = \frac{k}{1-kl}$ in $\mathbb{Q}/\mathbb{Z}$. Then we have $p = p'$ and $qq' \equiv 1 \pmod{p}$, so switching k and l corresponds with turning the 4-plat upside down, cf. §2.2.1, and $k = l$ implies $q = q'$.

For any integers k, l, let $\pi_1(k,l)$ denote the fundamental group of of $\mathbb{S}^3 \setminus J(k,l)$. By Proposition 1 of [14], for even l, say $l = 2n$, this group has a presentation

$$\pi_1(k, 2n) \cong \langle\, a, b \mid a w_k^n = w_k^n b \,\rangle \tag{4}$$

with

$$w_k = \begin{cases} (ab^{-1})^m (a^{-1}b)^m & \text{if } k = 2m, \\ (ab^{-1})^m ab (a^{-1}b)^m & \text{if } k = 2m+1. \end{cases} \tag{5}$$

We will sketch a proof here, as we need a little more information about the structure of $\pi_1(k,l)$. We will also prove that for $l = 2n+1$, the group has a presentation

$$\pi_1(k, 2n+1) \cong \langle\, a, b \mid a w_k^n b = w_k^{n+1} \,\rangle. \tag{6}$$

As in [29, Section 3.D], where this is made precise, we interpret Figure 4 as a knot, contained almost entirely in one plane, except for the crossings, and with the base point P at "the eye of the reader." For $0 \leq j \leq l+1$ in case $l > 0$ and for

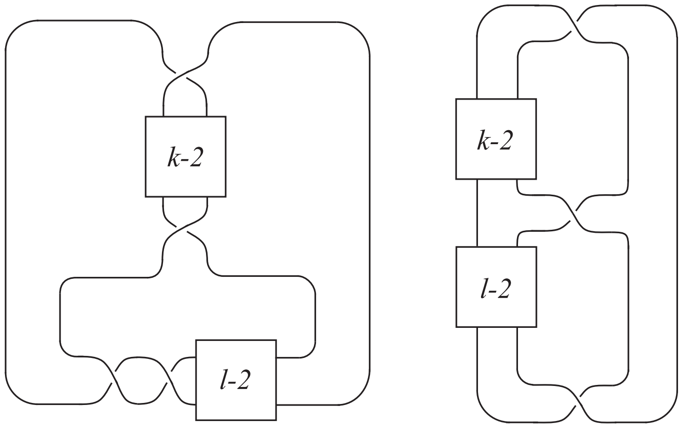

FIGURE 3. 4-plat presentation of $J(k, l)$ for $k, l \geq 2$

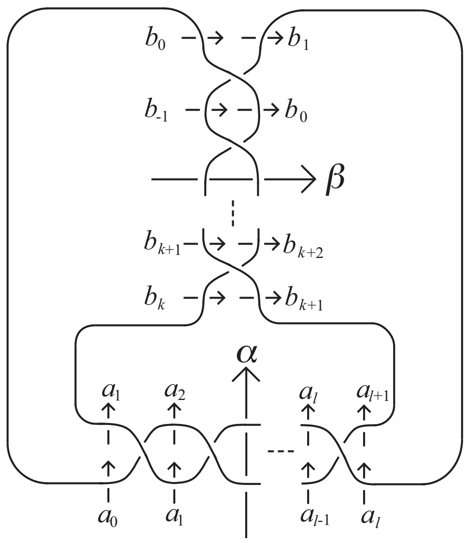

FIGURE 4. Generators for $\pi_1(k, l)$ with $k < 0 < l$

$l \leq j \leq 1$ in case $l < 0$, we let a_j be the loop based at P that consists of the line segment from P to the tail of the arrow labeled a_j, followed by the arrow itself and the segment from the head of the arrow to P. Similarly, for all appropriate j we let b_j be the loop associated to the arrow labeled b_j. The product xy of two loops x and y based at P is the compositum of the two loops, where we first follow x and then y. Set $a = a_0$, $b = b_1$, $\alpha = a_0a_1$, and $\beta = b_0b_1$. Then by induction (downwards if k or l is negative) we have $a_j = \alpha^{-d}a_{j-2d}\alpha^d$ for $d = \lfloor j/2 \rfloor$ and $b_j = \beta^{-d}b_{j-2d}\beta^d$ for $d = \lfloor j/2 \rfloor$ for each appropriate j. Using this and the identity $b_0 = a_0^{-1} = a^{-1}$, we can express b_j in terms of a and b for each j. Using $a_1 = b_k$ we can then also express a_j in terms of a and b for each j. We find $\beta = a^{-1}b$ and $\alpha = w_k$ with w_k as in (5), in terms of the meridians a and b. We are left with two relations in terms of a and b, namely $a_l = b_1$ and $a_{l+1} = b_{k+1}^{-1}$, which are dependent, as we have

$$a_l a_{l+1} = \alpha = a_0 a_1 = b_0^{-1} b_k = b_1 (b_0 b_1)^{-1} (b_k b_{k+1}) b_{k+1}^{-1} = b_1 \beta^{-1} \beta b_{k+1}^{-1} = b_1 b_{k+1}^{-1}.$$

It follows that the fundamental group is generated by the elements a and b with the relation $b_1 = a_l$. For even l, say $l = 2n$, this relation is $b = \alpha^{-n} a_0 \alpha^n$, or $w_k^n b = a w_k^n$. For odd l, say $l = 2n+1$, the relation is $b = \alpha^{-n} a_1 \alpha^n = \alpha^{-n} a_0^{-1} (a_0 a_1) \alpha^n = \alpha^{-n} a^{-1} \alpha^{n+1}$, or $a w_k^n b = w_k^{n+1}$. This shows that the fundamental group can indeed be presented as claimed.

Now let a_i', b_j', α', and β' be the analogous loops for the knot $J(l,k)$ and set $b' = b_1'$ and $a' = a_0'$. Then there is a natural isomorphism from $\pi_1(k,l)$ to $\pi_1(l,k)$ that sends a_j to b_j', b_i to a_i', and $\alpha = w_k(a,b)$ and $\beta = a^{-1}b$ to $\beta' = a'^{-1}b'$ and $\alpha' = w_l(a',b')$ respectively. This isomorphism is induced by turning the 4-plat associated to $J(k,l)$ upside down to obtain that of $J(l,k)$. The elements α and β will play an important role in the new model of the $\mathrm{PSL}_2(\mathbb{C})$-character variety of $J(k,l)$ that we will define later.

We leave it to the reader to check that the group presentations (1) coming from the Schubert normal form and the presentations (4) and (6) of $\pi_1(k,l)$ are equivalent in case kl is even. For even l an isomorphism is given by sending a to a and b to b, while for odd l (and thus even k) an isomorphism is given by sending a to a and b to b^{-1}.

We set $X(k,l) = X(\pi_1(k,l))$ and define $Y(k,l)$ similarly, as well as $X_0(k,l)$ and $Y_0(k,l)$ in case $J(k,l)$ is a hyperbolic knot.

2.4. **Newton Polygons and Algebraic curves.**

2.4.1. *Discrete valuations and Newton polygons.* In the proof of our main theorem we will make heavy use of valuations. A *non-archimedean* valuation on a field K is a map $v\colon K \to \mathbb{R} \cup \{\infty\}$ with $v(x) = \infty \Leftrightarrow x = 0$ that satisfies the ultrametric triangle inequality $v(x+y) \geq \min\big(v(x), v(y)\big)$ and $v(xy) = v(x) + v(y)$ for all $x, y \in K$. Given such a valuation v on K, the set $R_v = \{x \in K \;:\; v(x) \geq 0\}$ forms a subring of K that is a local ring with maximal ideal $\mathfrak{m}_v = \{x \in K \;:\; v(x) > 0\}$. For any $x, y \in K$ with $v(x) < v(y)$ we have $v(x+y) = v(x)$. For any real number α with $0 < \alpha < 1$ we obtain an absolute value $|\cdot|_v \colon K \to \mathbb{R}_{\geq 0}$ by setting $|x|_v = \alpha^{v(x)}$. For more details, see [9, Ch. 2] and [31, §I.1-2, §II.1-3].

An example of a non-archimedean valuation is the p-adic valuation v_p on $\mathbb{Q}$; for any nonzero integer a, the valuation $v_p(a)$ equals the number of factors p in a, and for any two nonzero integers a, b we have $v_p(a/b) = v_p(a) - v_p(b)$. By definition this valuation extends uniquely to a valuation, also denoted by v_p, on the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ at v_p, the field of p-adic numbers, containing the associated local ring $\mathbb{Z}_p$ of p-adic integers. We can also extend v_p, though not necessarily uniquely, to any finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$, and by taking limits also to any algebraic extension of $\mathbb{Q}$ or $\mathbb{Q}_p$. Note that for any such extension v of v_p we have $v(p^{1/n}) = 1/n$ for any nonzero integer n, so the values of a valuation are not necessarily integral.

Let v be a non-archimedean valuation on a field K and $f = \sum_{i=0}^{n} a_i x^i \in K[x]$ a nonzero polynomial. Then the *Newton polygon* of f at v is the lower convex hull of the $n+1$ points $\big(i, v(a_i)\big)$, where the point is at infinity if $a_i = 0$. Note that if $a_0 = a_1 = \ldots = a_{i-1} = 0$ and $a_i \neq 0$ for some $i > 0$, then the left-most segment of the Newton polygon is the vertical segent from $(0,\infty)$ to $\big(i, v(a_i)\big)$, which has horizontal length i. The following lemma tells us that the Newton polygon determines the valuations of the roots of f.

Lemma 2.4. *Let v be a non-archimedean valuation on an algebraically closed field K and $f \in K[x]$ a nonzero polynomial. Then for any rational number q, the number of roots of f in K with valuation q equals the horizontal length of the segment of the Newton polygon of f at v with slope $-q$ if such a segment exists, and it equals* 0 *otherwise.*

Proof. See [9, Prop. 2.9]. □

2.4.2. *Algebraic curves.* In this section we will assume that the ground field is algebraically closed. For the basic properties of algebraic varieties, in particular curves, and the notions of rational maps and morphisms between them, we refer the reader to [32, Ch. I-II]. The topology we use on algebraic varieties is the Zariski topology, which on curves is the cofinite topology. We stress the fact that a *rational map* $\varphi\colon C \dashrightarrow D$ of varieties is given by rational functions on C and not necessarily defined on the whole of C; the map φ is a *morphism* if it is regular everywhere on C and φ is called *birational* if it restricts to an isomorphism from a nonempty open subset of C to an open subset of D. In particular, two curves are birational if they are isomorphic up to a finite number of points.

Lemma 2.5. *Suppose $\varphi\colon C \to D$ is a birational morphism of curves. If D is smooth, then C is isomorphic to $\varphi(C)$.*

Proof. Let C' be a projective closure of C and let $\psi\colon U \to C \subset C'$ be a birational inverse of φ with $U \subset \varphi(C)$ open. Since $\varphi(C)$ is smooth and C' projective, the map ψ extends to a morphism $\hat{\psi}\colon \varphi(C) \to C'$ [10, Prop. I.6.8]. The composition $\hat{\psi} \circ \varphi\colon C \to C'$ is the identity on a dense open subset of C, so it is the identity on C. It follows that φ induces an isomorphism from C to $\varphi(C)$. □

We call a curve *hyperelliptic* if it is birational to a double cover of $\mathbb{P}^1$. Note that with this definition, all curves of genus 0 and 1 are hyperelliptic. For $i \in \{1,2\}$, let $\pi_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the projection on the i-th factor. If $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve, then for almost all $P \in \mathbb{P}^1$ the number of intersection points between $\mathbb{P}^1 \times \{P\}$ and C equals the degree $\deg \pi_2|_C$ of the map $\pi_2|_C\colon C \to \mathbb{P}^1$ induced by π_2; the *bidegree* of C is the pair of integers $(\deg \pi_2|_C, \deg \pi_1|_C)$. Two curves $C, C' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a,b) and (a',b') respectively have intersection number $ab' + a'b$.

Lemma 2.6. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth projective curve of bidegree (a,b) with $a, b > 0$. Then C is irreducible, its genus equals $(a-1)(b-1)$, and C is hyperelliptic if and only if $a \leq 2$ or $b \leq 2$.*

Proof. From $a, b > 0$ we find that C is connected by [10, Exc. III.5.6b]. Therefore, if C were not irreducible, some components would intersect in a singular point, contradicting smoothness of C. We conclude that C is irreducible. Its genus equals $(a-1)(b-1)$ by [10, Exc. III.5.6c]. If $a \leq 2$ or $b \leq 2$, then projection of C onto one of the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$ shows that C is either isomorphic to $\mathbb{P}^1$ or to a double cover of $\mathbb{P}^1$. In both cases C is hyperelliptic. If $\iota\colon C \to \mathbb{P}^1 \times \mathbb{P}^1$ denotes the embedding, then the canonical sheaf on C is isomorphic to $\iota^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a-2, b-2)$ [10, Prop. II.8.20 and Exm. II.8.20.3]. If $a, b > 2$, then this is very ample, so C is not hyperelliptic [10, Prop. IV.5.2]. This finishes the proof. □

Lemma 2.7. *Let D be a smooth projective irreducible curve over an algebraically closed field of characteristic not equal to 2, with genus $g(D)$ and function field $k(D)$. Let $h \in k(D)$ be a rational function on D and let a denote the number of points on D where h has odd valuation. If $a > 0$, then $k(D)[x]/(x^2 - h)$ is a function field, corresponding to a smooth projective irreducible curve C whose genus $g(C)$ equals $g(C) = 2g(D) - 1 + a/2$.*

Proof. There is a point where h has odd valuation, so h is not a square and $x^2 - h$ is irreducible. It follows that $k(D)[x]/(x^2 - h)$ is a function field, corresponding to some smooth projective irreducible curve C. The inclusion of function fields corresponds to a morphism $\varphi\colon C \to D$ of degree 2, which is separable as the characteristic is not equal to 2. The map φ ramifies at all points on D where h has odd valuation. For each such point Q there is a unique $P \in C$ with $\varphi(P) = Q$, at which the ramification index e_P satisfies $2 \leq e_P \leq \deg \varphi = 2$, so $e_P = 2$. From the theorem of Riemann-Hurwitz [10, Cor. IV.2.4] we find

$$2g(C) - 2 = \deg \varphi \cdot (2g(D) - 2) + \sum_{P \in C} (e_P - 1) = 2(2g(D) - 2) + a,$$

from which we get $g(C) = 2g(D) - 1 + a/2$. □

The following lemma is no more than a reformulation that we will use repeatedly.

Lemma 2.8. *Let $D \subset \mathbb{A}^2$ be a plane curve over an algebraically closed field, and P a smooth point on D corresponding with valuation v_P. Let h be a rational function on $\mathbb{A}^2$ that is regular on an open neighborhood $U \subset \mathbb{A}^2$ of P. Let $X \subset U$ be the vanishing locus of h on U. Then $v_P(h) > 0$ if and only if P is on X and $v_P(h) = 1$ if and only if X intersects D transversally at P.*

Proof. Let $\mathcal{O}_{\mathbb{A},P}$ and $\mathcal{O}_{D,P}$ be the local rings of P in $\mathbb{A}^2$ and D respectively. Since $\mathbb{A}^2$ is smooth at P, the curve D is locally principal at P, say given by $f = 0$ with f regular at P. Then there is an isomorphism $\mathcal{O}_{D,P} \cong \mathcal{O}_{\mathbb{A},P}/(f)$ of local rings. The point P lies on X if and only if h is contained in the maximal ideal of $\mathcal{O}_{\mathbb{A},P}$, so if and only if h is contained in the maximal of $\mathcal{O}_{D,P}$, i.e., $v_P(h) > 0$. The intersection multiplicity of D and X at P is given by the length of $\mathcal{O}_{\mathbb{A},P}/(f,h) \cong \mathcal{O}_{D,P}/(h)$, which equals $v_P(h)$. By definition this intersection is transversal when the multiplicity is 1, so when $v_P(h) = 1$. □

3. The standard model for the character varieties

For all integers k, l with l even, so that $J(k,l)$ is a knot, we will define a model for the $\mathrm{PSL}_2(\mathbb{C})$-character variety of $J(k,l)$ that is similar to the one often used in the literature. The following polynomials will be useful.

Definition 3.1. *Set $f_0 = 0$ and $f_1 = 1$. For all other $j \in \mathbb{Z}$, let $f_j \in \mathbb{Z}[u]$ be determined inductively (up and down) by the relation $f_{j+1} - uf_j + f_{j-1} = 0$. For all integers j we define g_j by $g_j = f_j - f_{j-1}$.*

For notational convenience, we merge the sequences $(f_j)_j$ and $(g_j)_j$ into a sequence $(\Phi_k)_k$ as follows.

Definition 3.2. *For each integer j we define $\Phi_{2j} = f_j$ and $\Phi_{2j-1} = g_j$. Furthermore, for each integer k we set $\Psi_k = \Phi_{k+1} - \Phi_{k-1}$.*

Lemma 3.3. *Let j be any integer. We have $f_{-j} = -f_j$ and $g_{-j} = g_{j+1}$. If $j \neq 0$ then the polynomial f_j has degree $|j| - 1$ and is odd or even, based on the parity of its degree. The polynomial g_j has degree $j - 1$ for $j > 0$ and degree $-j$ for $j \leq 0$. We also have $f_j(2) = j$ and $g_j(2) = 1$.*

Proof. The follows immediately by induction with respect to j, both upwards and downwards. □

Lemma 3.4. *In the ring $\mathbb{Z}[u][s]/(s^2 - us + 1) \cong \mathbb{Z}[s, s^{-1}]$ we have $u = s + s^{-1}$ and $f_j = (s^j - s^{-j})/(s - s^{-1})$ and $g_j = (s^j + s^{1-j})/(s + 1)$. We also have $f_{j-1}f_{j+1} = f_j^2 - 1$ and $g_j g_{j+1} = (u - 2)f_j^2 + 1$.*

Proof. The expression for f_j follows from induction, and the expression for g_j follows immediately. The last equations are easily checked in terms of s. □

Lemma 3.5. *Let k be any integer. Then we have $\Phi_{k+2} = u\Phi_k - \Phi_{k-2}$ and $\Psi_{k+2} = u\Psi_k - \Psi_{k-2}$. We also have $\Phi_k = (-1)^{k+1}\Phi_{-k}$ and $\Psi_k = (-1)^{k+1}\Psi_{-k}$, while $\deg \Phi_k = \lfloor (|k| - 1)/2 \rfloor$. Finally, we have*

$$\Psi_k = \begin{cases} (u-2)\Phi_k = (u-2)f_j & \text{if } k = 2j \text{ is even,} \\ \Phi_k = g_j & \text{if } k = 2j-1 \text{ is odd.} \end{cases}$$

Proof. The first statement follows from Definition 3.1 and the second from Lemma 3.3. The last statement follows from Definition 3.2 and the identity $g_{j+1} - g_j = (u - 2)f_j$, which is immediate from Definition 3.1. □

Lemma 3.6. *Suppose $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfy $\operatorname{tr} A = \operatorname{tr} B$. Set $y = \operatorname{tr} A^2$ and $r = \operatorname{tr} A^{-1}B$. Let k be any integer and set $m = \lfloor k/2 \rfloor$. Define $W_k = (AB^{-1})^m(A^{-1}B)^m$ if k is even and $W_k = (AB^{-1})^m AB(A^{-1}B)^m$ if k is odd. Then we have*

$$\operatorname{tr} W_k = \Phi_{-k}(r)\Psi_k(r)(y - r) + 2.$$

Proof. By Cayley-Hamilton we have $(\operatorname{tr} M) \cdot I = M + M^{-1}$ and $(\operatorname{tr} N) \cdot I = N + N^{-1}$ for all $M, N \in \mathrm{SL}_2(\mathbb{C})$. Taking traces after multiplying the former equation by N from the right and the latter by M from the left, we obtain

$$\operatorname{tr}\big(MN\big) = (\operatorname{tr} M)(\operatorname{tr} N) - \operatorname{tr}\big(M^{-1}N\big) = (\operatorname{tr} M)(\operatorname{tr} N) - \operatorname{tr}\big(MN^{-1}\big) \tag{7}$$

for all $M, N \in \mathrm{SL}_2(\mathbb{C})$. Set $c_{k,d} = \operatorname{tr}\big(W_k(A^{-1}B)^d\big)$ and

$$\gamma_{k,d} = \Phi_{-k}(r)\Psi_{k+2d}(r)(y - r) + f_{d+1}(r) - f_{d-1}(r).$$

The Lemma is equivalent to the special case $d = 0$ of the stronger statement that $c_{k,d} = \gamma_{k,d}$ for all integers k, d. We will prove by induction with respect to k that this is true for k and all integers d. We first use induction with respect to d for $-1 \leq k \leq 1$. we have $c_{0,0} = \operatorname{tr} I = 2 = \gamma_{0,0}$ and $c_{0,1} = \operatorname{tr}\big(A^{-1}B\big) = r = \gamma_{0,1}$ and $c_{1,-1} = \operatorname{tr}\big(A^2\big) = y = \gamma_{1,-1}$. Set $x = \operatorname{tr} A = \operatorname{tr} B$. Then by (7) we have $y = \operatorname{tr}\big(A^2\big) = (\operatorname{tr} A)^2 - \operatorname{tr} I = x^2 - 2$ and thus

$$c_{1,0} = \operatorname{tr} W_1 = \operatorname{tr}\big(AB\big) = (\operatorname{tr} A)(\operatorname{tr} B) - \operatorname{tr}\big(A^{-1}B\big) = x^2 - r = y - r + 2 = \gamma_{1,0}.$$

We also have $c_{-1,0} = \operatorname{tr}\big(BA\big) = \operatorname{tr}\big(AB\big) = \gamma_{1,0} = \gamma_{-1,0}$ and $c_{-1,1} = \operatorname{tr}\big(B^2\big) = (\operatorname{tr} B)^2 - \operatorname{tr} I = x^2 - 2 = y$ by (7). Also by (7), we have

$$\begin{aligned} c_{k,d+1} &= \operatorname{tr}\big(W_k(A^{-1}B)^{d+1}\big) = \operatorname{tr}\big(W_k(A^{-1}B)^d(A^{-1}B)\big) \\ &= \operatorname{tr}\big(W_k(A^{-1}B)^d\big)\big(\operatorname{tr} A^{-1}B\big) - \operatorname{tr}\big(W_k(A^{-1}B)^{d-1}\big) = rc_{k,d} - c_{k,d-1}. \end{aligned} \tag{8}$$

The sequence $(\gamma_{k,d})_d$ satisfies the same recursion, so by induction (increasing and decreasing) we find $c_{k,d} = \gamma_{k,d}$ for $-1 \le k \le 1$ and all integers d. Therefore, we get

$$\begin{aligned} c_{2,0} &= \operatorname{tr}\left(AB^{-1}A^{-1}B\right) = \left(\operatorname{tr} AB^{-1}A^{-1}\right)\left(\operatorname{tr} B\right) - \operatorname{tr}\left((AB^{-1}A^{-1})^{-1}B\right) \\ &= \left(\operatorname{tr}(B^{-1})\right)\left(\operatorname{tr} B\right) - \operatorname{tr}\left(ABA^{-1}B\right) = x^2 - c_{1,1} = y + 2 - \gamma_{1,1} = \gamma_{2,0}. \end{aligned}$$

Together with $c_{2,-1} = \operatorname{tr}\left(AB^{-1}\right) = \operatorname{tr}\left(B^{-1}A\right) = \operatorname{tr}\left((B^{-1}A)^{-1}\right) = \operatorname{tr}\left(A^{-1}B\right) = r = \gamma_{2,-1}$ this is the basis for the induction that shows $c_{2,d} = \gamma_{2,d}$ for all integers d, the induction step following again from (8). Now by (7) we have

$$\begin{aligned} c_{k+2,d} &= \operatorname{tr}\left(W_{k+2}(A^{-1}B)^d\right) = \operatorname{tr}\left((AB^{-1})W_k(A^{-1}B)^{d+1}\right) \\ &= \left(\operatorname{tr}(AB^{-1})\right)\left(\operatorname{tr}(W_k(A^{-1}B)^{d+1})\right) - \operatorname{tr}\left((AB^{-1})^{-1}W_k(A^{-1}B)^{d+1}\right) \\ &= rc_{k,d+1} - \operatorname{tr}\left(W_{k-2}(A^{-1}B)^{d+2}\right) = rc_{k,d+1} - c_{k-2,d+2}. \end{aligned}$$

From Lemma 3.5 it follows that we also have $\gamma_{k+2,d} = r\gamma_{k,d+1} - \gamma_{k-2,d+2}$ for all integers k and d. By induction with respect to k it follows that $c_{k,d} = \gamma_{k,d}$ for all integers k and d. □

Analogous to §2.2.2, for any integer k, any $\lambda_0 \in \mathbb{C}^*$, and $r_0 \in \mathbb{C}$, we let $W_k(\lambda_0, r_0)$ denote the right-hand side of (5) with $A(\lambda_0)$ and $B(\lambda_0, r_0)$ substituted for a and b. Then for any integers k, n, the assignments $a \mapsto A(\lambda_0)$ and $b \mapsto B(\lambda_0, r_0)$ can be extended to a representation $\rho \in \mathrm{R}(\pi_1(k, 2n))$ if and only if we have $A(\lambda_0)W_k(\lambda_0, r_0)^n = W_k(\lambda_0, r_0)^n B(\lambda_0, r_0)$, which results in four equations in λ_0 and r_0. Again these equations reduce to a single equation in r_0 and λ_0.

Proposition 3.7. *Let k, n be any integers. Consider the ring $\mathbb{Q}[r, \lambda, \lambda^{-1}]$ and let I denote the ideal generated by the four entries of the matrix $A(\lambda)W_k(\lambda, r)^n - W_k(\lambda, r)^n B(\lambda, r)$. Then I is generated by*

$$F_{k,n}(\lambda, r) = f_n(\operatorname{tr} W_k(\lambda, r)) \cdot F_{k,1}(\lambda, r) - f_{n-1}(\operatorname{tr} W_k(\lambda, r)) \tag{9}$$

with

$$F_{k,1}(\lambda, r) = -\Phi_{-k}(r)\Phi_{k-1}(r)(y - r) + 1,$$

and with $y = \lambda^2 + \lambda^{-2}$.

Proof. By Cayley-Hamilton we have $M^2 = tM - I$ for a matrix $M \in \mathrm{SL}_2(\mathbb{C})$ with trace t; by induction, both up and down, we find

$$M^j = f_j(t) \cdot M - f_{j-1}(t) \cdot I \tag{10}$$

for all $j \in \mathbb{Z}$. Completely analogous to Proposition 2.2, we find $F_{k,n} = (\lambda - \lambda^{-1})W_{12} + W_{22}$, where W_{ij} denotes the (i,j)-entry of $W_k(\lambda, r)^n$. Let w_{ij} be the (i,j)-entry of $W_k(\lambda, r)$ and set $t = \operatorname{tr}(W_k)$. Then from (10) we have $W_{12} = f_n(t)w_{12}$ and $W_{22} = f_n(t)w_{22} - f_{n-1}(t)$, which implies $F_{k,n} = f_n(t)F_{k,1} - f_{n-1}(t)$. From (10) we also find

$$W_k(\lambda, r) = (f_m(r)AB^{-1} - f_{m-1}(r)I)(f_m(r)A^{-1}B - f_{m-1}(r)I)$$

if $k = 2m$ is even and

$$W_k(\lambda, r) = (f_m(r)AB^{-1} - f_{m-1}(r)I)AB(f_m(r)A^{-1}B - f_{m-1}(r)I)$$

if $k = 2m + 1$ is odd, with $A = A(\lambda)$ and $B = B(\lambda, r)$. From this one easily checks that $F_{k,1} = (\lambda - \lambda^{-1})w_{12} + w_{22}$ is indeed as given. □

Recall that for all integers k, l, the $\mathrm{SL}_2(\mathbb{C})$- and $\mathrm{PSL}_2(\mathbb{C})$-character varieties of the fundamental group $\pi_1(k,l)$ of the complement of $J(k,l)$ in $\mathbb{S}^3$ are denoted by $X(k,l)$ and $Y(k,l)$ respectively.

Proposition 3.8. *Let k, l be any integers with l even. The variety $Y(k,l)$ is isomorphic to the subvariety $C(k,l)$ of $\mathbb{A}^2(r,y)$ defined by*

$$C(k,l)\colon\ f_n(t)\big(\Phi_{-k}(r)\Phi_{k-1}(r)(y-r)-1\big)+f_{n-1}(t)=0,$$

with $t=\Phi_{-k}(r)\Psi_k(r)(y-r)+2$ and $n=l/2$. The variety $X(k,l)$ is isomorphic to the double cover of $C(k,l)$ defined in $\mathbb{A}^2(r,x)$ by $y=x^2-2$.

Proof. Let $W_k(\lambda,r)$ be as in Proposition 3.7. Then $t=\operatorname{tr}W_k(\lambda,r)$ by Lemma 3.6. We conclude that $C(k,l)$ is the curve given by $F_{k,n}=0$ (in terms of r and y) in $\mathbb{A}^2(r,y)$. Completely analogous to §2.2.2, the varieties $X(k,l)$ and $Y(k,l)$ have models in $\mathbb{A}^2(r,x)$ and $\mathbb{A}^2(r,y)$ given by $F_{k,n}=0$ in terms of r and x and in terms of r and y respectively. The proposition follows. □

Note that if $kl=0$, then the variety $C(k,l)$ is empty. This reflects the fact that in those cases $J(k,l)$ is the trivial knot, so $\pi_1(k,l)$ is a free abelian group, which has no nonabelian representations. The following lemma will be useful later.

Lemma 3.9. *Suppose k, l are integers with l even. If $P=(r_0,y_0)\in C(k,l)(\overline{\mathbb{Q}})$ is a point with $\Psi_k(r_0)=0$, then k is even and $P=(2,2-\frac{4}{kl})$.*

Proof. By assumption the variety $C(k,l)$ is not empty, so we conclude $kl\neq 0$. Set $n=l/2$ and $t_0=\Phi_{-k}(r_0)\Psi_k(r_0)(y_0-r_0)+2$. Then by Proposition 3.8 we have

$$f_n(t_0)\big(\Phi_{-k}(r_0)\Phi_{k-1}(r_0)(y_0-r_0)-1\big)+f_{n-1}(t_0)=0. \tag{11}$$

From $\Psi_k(r_0)=0$ we get $t_0=2$, and by Lemma 3.3 we have $f_n(t_0)=n$ and $f_{n-1}(t_0)=n-1$. Suppose we had $\Phi_{-k}(r_0)=0$. Then the left-hand side of (11) equals $-n+(n-1)=-1$. From this contradiction we conclude $\Phi_{-k}(r_0)\neq 0$. If k were odd then we would have $0=\Psi_k(r_0)=\Phi_{-k}(r_0)\neq 0$ by Lemma 3.5, so we conclude that k is even and find $0=\Psi_k(r_0)=(2-r_0)\Phi_{-k}(r_0)$ by Lemma 3.5. This implies $r_0=2$. By Lemmas 3.3 and 3.5 we then have $\Phi_{-k}(r_0)=-\frac{1}{2}k$ and $\Phi_{k-1}(r_0)=1$, so the left-hand side of (11) equals $n(-\frac{1}{2}k(y_0-2)-1)+n-1$. Solving (11) for y_0 gives $y_0=2-\frac{2}{kn}=2-\frac{4}{kl}$. □

The models of $X(k,l)$ and $Y(k,l)$ described in Proposition 3.8, up to perhaps a linear transformation, are the standard models. Their usual projective closures in $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ are highly singular. Note that the trace $\operatorname{tr}(W_k)$ is linear in y for all nonzero integers k. We can exploit this to give a model of $Y(k,l)$ with a smooth completion in $\mathbb{P}^1\times\mathbb{P}^1$. This will be done in the next section.

4. A new model for the character varieties

In this section we introduce a new model for $Y(k,l)$. It does not respect integrality, but is geometrically nicer than the standard model in the sense that it is projective and all its irreducible components are smooth. The coordinates r and y from the previous section are the trace functions $t_{a^{-1}b}$ and t_{a^2} respectively. For the new model we will replace y by the trace function $t=t_{W_k}$, which is linear in y.

Let $D(k,l)$ be the variety in $\mathbb{P}^1_{\mathbb{Q}}(r)\times\mathbb{P}^1_{\mathbb{Q}}(t)$ that is the projective closure of the affine variety given by

$$\Phi_{k+1}(r)\Phi_{l-1}(t)=\Phi_{k-1}(r)\Phi_{l+1}(t). \tag{12}$$

Note that for $l = 2n$, expressed in terms of the polynomials f_j and g_j this is

$$\begin{aligned} g_{m+1}(r)g_n(t) &= g_m(r)g_{n+1}(t) \quad \text{if } k = 2m, l = 2n \\ f_{m+1}(r)g_n(t) &= f_m(r)g_{n+1}(t) \quad \text{if } k = 2m+1, l = 2n. \end{aligned}$$

Subtracting $\Phi_{k-1}(r)\Phi_{l-1}(t)$ from both sides of (12), and using Definition 3.2, we find that $D(k,l)$ is also given by the alternate equations

$$\Psi_k(r)\Phi_{l-1}(t) = \Phi_{k-1}(r)\Psi_l(t). \tag{13}$$

Remark 4.1. *Let k, n be any integers. For $k = n = 0$, the variety $D(k, 2n)$ is the full $\mathbb{P}^1 \times \mathbb{P}^1$, while for $k = \pm 1$ and $n \in \{0, k\}$, it is in fact empty. Suppose we are not in any of those cases. Then $D(k, 2n)$ has dimension 1, and from Lemma 3.5 one quickly finds the bidegree of $D(k, 2n)$. It equals $(\lfloor |k|/2 \rfloor, |n|)$ when $k \neq \pm 1$. For $k = \pm 1$ the bidegree equals $(0, kn - 1)$ if $kn > 0$ and it equals $(0, -kn)$ if $kn < 0$.*

In some sense it seems natural to include the line given by $t = \infty$ in $D(k, 2n)$ when $k = \pm 1$ and $kn > 0$; then $D(k, 2n)$ would be a curve of bidegree $(\lfloor |k|/2 \rfloor, |n|)$, as long as this differs from $(0, 0)$. Doing this is also natural in the sense that it would follow from a slightly different definition for $D(k, 2n)$ that gives an explicit equation on an affine chart that includes the line $t = \infty$. We have chosen not to do this in order to keep $D(k, 2n)$ birationally equivalent with the standard model $C(k, 2n)$ for $Y(k, 2n)$. Before we prove this, we state a few lemmas.

Lemma 4.2. *For every $j \in \mathbb{Z}$ the ideals (g_j, g_{j-1}), (f_j, f_{j-1}), (Φ_{j+1}, Φ_{j-1}), and (Ψ_j, Φ_{j-1}) of $\mathbb{Z}[u]$ all equal the unit ideal.*

Proof. From the identity $1 = f_{j-1}g_{j-1} - f_{j-2}g_j$ we find that the first ideal is the unit ideal. The identities in Lemma 3.4 show that the second ideal and the ideal $((u-2)f_i, g_i) = (\Psi_{2i}, \Phi_{2i-1})$ are unit ideals for any integer i. From $g_{i+1} = f_{i+1} - f_i$ it follows that $(\Psi_{2i+1}, \Phi_{2i}) = (g_{i+1}, f_i) = (f_{i+1}, f_i) = (1)$. This proves that the last ideal is the unit ideal both when j is odd and when j is even. The third ideal is of the form of the first or second ideal, depending on the parity of j, so it is also the unit ideal. □

Lemma 4.3. *Let k, l be any integers and $P = (r_0, t_0)$ a $\overline{\mathbb{Q}}$-point on the standard affine part of $\mathbb{P}^1 \times \mathbb{P}^1$. Then the following statements are equivalent.*

1. *We have $\Psi_k(r_0) = \Psi_l(t_0) = 0$.*
2. *The point P lies on $D(k,l)$ and $\Psi_k(r_0) = 0$.*
3. *The point P lies on $D(k,l)$ and $\Psi_l(t_0) = 0$.*

Proof. To show equivalence of (1) and (2), assume we have $\Psi_k(r_0) = 0$. From Lemma 4.2 we conclude $\Phi_{k-1}(r_0) \neq 0$, so (13) shows that P lies on $D(k,l)$ if and only if $\Psi_l(t_0) = 0$. Equivalence of (1) and (3) follows by symmetry. □

Proposition 4.4. *Suppose k, l are integers with l even and $kl \neq 0$. The map $\mathbb{A}^2(r, y) \to \mathbb{P}^1(r) \times \mathbb{P}^1(t)$ that sends (r, y) to $(r, \mathrm{tr}(W_k))$, with $\mathrm{tr}(W_k)$ as in Lemma 3.6, induces a birational morphism from $C(k,l)$ to $D(k,l)$.*

Proof. Let σ denote the map described. It is clearly well defined everywhere and therefore induces a morphism from $C(k,l)$ to its image. By Lemma 3.6, the map σ is given by $(r, y) \mapsto (r, \Phi_{-k}(r)\Psi_k(r)(y - r) + 2)$, which has a birational inverse, given by $(r, t) \mapsto (r, r + (t-2)\Phi_{-k}(r)^{-1}\Psi_k(r)^{-1})$. Note that Φ_{-k} divides Ψ_k, so σ induces an isomorphism from the open subset U of $\mathbb{A}^2(r, y)$ given by $\Psi_k(r) \neq 0$ to the open subset V of the standard affine part of $\mathbb{P}^1(r) \times \mathbb{P}^1(t)$ given by $\Psi_k(r) \neq 0$. These open sets are dense because $\Psi_k \neq 0$ for $k \neq 0$. Set $n = l/2$. By Proposition 3.7 the image $\sigma(C(k,l))$ is on V given by

$$f_n(t)\left(\frac{(t-2)\Phi_{k-1}(r)}{\Psi_k(r)} - 1\right) + f_{n-1}(t) = 0,$$

which is equivalent to the equation for $D(k,l)$ in (13) by Lemma 3.5. Therefore $U \cap C(k,l)$ is isomorphic with $V \cap D(k,l)$. Since $l \neq 0$, there are only finitely many t_0 with $\Psi_l(t_0) = 0$. Therefore, by Lemmas 3.9 and 4.3, the curves $C(k,l)$ and $D(k,l)$ contain no full components outside U and V respectively, so they are isomorphic outside a finite number of points, and therefore birationally equivalent. □

Remark 4.5. *We have already seen that* $Y(k,l)$ *is empty if* $kl = 0$. *Suppose* $|k| = 1$ *and* $l \notin \{0, 2k\}$ *or suppose* $k = l \in \{\pm 2\}$. *Then* $D(k,l)$ *consists of a finite number of lines (cf. Remark 4.1). By Proposition 4.4 this implies that* $C(k,l)$ *and* $Y(k,l)$ *consist of a number of curves of genus* 0. *The corresponding knots* $J(k,l)$ *are not hyperbolic in all these cases and we will not give them much further attention.*

Note that from Lemma 3.5 it follows that $D(k,l)$ and $D(-k,-l)$ are the same, reflecting the fact that $J(-k,-l)$ is the mirror image of $J(k,l)$.

The symmetry of the equation for $D(k,l)$ in (12) shows that the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ that sends (r,t) to (t,r), induces an isomorphism from $D(k,l)$ to $D(k,l)$. Since r and t are the traces of the elements β and α in the fundamental group of $\mathbb{S}^3 \setminus J(k,l)$ respectively, as described in Figure 4, it follows from the discussion at the end of §2.3 that this isomorphism is induced by turning upside down the 4-plat representation as in Figure 2, which also switches α and β. In particular this applies when $k = l \neq 0$, in which case $D(l,l)$ contains an irreducible component given by $r = t$. This means that $Y(l,l)$ is reducible for $|l| > 2$. The reducibility of $Y(l,l)$ for $|l| > 2$ was already known from [22], [28], as for the associated two-bridge knot $K(p,q)$ we have $q^2 \equiv 1 \pmod{p}$. We can now identify the component given by $r = t$.

Proposition 4.6. *Suppose* l *is an even integer and* $|l| > 2$. *Then under the birational equivalence between* $Y(l,l)$ *and* $D(l,l)$, *the irreducible component* $Y_0(l,l)$ *corresponds to the line given by* $r = t$.

Proof. The automorphism of $D(l,l)$ that sends (r,t) to (t,r) is induced by turning upside down the 4-plat presentation in Figure 2. By [22, proof of Prop. 5.5], this involution acts trivially on the component $Y_0(l,l)$ of $Y(l,l)$. This implies that $Y_0(l,l)$ corresponds to the line given by $r = t$. □

Proof of Theorem 1.4. Let $\rho\colon \pi_1\big(\mathbb{S}^3 \setminus J(k,l)\big) \to \mathrm{SL}_2(\mathbb{C})$ denote a lift of the discrete faithful representation (cf. end of §2.1.1). By definition the trace field $F\big(J(k,l)\big)$ of $J(k,l)$ is generated by the traces of the elements in the image of ρ, so it equals the field of definition of the point χ on $X(k,l)$ associated to ρ. The images of meridians under ρ are parabolic (this follows from [33, Ch. 5], cf. [28, §2] and [25, §1]), so their traces equal ± 2. Therefore, in terms of the coordinates r, x as in Proposition 3.8, the point χ satisfies $x = \pm 2$ and maps to the point $(r_0, 2)$ on $C(k,l) \subset \mathbb{A}^2(r,y)$ for some $r_0 \in \mathbb{C}$. The trace field then equals $\mathbb{Q}(r_0)$. Substituting $y = 2$ in the equation for $C(k,l)$ gives a polynomial with root r_0 of degree $-kl/2$ if $kl < 0$ and degree $kl/2 - 1$ if $kl > 0$. This proves the first two bounds. If $k = l$, then the canonical component of $Y(l,l)$ corresponds by Propositions 4.4 and 4.6 to the component of $C(k,l)$ given by $r = \Phi_{-k}(r)\Psi_k(r)(y-r)+2$. Substituting $y = 2$ and taking out a factor $r-2$ gives an equation of degree $|l|-1$, which proves the final upper bound. The first two bounds also follow immediately from [25, §3]. □

Based on Proposition 4.6, we give the following definition.

Definition 4.7. *For each nonzero even integer l, let $D_0(l,l)$ denote the component of $D(l,l)$ given by $r = t$ and let $D_1(l,l)$ denote the projective closure of the scheme-theoretic complement of $D_0(l,l)$ in $D(l,l)$; if $|l| > 2$, then we denote the scheme-theoretic complement of $Y_0(l,l)$ in $Y(l,l)$ by $Y_1(l,l)$ and the scheme-theoretic complement of $X_0(l,l)$ in $X(l,l)$ by $X_1(l,l)$.*

Note that $D_1(2n,2n)$ is given by $(g_{n+1}(r)g_n(t) - g_n(r)g_{n+1}(t))/(t-r) = 0$ for any nonzero integer n. For $|n| = 1$ (the trefoils, which are nonhyperbolic), we see that $D_1(2n,2n)$ is empty; for $|n| > 1$ it is of bidegree $(|n|-1, |n|-1)$.

5. Smoothness and Irreducibility of the character varieties

In this section we will prove the following theorem, covering all hyperbolic knots of the form $J(k,l)$.

Theorem 5.1. *Let l be an even integer with $|l| \geq 2$. If k is an integer with $k \neq l$ and $|k| \geq 2$, then $D(k,l)$ is smooth over $\mathbb{Q}$. If $|l| > 2$, then $D_1(l,l)$ is smooth over $\mathbb{Q}$.*

We split the proof of the first part of Theorem 5.1 into three cases, based on the parity of k and the sign of kl in case k is even. Theorem 5.1 will be proved at the end of this section as a corollary of Propositions 5.8, 5.11, 5.20, and 5.21. The approach is the same for all cases, but the details are different. We first sketch the idea behind our approach.

Definition 5.2. *For each integer k we set $h_k = \Phi_{k+1}/\Phi_{k-1}$.*

Suppose k, l are integers and $P = (r_0, t_0)$ is a singular point on the affine part of $D(k,l)$. We show that this implies $\Phi_{k-1}(r_0) \neq 0$ and $\Phi_{l-1}(t_0) \neq 0$. Then $D(k,l)$ can be given around P by $h_k(r) = h_l(t)$. The fact that P is a singular point is then equivalent with the fact that r_0 and t_0 are critical points for h_k and h_l respectively. We show that for each k, the values of h_k at its critical points are all different from each other, and they are also different from the values of h_l at all its critical points when $k \neq l$. This is done using complex absolute values or p-adic valuations, depending on the case. The equation $h_k(r_0) = h_l(t_0)$ then implies $k = l$ and $r_0 = t_0$. Indeed, for $k = l$ the component $D_0(l,l)$ given by $r = t$ intersects the curve $D_1(l,l)$ in singular points of $D(l,l)$.

Definition 5.3. *For every $n \in \mathbb{Z}$, set $F_n = f'_{n+1}f_n - f_{n+1}f'_n$ and $G_n = g'_{n+1}g_n - g_{n+1}g'_n$.*

Note that F_n and G_n are the numerators of the derivatives of f_{n+1}/f_n and h_{2n}. We first state some facts.

Lemma 5.4. *For every $n \in \mathbb{Z}$ the following statements hold.*

1. *If $n \neq 0$, then the polynomial f_n is separable.*
2. *The polynomials F_n and G_n have leading coefficient ± 1.*

(3) *We have* $(u+2)G_n = f_{2n} + 2n = \frac{s^{2n}-s^{-2n}}{s-s^{-1}} + 2n$ *in* $\mathbb{Z}[u][s]/(s^2-us+1)$.
(4) *We have* $G_n(2) = n$ *and* $G_n(-2) = \frac{1}{3}n(4n^2-1)$.
(5) *For any field* $\mathbb{F}$ *with characteristic not dividing* $2n-1$, *the polynomial* g_n *is separable over* $\mathbb{F}$ *and we have* $(G_n, g_n) = (1)$ *in* $\mathbb{F}[u]$.

Proof. Set $h = (s^{n+1} - s^{n-1})f_n \in \mathbb{Z}[u][s]/(s^2-us+1) \cong \mathbb{Z}[s,s^{-1}]$. Then we have $h = s^{2n}-1$, which is separable, as $s\frac{dh}{ds} - 2nh = 2n$ is a nonzero constant for $n \neq 0$. We conclude that f_n does not have multiple factors either, which proves (1).

The polynomials g_n and g_{n+1} are monic, while their degrees differ by 1. This implies that the leading terms of $g'_{n+1}g_n$ and $g'_n g_{n+1}$ also differ by 1. Therefore, their difference G_n indeed has leading coefficient ± 1. The same argument applies to F_n, which proves (2).

The identity in (3) is easily verified in $\mathbb{Z}[s,s^{-1}]$. Note that we have $g'_n = \frac{dg_n}{ds} / \frac{du}{ds} = \frac{dg_n}{ds} \cdot \frac{s^2}{s^2-1}$.

One can prove (4) by dividing the identity of (3) by $u+2 = s^{-1}(s+1)^2$, setting $s = \pm 1$ and applying l'Hôpital's rule. Alternatively, it follows from induction that we have $g_n(-2) = (-1)^{n-1}(2n-1)$, while by Lemma 3.3 we have $g_n(2) = 1$. From $g'_{n+1} = [ug_n - g_{n-1}]' = ug'_n - g'_{n-1} + g_n$ we then find by induction that $g'_n(2) = \frac{1}{2}n(n-1)$, while we have $g'_{n+1}(-2) = (-1)^n \frac{1}{6}n(n-1)(2n-1)$. It follows that $G_n(\pm 2)$ is as given.

For (5), let $\mathbb{F}$ be a field with characteristic not dividing $2n-1$. By Lemma 3.4 we have $(s+1)s^{n-1}g_n = s^{2n-1}+1$ and the reduction of this polynomial to $\mathbb{F}$ is separable. Then the reduction of the polynomial g_n has no multiple factors either, so g_n is separable over $\mathbb{F}$. The ideal $(G_n, g_n) \subset \mathbb{F}[u]$ contains $g'_{n+1}g_n - G_n = g_{n+1}g'_n$. By Lemma 4.2, the polynomials g_n and g_{n+1} have no roots in common, and as g_n is separable over $\mathbb{F}$, it also has no roots in common with g'_n, so in $\mathbb{F}[u]$ we find $(1) = (g_n, g_{n+1}g'_n) = (G_n, g_n)$, which finishes the proof of (5). □

For each integer k, set $\Delta_k = \Phi'_{k+1}\Phi_{k-1} - \Phi_{k+1}\Phi'_{k-1}$. Note that for even k, say $k = 2m$, we have $\Delta_k = G_m$, while for odd k, say $k = 2m+1$, we have $\Delta_k = F_m$.

Lemma 5.5. *Let* k *and* l *be any integers with* l *even. Suppose* $P = (r_0, t_0)$ *is a singular* $\overline{\mathbb{Q}}$*-point of the standard affine part of* $D(k,l)$*. Then we have* $\Phi_{k-1}(r_0) \neq 0 \neq \Phi_{l-1}(t_0)$ *and* $\Delta_k(r_0) = \Delta_l(t_0) = 0$.

Proof. Set $F = \Phi_{k+1}(r)\Phi_{l-1}(t) - \Phi_{k-1}(r)\Phi_{l+1}(t)$ and $F_x = \partial F/\partial x$ for $x = r, t$. Then we have $F(P) = F_r(P) = F_t(P) = 0$, so also

$$0 = \Phi'_{l-1}(t_0)F(P) - \Phi_{l-1}(t_0)F_t(P) = \Phi_{k-1}(r_0)\Delta_l(t_0)$$

and

$$0 = \Phi'_{l+1}(t_0)F(P) - \Phi_{l+1}(t_0)F_t(P) = \Phi_{k+1}(r_0)\Delta_l(t_0).$$

By Lemma 4.2 we can not have $\Phi_{k-1}(r_0) = \Phi_{k+1}(r_0) = 0$, so we have $\Delta_l(t_0) = 0$ and, similarly, $\Delta_k(r_0) = 0$. Since l is even, say $l = 2n$, we have $\Delta_l = G_n$. From Lemma 5.4(5) we conclude $\Phi_{l-1}(t_0) = g_n(t_0) \neq 0$. If we had $\Phi_{k-1}(r_0) = 0$, then $F(P) = 0$ would imply $\Phi_{k+1}(r_0) = 0$, which contradicts Lemma 4.2. We conclude $\Phi_{k-1}(r_0) \neq 0$. □

The following lemma will be used to prove smoothness at infinity.

Lemma 5.6. *Let* $e, f \in \mathbb{Z}[r]$ *and* $g, h \in \mathbb{Z}[t]$ *be nonzero separable polynomials, and assume that* $\deg e - \deg f = \pm 1$ *and* $\deg g - \deg h = \pm 1$*. Let* $C \subset \mathbb{P}^1(r) \times \mathbb{P}^1(t)$ *be* *the projective closure of the affine curve given by $e(r)g(t) = f(r)h(t)$. Then C is smooth at its points at infinity and the two lines at infinity intersect C transversally everywhere.*

Proof. Set $r' = r^{-1}$ and $t' = t^{-1}$ in the function field $\overline{\mathbb{Q}}(r,t)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ over $\overline{\mathbb{Q}}$. Let L be the line at infinity given by $r' = 0$. By symmetry between r and t it suffices to consider the points in $L \cap C$. This means it suffices to check all points on C with $r' = 0$ in the affine patches with coordinates (r', t) and (r', t'). Set $a = \deg e$ and $b = \deg g$. By symmetry between (e, g) and (f, h) we may assume $\deg f = a + 1$. Set $e'(r') = r'^{\deg e} e(1/r')$ and define f', g', h' similarly. Note that e', f', g', h' do not vanish at 0. Then on the affine patch with coordinates (r', t), the curve C is given by $r'e'(r')g(t) = f'(r')h(t)$. Now first consider the case $\deg h = b + 1$. Then C is of bidegree $(a + 1, b + 1)$. The line L is of bidegree $(1, 0)$, so the intersection number $L \cdot C$ equals $b+1$, when counting the intersection points with multiplicities. For each root τ of $h(t)$ there is a point $(r', t) = (0, \tau)$ on $L \cap C$, so there are at least $b + 1$ different points on $L \cap C$. This implies that all intersection multiplicities are 1, which shows that all points on $L \cap C$ are nonsingular and all intersections are transversal. Now consider the case $\deg h = b - 1$. Then C is of bidegree $(a + 1, b)$, so we have $L \cdot C = b$. On the patch with coordinates (r', t'), the curve C is given by $r'e'(r')g'(t') = t'f'(r')h'(t')$. Now if $h(0) \neq 0$, then $\deg h'(t') = b - 1$, and for each of the b roots τ of $t'h'(t')$ there is a point $(r', t') = (0, \tau)$ on $L \cap C$. If $h(0) = 0$, then h has a simple root at 0 as h is separable, so $\deg h'(t') = b - 2$ and there are also b points on $L \cap C$, namely $(r', t) = (0, 0)$ and the $b - 1$ points $(0, \tau)$ for any root τ of $t'h'(t')$. In either case we find that all intersection multiplicities are 1, so all points on $L \cap C$ are nonsingular and the intersections are transversal. □

In the case that k is even and kl is negative we use the following lemma. Recall that we have $h_{2n} = \Phi_{2n+1}/\Phi_{2n-1} = g_{n+1}/g_n$.

Lemma 5.7. *Let n be any nonzero integer, and $\omega \in \mathbb{C}$ a root of G_n. If $n > 0$, then $|h_{2n}(\omega)| > 1$, and if $n < 0$, then $|h_{2n}(\omega)| < 1$.*

Proof. Note that $h_{2n}(\omega)$ is well defined, as $g_n(\omega) \neq 0$ by Lemma 5.4(5). Assume $n > 0$, and choose a $\sigma \in \mathbb{C}^*$ such that $\omega = \sigma + \sigma^{-1}$. Then from Lemma 5.4(3) we find $\sigma^{2n} - \sigma^{-2n} = -2n(\sigma - \sigma^{-1})$, which shows that $\sigma^{2n} - \sigma^{-2n}$ and $\sigma - \sigma^{-1}$ are in opposite half-planes (upper and lower half-plane, both including the real line). Note that for each $z \in \mathbb{C}^*$, the values of $z, z - z^{-1}$, and $z - \overline{z}$ are all in the same half-plane, so we conclude that $\sigma^{2n} - \overline{\sigma}^{2n}$ and $\sigma - \overline{\sigma}$ are in opposite half-planes. Since both these values are purely imaginary, we conclude $(\sigma - \overline{\sigma})(\sigma^{2n} - \overline{\sigma}^{2n}) \geq 0$, with equality if and only if σ^{2n} is real. Set $\alpha = \sigma\overline{\sigma} = |\sigma|^2 > 0$. Then α and α^{2n} either both exceed 1, or they both do not, and we have $(\alpha - 1)(\alpha^{2n} - 1) \geq 0$ in either case, with equality if and only if $\alpha = 1$. Now we have

$$
\begin{aligned}
|\sigma^{2n+1} + 1|^2 - |\sigma^{2n} + \sigma|^2 &= (\sigma^{2n+1} + 1)(\overline{\sigma}^{2n+1} + 1) - (\sigma^{2n} + \sigma)(\overline{\sigma}^{2n} + \overline{\sigma}) \\
&= (\alpha - 1)(\alpha^{2n} - 1) + (\sigma - \overline{\sigma})(\sigma^{2n} - \overline{\sigma}^{2n}) \geq 0, \qquad (14)
\end{aligned}
$$

with equality if and only if $|\sigma|^2 = \alpha = 1$ and σ^{2n} is real, so if and only if $\sigma^{2n} = \pm 1$. If $\sigma^{2n} = \pm 1$, then from $\sigma^{2n} - \sigma^{-2n} = -2n(\sigma - \sigma^{-1})$ we find $\sigma = \sigma^{-1}$, so $\sigma = \pm 1$, and $\omega = \pm 2$. From $G_n(2) = n$ and $G_n(-2) = \frac{1}{3}n(4n^2 - 1)$ (see Lemma 5.4) we conclude that the inequality in (14) is strict, and $|\sigma^{2n+1} + 1| > |\sigma^{2n} + \sigma|$. As we have $g_n(\omega) = (\sigma^n + \sigma^{1-n})/(\sigma + 1)$, we get

$$|h_{2n}(\omega)| = \left|\frac{\sigma^{2n+1}+1}{\sigma^{2n}+\sigma}\right| > 1.$$

The proof for $n < 0$ is similar. In that case $\sigma - \overline{\sigma}$ and $\sigma^{2n} - \overline{\sigma}^{2n}$ are in the same half-planes, and $(\alpha - 1)(\alpha^{2n} - 1) \leq 0$. □

We now have all tools to handle the case that k is even and kl is negative. This is done in the following proposition.

Proposition 5.8. *Let k, l be any even integers with $kl < 0$. Then $D(k, l)$ is smooth over $\mathbb{Q}$.*

Proof. Set $m = k/2$ and $n = l/2$. The curve $D(k, l)$ is the same as $D(-k, -l)$, so without loss of generality we assume $l > 0$ and $k < 0$. We will argue over $\mathbb{C}$. Assume $P = (r_0, t_0)$ is a singular point of the standard affine part of $D(k, l)$ with $r_0, t_0 \in \mathbb{C}$. By Lemma 5.5 we have $g_m(r_0) \neq 0 \neq g_n(t_0)$, so we may rewrite $F(P) = 0$ as $h_k(r_0) = h_l(t_0)$. This contradicts the fact that from Lemma 5.7 we have $|h_k(r_0)| < 1 < |h_l(t_0)|$, so there is no singular point on the affine part of $D(k, l)$. The points at infinity are smooth by Lemma 5.6. □

We will see that in the remaining cases (k is odd or kl is positive) we can use non-archimedean places instead of complex absolute values. We use the following lemmas.

Lemma 5.9. *For every $n \in \mathbb{Z}$, we have the following identities*

$$2 - u = g_{n+1}^2 + g_n^2 - u g_n g_{n+1}, \tag{15}$$

$$(4 - u^2)G_n = (2n+1)g_n^2 + (2n-1)g_{n+1}^2 - 2nu g_n g_{n+1}, \tag{16}$$

$$(4 - u^2)G_n = g_n^2 - g_{n+1}^2 - 2n(u - 2), \tag{17}$$

$$(u^2 - 4)F_n = f_{n+1}^2 - f_n^2 - (2n + 1). \tag{18}$$

Proof. All these identities can be verified in $\mathbb{Z}[u][s]/(s^2 - us + 1) \cong \mathbb{Z}[s, s^{-1}]$. Note that we have $g_n' = \frac{dg_n}{ds} / \frac{du}{ds} = \frac{dg_n}{ds} \cdot \frac{s^2}{s^2-1}$, and something similar for f_n'. Equation (15) also follows from the last equation of Lemma 3.4 and the relation $tf_n = f_{n+1} + f_{n-1}$. Equation (17) also follows by subtracting $2n$ times the equation (15) from (16). □

It turns out that for the non-archimedean places it is more useful to look at the values of $h_l^2 - 1$ than those of h_l, which we used in the case that k is even and kl is negative. For any integer n and any root ω of G_n we have $g_n(\omega) \neq 0$ by Lemma 5.4(5); dividing equation (17) by $g_n(\omega)^2$, we get

$$h_{2n}(\omega)^2 - 1 = \left(\frac{g_{n+1}(\omega)}{g_n(\omega)}\right)^2 - 1 = \frac{2n(2-\omega)}{g_n(\omega)^2}. \tag{19}$$

Recall from §2.4 that for any prime p, the discrete valuation on $\mathbb{Q}$ associated to p is denoted by v_p and satisfies $v_p(p) = 1$. We scale each discrete valuation v on any number field so that it restricts to v_p on $\mathbb{Q}$ for some prime p, i.e., such that $v(p) = 1$.

Lemma 5.10. *Let n be any integer, and p a prime dividing $2n$. Let K be a number field containing a root ω of G_n. Let v be a valuation on K with $v(p) = 1$. Then $v(g_n(\omega)) = 0$.*

Proof. By Lemma 5.4(2) the polynomial G_n is monic, so ω is an algebraic integer. Let $\mathfrak{p}$ be the prime associated with v, and $\mathbb{F}_\mathfrak{p}$ its residue field. Then the characteristic p of $\mathbb{F}_\mathfrak{p}$ does not divide $2n-1$, so by Lemma 5.4(5) the reduction of $g_n(\omega)$ to $\mathbb{F}_\mathfrak{p}$ is not 0. This implies $v(g_n(\omega)) = 0$. □

From Lemma 5.10 we find that if ω is a root of G_n, and v is some extension of the valuation associated to a prime dividing $2n$, then the valuation at v of the element in (19) equals $v(2n) + v(\omega - 2)$. The proof of the following proposition shows that for odd k, in order to show that $D(k, 2n)$ is smooth, it suffices to note that this valuation is at least 1.

Proposition 5.11. *Let k, l be any nonzero integers with k odd, l even, and $|k| \geq 2$. Then the curve $D(k,l)$ is smooth over $\mathbb{Q}$.*

Proof. Set $m = (k-1)/2$ and $n = l/2$, so that $k = 2m+1$ and $l = 2n$. Assume $P = (r_0, t_0)$ is a singular point over $\overline{\mathbb{Q}}$ of the standard affine part of $D(k,l)$. Let K be the number field $\mathbb{Q}(r_0, t_0)$, and let v be the valuation on K associated to a prime above 2, normalized so that $v(2) = 1$. By Lemma 5.5 we have $f_m(r_0) \neq 0 \neq g_n(t_0)$ and $F_m(r_0) = G_n(t_0) = 0$. From Lemma 5.10 we then conclude $v(g_n(t_0)) = 0$. Now around P the curve $D(k,l)$ is given by $f_{m+1}(r)/f_m(r) = g_{n+1}(t)/g_n(t)$, which by (19) and (18) of Lemma 5.9 implies

$$\frac{2m+1}{f_m(r_0)^2} = \frac{2m+1+(r_0^2-4)F_m(r_0)}{f_m(r_0)^2} = \frac{f_{m+1}(r_0)^2 - f_m(r_0)^2}{f_m(r_0)^2}$$
$$= \left(\frac{f_{m+1}(r_0)}{f_m(r_0)}\right)^2 - 1 = \left(\frac{g_{n+1}(t_0)}{g_n(t_0)}\right)^2 - 1 = \frac{2n(2-t_0)}{g_n(t_0)^2}.$$

This contradicts the fact that the valuation at v of the left-hand side is at most 0, while the valuation of the right-hand side is at least 1. We conclude that no singular point P exists on the affine part. By Lemma 5.6 there are also no singular points at infinity. □

The only remaining case is the case that k is even and kl is positive. We deal with this case by investigating the possible values of the valuation of the expression in (19) at some valuation extending v_p for some prime p dividing $2n$.

Lemma 5.12. *Let n be a positive integer and p a prime dividing n and set $e = v_p(n)$. Then for any integer $j \geq 0$ we have $v_p\left(\binom{n}{p^j}\right) = \max(e-j, 0)$ and for any $0 < k < p^j$ we have $v_p\left(\binom{n}{k}\right) > e - j$.*

Proof. For $j > e$ the statement is trivial, as $\binom{n}{k}$ is an integer, so we may assume $j \leq e$. Let l be any integer satisfying $1 \leq l \leq p^e$, and write $\binom{n}{l}$ as

$$\binom{n}{l} = \frac{n}{l} \cdot \prod_{i=1}^{l-1} \frac{n-i}{i}.$$

For all i with $1 \leq i < p^e$ we have $v_p(i) < v_p(n)$, so $v_p(n-i) = v_p(i)$ and $v_p\left(\frac{n-i}{i}\right) = 0$. Therefore, we have $v_p\left(\binom{n}{l}\right) = v_p(n) - v_p(l)$. Applying this to $l = k$ and $l = p^j$, we obtain the statement, as $v_p(k) < j = v_p(p^j)$. □

Lemma 5.13. *Let n be a positive integer and p a prime dividing $2n$. Let K be a number field and v a valuation on K with $v(p) = 1$. Let $\alpha \in K$ satisfy $v(\alpha) = 0$*

and set $e = v(4n)$. If $p \neq 2$, then also assume $v(2\alpha^{2n+1} + \alpha^2 + 1) = 0$. Then the Newton polygon of

$$(S+\alpha)^{4n} + 2n\left((S+\alpha)^{2n+1} - (S+\alpha)^{2n-1}\right) - 1 = \sum_{i=0}^{4n} b_i S^i$$

at v is the lower convex hull of the points

$$\begin{cases} (0, v(b_0)), (1, e), (p, e-1), \ldots, (p^j, e-j), \ldots, (p^e, 0), (4n, 0) & \text{if } p \neq 2, \\ (0, v(b_0)), (1, v(b_1)), (2, v(b_2)), (3, e-1), (4, e-2), \ldots, (2^e, 0), (4n, 0) & \text{if } p = 2. \end{cases}$$

Proof. The Newton polygon is the lower convex hull of all the points $(i, v(b_i))$ for $0 \leq i \leq 4n$. It suffices to show that for each point (i, a) in the given sequences we have $a = v(b_i)$, while for each k for which there is no point (k, a) in the sequence, there is a pair (i_1, a_1), (i_2, a_2) of consecutive points with $i_1 < k < i_2$, such that $v(b_k) \geq \max(a_1, a_2)$; this would certainly imply that the point $(k, v(b_k))$ is not below the line segment through (i_1, a_1) and (i_2, a_2). Note that for $k \geq 1$ we have

$$b_k = \binom{4n}{k}\alpha^{4n-k} + 2n\left(\binom{2n+1}{k}\alpha^{2n+1-k} - \binom{2n-1}{k}\alpha^{2n-1-k}\right). \tag{20}$$

Suppose $p \neq 2$, and let (i, a) be a point in the corresponding given sequence. If $i = 0$, then $a = v(b_0)$ by definition. We have $b_1 = 2n\alpha^{2n-2}(2\alpha^{2n+1} + \alpha^2 + 1 + 2n(\alpha^2 - 1))$. By hypothesis we have $v(2\alpha^{2n+1} + \alpha^2 + 1) = 0$ and as $v(2n(\alpha^2 - 1))$ is positive, the valuation of the last factor of b_1 is zero. Therefore, if $i = 1$, then $v(b_i) = v(2n) = v(4n) = e = a$, as needed. If $i = p^j$ for $1 \leq j \leq e$, then by Lemma 5.12 the valuation of the first term in (20) for $k = i$ equals $e - j$, while the valuation of the second term is at least $v(2n) = e$, so we get $v(b_i) = e - j = a$, as needed. If $i = 4n$, then $b_i = 1$, so $v(b_i) = 0 = a$, as needed. Suppose $k \leq 4n$ is an integer for which there is no point (k, a) in the sequence. If $k > p^e$, then all we need to note is that $v(b_k) \geq 0$. If $k \leq p^e$, then there is a $j \in \{1, 2, \ldots, e\}$ such that $p^{j-1} < k < p^j$, in which case the first term of (20) has valuation at least $e - j + 1$ by Lemma 5.12, while the second term has valuation at least e, so we have $v(b_k) \geq e - j + 1 = \max(e - j + 1, e - j)$, which is exactly what we wanted to show.

Now suppose $p = 2$, and let (i, a) be a point in the corresponding given sequence. If $0 \leq i \leq 2$, then $a = v(b_i)$ by definition. Note that

$$b_3 = \frac{2n\alpha^{2n-4}}{3}\left(2(8n^2 - 6n + 1)\alpha^{2n+1} + n(4n^2 - 1)\alpha^2 - (n-1)(4n^2 - 8n + 3)\right).$$

The first term between the parentheses has valuation 1, while of the second and third term, exactly one has valuation 1, and the other has valuation 0, as exactly one of n and $n-1$ is even. We conclude that the expression between the parentheses has valuation 0, so if $i = 3$, then $v(b_i) = v(2n) = e - 1 = a$, as needed. If $i = 2^j$ for $2 \leq j \leq e$, then by Lemma 5.12 the valuation of the first term in (20) for $k = i$ equals $e - j$, while the valuation of the second term is at least $v(2n) = e - 1 > e - j$, so we get $v(b_i) = e - j = a$, as needed. If $i = 4n$, then $b_i = 1$, so $v(b_i) = 0 = a$, as needed. Suppose $k \leq 4n$ is an integer for which there is no point (k, a) in the sequence. If $k > 2^e$, then all we need to note is that $v(b_k) \geq 0$. If $k \leq 2^e$, then there is a $j \in \{3, 4, \ldots, e\}$ such that $2^{j-1} < k < 2^j$, in which case the first term of (20) has valuation at least $e - j + 1$ by Lemma 5.12, while the second term has valuation at least $e - 1 \geq e - j + 1$, so we have $v(b_k) \geq e - j + 1 = \max(e - j + 1, e - j)$, which is exactly what we wanted to show. This finishes the proof. ∎

Lemma 5.14. *Let n be any positive integer and p a prime. Set $e = v_p(4n)$. Then the Newton polygon of*

$$(S+1)^{4n} + 2n\left((S+1)^{2n+1} - (S+1)^{2n-1}\right) - 1$$

at v_p has vertices

$$\begin{cases} (0,\infty),(1,e),\ldots,(p^j,e-j),\ldots,(p^e,0),(4n,0) & \text{if } p \neq 2, \\ (0,\infty),(1,e+1),(4,e-2),\ldots,(2^j,e-j),\ldots,(2^e,0),(4n,0) & \text{if } p = 2. \end{cases}$$

Proof. The terms of lowest degree in the polynomial are $0S^0 + 8nS^1 + 4n(4n-1)S^2$. If p divides $2n$, then we can apply Lemma 5.13 with $\alpha = 1$, and the result follows immediately from $b_0 = 0$, $b_1 = 8n$, and $b_2 = 4n(4n-1)$. If p does not divide $2n$, then $p \neq 2$, and $e = 0$, and $v_p(b_1) = 0$. It follows that the Newton polygon has vertices $(0,\infty)$, $(1,0)$, and $(4n,0)$, exactly as claimed. □

Proposition 5.15. *Let n be any positive integer and p a prime. Let K be a number field containing a root ω of G_n. Let v be a valuation on K with $v(p) = 1$. Then we have $0 \leq v(\omega - 2) \leq 1$ and $v(h_{2n}(\omega)^2 - 1) \leq v(2n) + 1$. If p divides $2n$, then we also have $v(2n) \leq v(h_{2n}(\omega)^2 - 1)$. Moreover, if $p \neq 3$ or $v(n) = 0$, then the upper bounds in the first two inequalities are strict.*

Proof. By Lemma 5.4(2) the root ω is an algebraic integer, so we have $v(2-\omega) \geq 0$ and $v(g_n(\omega)) \geq 0$. From (19) we know $h_{2n}(\omega)^2 - 1 = 2n(2-\omega)g_n(\omega)^{-2}$, so $v(\omega - 2) \leq 1$ implies $v(h_{2n}(\omega)^2 - 1) \leq v(2n) + 1$ and if the former inequality is strict, then so is the latter. Also, if p divides $2n$, then by Lemma 5.10 we have $v(g_n(\omega)) = 0$, so $v(h_{2n}(\omega)^2 - 1) = v(2n) + v(2-\omega) \geq v(2n)$. Therefore it suffices to show that $v(\omega - 2) \leq 1$, and that this inequality is strict in the claimed cases. Let L be a finite field extension of K containing a root σ of $s^2 - \omega s + 1 = 0$, and extend v to L. Then $\omega = \sigma + \sigma^{-1}$, so σ is a root of $f = s^{4n} + 2n(s^{2n+1} - s^{2n-1}) - 1$ by Lemma 5.4(3). This implies that $\sigma - 1$ is a root of the polynomial in Lemma 5.14, which we will denote by F. First consider the case $p = 2$. The polynomial f has roots 1 and -1 of multiplicity 1 and 3 respectively, corresponding to roots 0 and -2 of F, which in turn correspond to the line segments of the Newton polygon from $(0,\infty)$ to $(1,e+1)$ and from $(1,e+1)$ to $(4,e-2)$ respectively by Lemma 2.4. If σ were one of these roots of f, then we would have $\omega = \pm 2$, which contradicts $G_n(2) = n$ and $G_n(-2) = \frac{1}{3}n(4n^2 - 1)$ by Lemma 5.4. The root $\sigma - 1$ of F therefore corresponds to another segment of the Newton polygon of F, all of which have slope between $-\frac{1}{4}$ and 0, so we have $0 \leq v(\sigma - 1) \leq \frac{1}{4}$ by Lemma 2.4, and thus $v(\omega - 2) = v(\sigma^{-1}(\sigma - 1)^2) = 2v(\sigma - 1) \leq \frac{1}{2} < 1$. Now consider the case $p > 2$. We still have $\sigma - 1 \neq 0$, so the root $\sigma - 1$ of F corresponds to a nonvertical segment of the Newton polygon of F. These segments all have slope equal to 0 or $-1/(p^j - p^{j-1})$ for some $1 \leq j \leq v(n)$, so we have $v(\sigma - 1) \leq \frac{1}{p^j - p^{j-1}}$, and $0 \leq v(\omega - 2) = 2v(\sigma - 1) \leq \frac{2}{p^j - p^{j-1}} \leq 1$, where the equality follows as it did in case $p = 2$. The last inequality is strict unless $p = 3$ and $j = 1$, in which case $v(n) > 0$. This proves the proposition. □

If k and l are even and kl is positive, and k and l are not equal and do not differ by a factor 3, then the results above are sufficient to show that there exists a prime p such that the values at critical points of h_k are different from those of h_l. This would show that $D(k,l)$ is smooth over $\mathbb{Q}$. The following results allow us to also handle the case that k and l differ by a factor 3.

Lemma 5.16. *Let n be any positive integral multiple of 3. Let K be the number field $\mathbb{Q}(i) = \mathbb{Q}[x]/(x^2+1)$ and let v be the unique valuation on K satisfying $v(3) = 1$. Set $e = v(n)$. Then the Newton polygon of*

$$(S+i)^{4n} + 2n\left((S+i)^{2n+1} - (S+i)^{2n-1}\right) - 1$$

at v has vertices

$$(0, e), (3, e-1), \ldots, (3^j, e-j), \ldots, (3^e, 0), (4n, 0).$$

Proof. This follows immediately from Lemma 5.13 with $\alpha = i$ and $p = 3$. □

Proposition 5.17. *Let n be any positive integral multiple of 3. Let K be a number field containing a root ω of G_n. Let v be a valuation on K with $v(3) = 1$. Then $v(2n) \le v(h_{2n}(\omega)^2 - 1 - n) < v(2n) + 1$.*

Proof. From (19) we deduce $h_{2n}(\omega)^2 - 1 - n = ng_n(\omega)^{-2}A$ with $A = 4 - 2\omega - g_n(\omega)^2$. By Lemma 5.10 we have $v(g_n(\omega)) = 0$, so $v(h_{2n}(\omega)^2 - 1 - n) = v(n) + v(A)$. As ω is an algebraic integer, we have $v(A) \ge 0$, so it suffices to show $v(A) < 1$. Let L be a finite field extension of K containing a square root i of -1 and a root σ of $s^2 - \omega s + 1 = 0$, and extend v to L. Let R denote the discrete valuation ring of L associated to v, and $\mathfrak{m}$ its maximal ideal. For each $\varepsilon \in \{\pm 1\}$ we have $-(\sigma+1)^2 A = X(\varepsilon) + Y(\varepsilon) + Z(\varepsilon)$ with $X(\varepsilon) = 2\sigma^{-1}(\sigma - \varepsilon)^2(\sigma^2+1)$, $Y(\varepsilon) = 3\varepsilon(\sigma - \varepsilon)^2$, and $Z(\varepsilon) = \varepsilon(\sigma^{2n} + \varepsilon)(\sigma^{2-2n} + \varepsilon)$.

We have $\omega = \sigma + \sigma^{-1}$, so σ is a root of $f = s^{4n} + 2n(s^{2n+1} - s^{2n-1}) - 1$ by Lemma 5.4(3). This implies that $\sigma - i$ is a root of the polynomial in Lemma 5.16. Since the slopes of the Newton polygon of this polynomial are between $-\frac{1}{3}$ and 0 by Lemma 5.16, we have $v(\sigma - i) \le \frac{1}{3}$ by Lemma 2.4. Replacing i by $-i$ temporarily, we also find $v(\sigma + i) \le \frac{1}{3}$, so we get $v(\sigma^2+1) = v(\sigma+i) + v(\sigma-i) \le \frac{2}{3} < 1$. From $f(\sigma) = 0$ we get $(\sigma^{2n} - 1)(\sigma^{2n}+1) = -2n\sigma^{2n-1}(\sigma^2 - 1)$. As the elements $\sigma^{2n} - 1$ and $\sigma^{2n} + 1$ differ by 2, which is a unit in R, at least one of them is also a unit, with valuation 0, so we conclude

$$\begin{aligned}\max\left(v(\sigma^{2n}+1), v(\sigma^{2n}-1)\right) &= v(\sigma^{2n}+1) + v(\sigma^{2n}-1)\\ &= v\left((\sigma^{2n}+1)(\sigma^{2n}-1)\right) = v\left(-2n\sigma^{2n-1}(\sigma^2-1)\right)\\ &= v(n) + v(\sigma^2-1) \ge 1 + v(\sigma^2-1) \ge 1 + v(\sigma+1).\end{aligned}$$

Suppose first that $\sigma^{2n} - 1$ is a unit, and thus that $v(\sigma^{2n}+1) \ge 1$. Since $\sigma^{2n} - 1$ is a multiple of $\sigma^2 - 1$ in R, we find that $\sigma^2 - 1$ is also a unit, so $v(\sigma+1) = v(\sigma-1) = 0$. We get $v(X(1)) = v(\sigma^2+1) < 1$, while $v(Y(1)), v(Z(1)) \ge 1$, so we obtain $v(A) = v(-(\sigma+1)^2 A) = v(X(1) + Y(1) + Z(1)) = v(X(1)) < 1$ and we are done. Hence we may assume that $\sigma^{2n} - 1$ is not a unit, so $\sigma^{2n} + 1$ is a unit and we have $v(\sigma^{2n} - 1) \ge 1 + v(\sigma+1)$. Since $\sigma^{2-2n} - 1$ is a multiple of $\sigma+1$ we also have $v(\sigma^{2-2n} - 1) \ge v(\sigma+1)$ and thus $v(Z(-1)) \ge (1 + v(\sigma+1)) + v(\sigma+1) \ge 1 + 2v(\sigma+1)$. We also have $v(Y(-1)) = 1 + 2v(\sigma+1)$ and $v(X(-1)) = v(\sigma^2+1) + 2v(\sigma+1) < 1 + 2v(\sigma+1)$. This yields $v(A) = v(-(\sigma+1)^2 A) - 2v(\sigma+1) = v(X(-1) + Y(-1) + Z(-1)) - 2v(\sigma+1) = v(X(-1)) - 2v(\sigma+1) < 1$, which finishes the proof. □

Lemma 5.18. *Let n be any positive integral multiple of 3. Let K be the number field $\mathbb{Q}[x]/(x^2-3)$, let β be the image of x in K, and let v be the unique valuation on K satisfying $v(3) = 1$. Set $e = v(n)$ and $\alpha = -2 + \beta$. Then the Newton polygon of*

$$(S+\alpha)^{4n} + 2n\left((S+\alpha)^{2n+1} - (S+\alpha)^{2n-1}\right) - 1$$

at v has vertices

$$(0, e + \tfrac{3}{2}), (1, e), (3, e-1), \ldots, (3^j, e-j), \ldots, (3^e, 0), (4n, 0).$$

Proof. By Lemma 5.13 it suffices to check $v(a) = 0$ with $a = 2\alpha^{2n+1} + \alpha^2 + 1$, and $v(b_0) = e + \frac{3}{2}$ with $b_0 = \alpha^{4n} + 2n(\alpha^{2n+1} - \alpha^{2n-1}) - 1$. Note that we have $\alpha = 1 + \gamma$ with $\gamma = \beta - \beta^2$, while β generates the ideal $\mathfrak{p}$ to which v is associated. It follows that $\alpha \equiv 1 \pmod{\mathfrak{p}}$, so $a \equiv 1 \pmod{\mathfrak{p}}$, and indeed $v(a) = 0$. Expanding the powers of $\alpha = 1 + \gamma$ gives $b_0 = \sum_{i=1}^{4n} c_i\gamma^i$ with $c_i = \binom{4n}{i} + 2n\binom{2n+1}{i} - 2n\binom{2n-1}{i}$. We claim that for $i \geq 4$ we have $v\big(\binom{4n}{i}\big) \geq e + 2 - i/2$. Write $\binom{4n}{i} = 4n \cdot \frac{1}{i!}(4n-1)\cdots(4n-i+1)$ and note that the product of at least three consecutive integers is divisible by 3. As $v(4!) = v(5!) = 1$, we conclude that for $i = 4, 5$ we have $v\big(\binom{4n}{i}\big) \geq v(n) = e \geq e + 2 - i/2$. Suppose $i \geq 6$ and let $j \geq 2$ be the integer satisfying $3^{j-1} \leq i < 3^j$. Since $v\big(\binom{4n}{i}\big)$ is an integer, we have $v\big(\binom{4n}{i}\big) \geq e + 1 - j \geq e + 2 - i/2$ by Lemma 5.12, where the last inequality follows from $i \geq 6$ for $j = 2$ and from $i \geq 3^{j-1} \geq 2j + 2$ for $j \geq 3$. This proves the claim, and as the last two terms of c_i have valuation at least $v(2n) = e$, we find $v(c_i) \geq e + 2 - i/2$ for $i \geq 4$. This gives $v(c_i\gamma^i) \geq e + 2 - i/2 + i \cdot v(\gamma) = e + 2$ for $i \geq 4$, and therefore $v(\sum_{i=4}^{4n} c_i\gamma^i) \geq e + 2$. We also have

$$c_1\gamma + c_2\gamma^2 + c_3\gamma^3 = 4n\big((140\beta - 252)n^2 - (144\beta - 264)n + 33\beta - 63\big),$$

which has valuation $e + \frac{3}{2}$, as there is a unique term with lowest valuation $\frac{3}{2}$ inside the parentheses, namely 33β. We conclude $v(b_0) = e + \frac{3}{2}$. □

Proposition 5.19. *Let n be any positive integral multiple of 3. Let K be a number field containing a root ω of G_n. Let v be a valuation on K with $v(3) = 1$. Then $v(n) \leq v(h_{2n}(\omega)^2 - 1 - 3n) < v(n) + 1$ or $v(h_{2n}(\omega)^2 - 1 - 3n) \geq v(n) + 2$.*

Proof. From (19) we deduce $h_{2n}(\omega)^2 - 1 - 3n = ng_n(\omega)^{-2}A$ with $A = -2(\omega + 4) + 3(4 - g_n(\omega)^2)$. By Lemma 5.10 we have $v(g_n(\omega)) = 0$, so $v(h_{2n}(\omega)^2 - 1 - 3n) = v(n) + v(A)$. As we clearly have $v(A) \geq 0$, it suffices to show $v(A) < 1$ or $v(A) \geq 2$. Let L be a finite field extension of K containing a square root β of 3 and a root σ of $s^2 - \omega s + 1 = 0$. Set $\alpha = -2 + \beta$ and $\overline{\alpha} = -2 - \beta = \alpha^{-1}$, and extend v to L. Let F denote the polynomial in Lemma 5.18, and set $f(s) = s^{4n} + 2n(s^{2n+1} - s^{2n-1}) - 1$, so that $F(S) = f(S + \alpha)$. From Lemma 2.4 and the slopes of the Newton polygon of F given in Lemma 5.18, we conclude that there is a unique root S_0 of F with $v(S_0) = \frac{3}{2}$. Then $s_0 = S_0 + \alpha$ is a root of f, and as f is anti-reciprocal, so is $s_1 = s_0^{-1}$ and both are units in the ring of integers of L. Set $S_1 = s_1 - \alpha$. Then S_1 is root of F and from the identity $S_1 = -\overline{\alpha}s_0^{-1}S_0 - 2\beta$ and the inequality $v(2\beta) = \frac{1}{2} < v(\overline{\alpha}s_0^{-1}S_0)$ we conclude $v(S_1) = \frac{1}{2}$. Note that $S_2 = 3 - \beta$ is also a root of F, corresponding to the root 1 of f, and with $v(S_2) = \frac{1}{2}$. By Lemma 2.4 and Lemma 5.18 there are only three roots z of F with $v(z) \geq \frac{1}{2}$, so all roots z of F, other than S_0, S_1, S_2, satisfy $v(z) < \frac{1}{2}$.

Now $\omega = \sigma + \sigma^{-1}$, so by Lemma 5.4(3), the element σ is a root of f, and therefore $\sigma - \alpha$ is a root of F. First suppose the inequality $v(\sigma - \alpha) < \frac{1}{2} = v(2\beta)$ holds. Then we also have $v(\sigma - \overline{\alpha}) = v(\sigma - \alpha + 2\beta) < \frac{1}{2}$, and thus $v(\omega + 4) = v\big(\sigma^{-1}(\sigma - \alpha)(\sigma - \overline{\alpha})\big) < 1$. From $0 \leq v(2(\omega + 4)) < 1 \leq v(3(4 - g_n(\omega)^2))$ we conclude $v(A) = v(\omega + 4) < 1$ and we are done.

Now suppose $v(\sigma - \alpha) \geq \frac{1}{2}$, then $\sigma - \alpha = S_i$ for some i with $0 \leq i \leq 2$. For $i = 2$ we get $\sigma = S_2 + \alpha = 1$ and thus $\omega = 2$, so $G_n(\omega) = n \neq 0$ by Lemma

5.4(4). From this contradiction we conclude $\sigma = s_i$ for $i = 0$ or $i = 1$, so that $\omega + 4 = \sigma + \sigma^{-1} + 4 = s_0 + s_1 + 4 = -\overline{\alpha} S_0 S_1$. This implies $v(\omega + 4) = \frac{3}{2} + \frac{1}{2} = 2$. We rewrite A as

$$A = -2(\omega + 4) + 9 - 3(\omega - 2)\left(\frac{g_n(\omega)^2 - 1}{\omega - 2}\right). \tag{21}$$

From Lemma 3.3 we know that $d(t) = (g_n(t) - 1)/(t - 2)$ is a polynomial, so $v((g_n(\omega)^2 - 1)/(\omega - 2)) = v((g_n(\omega) + 1)d(\omega)) \geq 0$. From $\omega - 2 = (\omega + 4) - 6$ we get $v(\omega - 2) = v(6) = 1$, so the last term of (21) has valuation at least 2, while $v(-2(\omega + 4)) = v(9) = 2$. We conclude $v(A) \geq 2$, which finishes the proof. □

Proposition 5.20. *Let k, l be any even integers with $k \neq l$ and $kl > 0$. Then $D(k, l)$ is smooth over $\mathbb{Q}$.*

Proof. The curve $D(k, l)$ is the same as $D(-k, -l)$, so without loss of generality we assume $k, l > 0$. Set $m = k/2$ and $n = l/2$ and $F = g_{m+1}(r)g_n(t) - g_m(r)g_{n+1}(t)$. Assume $P = (r_0, t_0)$ is a singular point over $\overline{\mathbb{Q}}$ of the standard affine part of $D(k, l)$. Let K be the number field $\mathbb{Q}(r_0, t_0)$. By Lemma 5.5 we have $G_n(t_0) = 0$ and $G_m(r_0) = 0$, and $D(k, l)$ is given around P by $h_k(r) = h_l(t)$. Set $c = h_k(r_0)^2 - 1$ and $d = h_l(t_0)^2 - 1$. Let p be any prime such that $v_p(m) \neq v_p(n)$. Set $e = v_p(n) - v_p(m)$. By symmetry we may assume $e \geq 1$. Let $\mathfrak{p}$ be a prime of K above p, and let v be the valuation on K associated to $\mathfrak{p}$, normalized so that v restricts to v_p on $\mathbb{Q}$. By Lemma 5.15 we have $v(c) \leq v(2m) + 1 \leq v(2m) + e = v(2n) \leq v(d)$. From $c = d$ we conclude that all inequalities are equalities, so $e = 1$ and by Lemma 5.15 we have $p|m$ and $p = 3$, and thus $n = 3m$. Proposition 5.17 shows $v(2m) + 1 = v(2n) \leq v(d - n) < v(2n) + 1 = v(2m) + 2$, while from Proposition 5.19 we get $v(c - 3m) < v(2m) + 1$ or $v(c - 3m) \geq v(2m) + 2$. This contradicts the equality $c - 3m = d - n$, and we conclude that no singular point P exists on the affine part. By Lemma 5.6 there are also no singular points at infinity. □

We have now proved the first statement of Theorem 5.1, split over several Propositions. To prove the last statement, we set set $H_n = g''_{n+1} g_n - g_{n+1} g''_n$ for each integer n, where derivatives are taken with respect to u. In $\mathbb{Z}[u][s]/(s^2 - us + 1) \cong \mathbb{Z}[s, s^{-1}]$ one checks

$$\frac{1}{2}(u - 2)(u + 2)^2 H_n = (n - 1)f_{2n+1} + f_{2n} - (n + 1)f_{2n-1} - nu + 2n. \tag{22}$$

Recall that for even $l \neq 0$, the curve $D_1(l, l)$ is the projective closure of the scheme-theoretic complement in $D(l, l)$ of the line given by $r = t$.

Proposition 5.21. *Let l be any even integer with $|l| \geq 4$. Then the curve $D_1(l, l)$ is smooth over $\mathbb{Q}$.*

Proof. Set $n = l/2$ and $F = g_{n+1}(r)g_n(t) - g_n(r)g_{n+1}(t)$ and $G = F/(t - r)$. Then $D_1(l, l)$ is defined by $G(r, t) = 0$. Any singular point of $D_1(l, l)$ is also a singular point of $D(l, l)$. By Lemma 5.6 we find that $D(l, l)$ is smooth at all points at infinity, so $D_1(l, l)$ is as well. Assume $P = (r_0, t_0)$ is a singular point of the standard affine part of $D_1(l, l)$. Then P is also a singular point of $D(l, l)$. By Lemma 5.5 we then have $G_n(t_0) = 0$ and $G_n(r_0) = 0$, and we may rewrite $F(P) = 0$ as $h_l(r_0) = h_l(t_0)$. Then from (16) of Lemma 5.9 we have

$$2nr_0 = (2n-1)h_l(r_0) + (2n+1)h_l(r_0)^{-1} = (2n-1)h_l(t_0) + (2n+1)h_l(t_0)^{-1} = 2nt_0,$$

which implies $r_0 = t_0$. Set $F_t = \partial F/\partial t$ and $F_{t^2} = \partial F_t/\partial t$ and $G_t = \partial G/\partial t$. Then we have $G(r_0, r_0) = F_t(r_0, r_0) = G_n(r_0)$, where the first equality can be viewed as an algebraic version of l'Hôpital's rule applied to $\lim_{t\to r_0} G(r_0, t)$. That same rule also gives $G_t(r_0, r_0) = \frac{1}{2}F_{t^2}(r_0, r_0) = \frac{1}{2}H_n(r_0)$. The fact that $D_1(l, l)$ is singular at P implies $0 = G(P) = G_n(r_0)$ and $0 = G_t(P) = \frac{1}{2}H_n(r_0)$. From Lemma 5.4(3) and (22) we then deduce

$$\begin{aligned} 0 &= (r_0 + 2)\left(((n-1)r_0 + 1)\, G_n(r_0) - \frac{1}{2}(r_0^2 - 4)H_n(r_0)\right) \\ &= n\,(2f_{2n-1}(r_0) + (2n-1)r_0)\,, \end{aligned}$$

so we get $r_0 = 2(f_{2n-1}(r_0)+n)$. This implies $v(r_0) \geq 1$ for any valuation v of $\mathbb{Q}(r_0)$ with $v(2) = 1$, which contradicts the inequality $v(r_0-2) < 1$ from Proposition 5.15. We conclude that $D_1(l, l)$ has no singular points. □

Proof of Theorem 5.1. The first statement follows immediately from Propositions 5.8, 5.11, 5.20, while the last statement follows from Proposition 5.21. □

6. Genera of the irreducible components

The following theorem tells us the number of irreducible components of $Y(k, l)$ in all cases that $J(k, l)$ is a hyperbolic knot. Recall that $Y_0(l, l)$ and $Y_1(l, l)$ were defined in Definition 4.7.

Theorem 6.1. *Let k, l be any nonzero integers with l even, $|k| \geq 2$, and $k \neq l$.*

1. *The curve $D(k, l)$ is a smooth, projective, geometrically irreducible curve of bidegree $(\lfloor |k|/2 \rfloor, |l|/2)$ containing an open subset that is isomorphic to $Y(k, l) = Y_0(k, l)$.*
2. *If $|l| > 2$, then $D_1(l, l)$ is a smooth, projective, geometrically irreducible curve of bidegree $(|l|/2 - 1, |l|/2 - 1)$ containing an open subset that is isomorphic to $Y_1(l, l)$. The curve $Y(l, l)$ consists of two geometrically irreducible components, namely $Y_0(l, l)$ and $Y_1(l, l)$.*

Proof. The curves $D(k, l)$ and $D(l, l)$ are projective by construction. By Theorem 5.1 the curve $D(k, l)$ is smooth and its bidegree is given in Remark 4.1. Every smooth projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a, b) with $a, b > 0$ is geometrically irreducible by Lemma 2.6, so $D(k, l)$ is geometrically irreducible. By Lemma 2.5 and Proposition 4.4 the curve $C(k, l)$ is isomorphic to an open subset of $D(k, l)$. Since $Y(k, l)$ is isomorphic to $C(k, l)$, we conclude that $Y(k, l)$ is isomorphic to an open subset of $D(k, l)$, so $Y(k, l)$ is geometrically irreducible and smooth as well, and therefore equal to $Y_0(k, l)$. Suppose $|l| > 2$. By Theorem 5.1 the curve $D_1(l, l)$ is smooth and its bidegree is given at the end of §4. By Lemma 2.6 the curve $D_1(l, l)$ is geometrically irreducible, so $D(l, l)$ consists of two irreducible components, namely $D_0(l, l)$ and $D_1(l, l)$, cf. end of §4. By Proposition 4.4, the curve $C(l, l)$ is birationally equivalent to $D(l, l)$, so it also has two components, one of which is isomorphic to a subset of $D_1(l, l)$ by Lemma 2.5. Since $Y(l, l)$ is isomorphic to $C(l, l)$, the curve $Y(l, l)$ also has two components, so $Y_1(l, l)$ is irreducible and the components are $Y_0(l, l)$ and $Y_1(l, l)$. Since $Y_0(l, l)$ corresponds to $D_0(l, l)$ by Proposition 4.6, it is $Y_1(l, l)$ that is isomorphic to a subset of $D_1(l, l)$. □

It is now easy to find the genus of the components of $Y(k, l)$.

Theorem 6.2. *Let k, l be any nonzero integers with l even, $|k| \geq 2$, and $k \neq l$.*

(1) *The curve* $Y(k,l) = Y_0(k,l)$ *has geometric genus* $(\lfloor |k|/2 \rfloor - 1)(|l|/2 - 1)$ *and is hyperelliptic if and only if* $|k| \le 5$ *or* $|l| \le 5$.
(2) *If* $|l| > 2$, *then the curve* $Y_0(l,l)$ *has genus* 0 *and the curve* $Y_1(l,l)$ *has genus* $(|l|/2 - 2)^2$ *and is hyperelliptic if and only if* $|l| \le 6$.

Proof. By Theorem 6.1 the curves $Y(k,l)$ and $Y_0(k,l)$ are both birationally equivalent to $D(k,l)$, which is a smooth irreducible curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(\lfloor |k|/2 \rfloor, |l|/2)$. Statement (1) therefore follows from Lemma 2.6. By Proposition 4.6 the curve $Y_0(l,l)$ is birationally equivalent to a line, so it has genus 0. By Theorem 6.1 the curve $Y_1(l,l)$ is birationally equivalent to $D_1(l,l)$, which is smooth of bidegree $(|l|/2 - 1, |l|/2 - 1)$. Therefore, statement (2) follows from Lemma 2.6. □

Proof of Theorem 1.1. This follows immediately from Theorem 6.2. □

Our next goal is to investigate the ramification of the map from $X(k,l)$ to $Y(k,l)$, which we will then use to compute the genus of the irreducible components of $X(k,l)$. The component $X_0(k,l)$ lies above $Y_0(k,l)$. For $|l| > 2$ we know that $Y(l,l)$ consists of two irreducible components, so $X(l,l)$ consists of at least two components.

Lemma 6.3. *Let* m, n *be any nonzero integers. Consider the function* $h = (r - 2)\big(2 - t + (r^2 - 4) f_m(r)^2\big)$ *on* $D(2m, 2n)$. *Then* h *is regular and nonvanishing at all singular points of* $D(2m,2n)$, *and has odd valuation at exactly* $2|mn| + 2|m| + 2|n| - 2a$ *nonsingular points of* $D(2m,2n)$, *with* $a = 2$ *for* $mn > 0$ *and* $a = 1$ *for* $mn < 0$. *If* $m = n$, *then exactly* $2|n|$ *of these points lie on the line determined by* $r = t$.

Proof. Set $d = 2 - t + (r^2 - 4) f_k(r)^2$, so that $h = (r-2)d$. Let M and C denote the vanishing locus of $r - 2$ and d respectively. From $g_m(2) = g_{m+1}(2) = 1$, we see that there are $|n|$ points in the affine part of the intersection $M \cap D(2m,2n)$, namely $(2, \tau)$ for each root τ of $g_{n+1}(t) - g_n(t)$. As $D(2m,2n)$ has bidegree $(|m|, |n|)$, we have $M \cdot D(2m,2n) = |n|$, which shows that all intersection multiplicities are trivial, so all intersection points are smooth, and the intersections are transversal. This implies that for each Q of these $|n|$ points we have $v_Q(r-2) = 1$ by Lemma 2.8. The points in the standard affine part of the intersection $C \cap D(2m,2n)$ correspond to the roots of $F(r,T)$ with $F(r,t) = g_m(r) g_{n+1}(t) - g_{m+1}(r) g_n(t)$ and $T = (r^2-4) f_m(r)^2 + 2$. The degree of T equals $2|m|$. From Lemma 3.3 we find that the degree of $F(r,T)$ as a polynomial in r equals $2|mn| + |m| + 1 - a$. We now show that $F(r,T)$ is separable. Consider the extension $\mathbb{Z}[r][q]/(q^2 - rq + 1) \cong \mathbb{Z}[q, q^{-1}]$. Then we have $r = q + q^{-1}$ and from $f_m(r) = (q^m - q^{-m})/(q - q^{-1})$ we find $T = q^{2m} + q^{-2m}$. This yields $F(r,T) = q^{1-m-2mn}(q^{2m} - 1)(q^{4mn-1} - 1)/(q+1)$, and as $\gcd(2m, 4mn-1) = 1$, we find the only multiple factor of $F(r,T)$ in $\mathbb{Z}[q,q^{-1}]$ is $(q-1)^2 = q(r-2)$, which corresponds to the single root $r = 2$. We conclude that $F(r,T)$ is indeed separable. This shows that all $2|mn| + |m| + 1 - a$ intersection points R in the affine part $C \cap D(2m,2n)$ are transversal intersections, so they are smooth points of $D(2m,2n)$, and we have $v_R(d) = 1$ by Lemma 2.8. If h vanishes, then either $r - 2$ or d does. The only point where both $r - 2$ and d vanish is $P = (2,2)$, where the valuation $v_P(h) = v_P(r-2) + v_P(d) = 2$ is even. At the remaining $(|n| - 1) + (2|mn| + |m| - a) = 2|mn| + |m| + |n| - 1 - a$ points S where $r - 2$ or d vanishes, the valuation $v_S(h) = v_S(r-2) + v_S(d) = 1$ is odd.

Let L_r and L_t denote the lines given by $r = \infty$ and $t = \infty$ respectively. Lemma 5.6 tells us that L_r and L_t intersect $D(2m, 2n)$ transversally everywhere, so $1/r$ is a uniformizer at every point in $L_r \cap D(2m, 2n)$, while $1/t$ is a uniformizer at every point in $L_t \cap D(2m, 2n)$. This shows $v_S(r^i t^j) = -i$ for every point S in $L_r \cap D(2m, 2n)$ that is not on L_t, while $v_S(r^i t^j) = -j$ for every point S in $L_t \cap D(2m, 2n)$ that is not on L_r, and $v_S(r^i t^j) = -i-j$ for the unique point in $L_r \cap L_t \cap D(2m, 2n)$, if it exists. We obtain $v_S(h) = -2|m| - 1$ and $v_S(h) = -1$ and $v_S(h) = -2|m| - 1$ for these three cases respectively. There are $|n|$ points in $L_r \cap D(2m, 2n)$ and $|m|$ points in $L_t \cap D(2m, 2n)$, while the overlap $L_r \cap L_t \cap D(2m, 2n)$ contains a point if and only if $mn > 0$. This gives a total of $|m| + |n| + 1 - a$ points S at infinity, all with $v_S(h)$ odd. Together with the affine points this makes $2|mn| + 2|m| + 2|n| - 2a$ points where h has odd valuation. Suppose $m = n$. Then $2|n|$ of these points lie on the line given by $r = t$, namely the point in $L_r \cap L_t$, and the $2|n| - 1$ points (r_0, r_0) for all roots $r_0 \neq 2$ of $T - r$. □

Lemma 6.4. *Let m, n be any nonzero integers with $m \notin \{-1, 0\}$. Consider the function $h = t - 2 + (r + 2)g_{m+1}(r)^2$ on $D(2m + 1, 2n)$. Then h has odd valuation at exactly $|2m + 1| \cdot |n| + |n| + 2|m| - 2a$ points of $D(2m + 1, 2n)$, with $a = 1$ when $n > 0$ and $a = 2$ when $m, n < 0$ and $a = 0$ when $n < 0 < m$.*

Proof. Let C denote the vanishing locus of h. The points in the standard affine part of the intersection $C \cap D(2m, 2n)$ correspond to the roots of $F(r, T)$ with $F(r, t) = f_m(r)g_{n+1}(t) - f_{m+1}(r)g_n(t)$ and $T = 2 - (r + 2)g_{m+1}(r)^2$. The degree of T equals $|2m + 1|$. From Lemma 3.3 we find that the degree of $F(r, T)$ equals $|2m + 1| \cdot |n| + |m| - a$. We now show that $F(r, T)$ is separable. Consider the extension $\mathbb{Z}[r][q]/(q^2 - rq + 1) \cong \mathbb{Z}[q, q^{-1}]$. Then we have $r = q + q^{-1}$ and $T = -q^{2m+1} - q^{-2m-1}$. This yields $F(r, T) = -q^{2mn+m+n-1}(q^{2m+1} + 1)(q^{2(2m+1)n-1} - 1)/(q^2 - 1)$, and as $\gcd(2(2m + 1), 2(2m + 1)n - 1) = 1$, we find that $F(r, T)$ has no multiple factors in $\mathbb{Z}[q, q^{-1}]$, so $F(r, T)$ is indeed separable. This shows that all $|2m + 1| \cdot |n| + |m| - a$ intersection points R in the affine part $C \cap D(2m + 1, 2n)$ are transversal intersections, and we have $v_R(h) = 1$ by Lemma 2.8.

Let L_r and L_t denote the lines given by $r = \infty$ and $t = \infty$ respectively. As in the proof of Lemma 6.3, the valuation $v_S(h)$ is odd at every point S at infinity. There are $|m| + |n| - a$ points at infinity, so we get a total of $|2m + 1| \cdot |n| + |n| + 2|m| - 2a$ points S with $v_S(h)$ odd. □

We now have enough information to compute the genus of the irreducible components of $X(k, l)$ for any k, l for which $J(k, l)$ is a hyperbolic knot. Recall that if l is an even integer with $|l| > 2$, then $X_1(l, l)$ is the scheme-theoretic complement of $X_0(l, l)$ in $X(l, l)$.

Theorem 6.5. *Suppose l is a nonzero even integer, say $l = 2n$. If $k \neq l$ is an integer satisfying $|k| \geq 2$, then $X(k, l)$ is irreducible and the genus of $X_0(k, l)$, its only irreducible component, equals*

$$3|mn| - |m| - a|n| + b,$$

with $m = \lfloor k/2 \rfloor$ and

$$a = \begin{cases} 4 & \text{if } k \text{ is odd and } k < 0, \\ 1 & \text{otherwise.} \end{cases} \qquad b = \begin{cases} 2 & \text{if } k \text{ is odd and } k < 0 < l, \\ 1 & \text{if } k \text{ is odd and } l < 0, \\ -1 & \text{if } k \text{ is even and } kl > 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $|l| > 2$, then $X(l,l)$ has two components, namely $X_0(l,l)$ of genus $|n|-1$ and $X_1(l,l)$ of genus $3n^2 - 7|n| + 5$.

Proof. By Theorem 6.1 the curve $Y(k,l)$ is geometrically irreducible for $k \neq l$, and the curves $Y_0(l,l)$ and $Y_1(l,l)$ are irreducible if $|l| > 2$. Smooth projective completions of these curves are $D(k,l)$, $D_0(l,l)$, and $D_1(l,l)$ respectively. Their genera are given in Theorem 6.2. The double cover $X(k,l)$ of $Y(k,l)$ is given by $y = x^2 - 2$. For k odd, so $k = 2m+1$, with $|k| > 2$, this is equivalent to $t - 2 = g_{m+1}(r)^2(x^2 - 2 - r)$ by Lemma 3.6, or $(g_{m+1}(r)x)^2 = h$ with h as in Lemma 6.4; the fact that $X(k,l)$ is irreducible and the value of its genus now follow from Lemmas 2.7 and 6.4. Now assume k is a nonzero even number, so $k = 2m$. Then the double cover $X(k,l)$ of $Y(k,l)$ is given by $t-2 = (2-r)f_m(r)^2(x^2-2-r)$ by Lemma 3.6, or $\big((r-2)f_k(r)x\big)^2 = h$ with h as in Lemma 6.3. If $k \neq l$, then the fact that $X(k,l)$ is irreducible and the value of its genus follow immediately from Lemmas 2.7 and 6.3. If $k = l$ and $|l| > 2$, then we apply Lemmas 2.7 and 6.3 to both irreducible components of $Y(l,l)$ to obtain the final statement. □

Proof of Theorem 1.2. This follows immediately from Theorem 6.5. □

Note that from Theorem 6.5 we can find all hyperbolic knots in the family $J(k,l)$ for which the genus of $X_0(k,l)$ equals 1. Up to switching k and l and changing sign of both k and l, these are $J(4,4)$ and $J(2,3)$ and $J(-2,2)$, the former of which is the 7_4 knot (see [29, page 391]), and the latter two of which are the figure-eight knot.

7. Commensurability classes

Recall that a compact orientable 3-manifold M is *fibered* if it is homeomorphic to a surface bundle over $\mathbb{S}^1$. One of the most intriguing open conjectures today is Thurston's virtual fibration conjecture.

Conjecture 7.1 (Thurston). *Every finite-volume hyperbolic 3-manifold has a finite cover that is fibered.*

Two manifolds are *commensurable* if they share a common finite cover. Since any finite cover of a fibered manifold is fibered, if one manifold is commensurable to a fibered manifold, then their common cover is also fibered. It follows that Thurston's conjecture is equivalent to stating that every finite-volume hyperbolic 3-manifold is commensurable to a fibered manifold.

As knot complements rarely cover each other, it is too much to hope that every knot complement has a finite cover that is a fibered knot complement. However, it is natural to ask whether any knot complement in $\mathbb{S}^3$ is commensurable to a fibered knot complement in $\mathbb{S}^3$. Reid and Walsh [23] have answered this question negatively for nonfibered hyperbolic two-bridge knot complements by showing that these are the unique knot complements in $\mathbb{S}^3$ in their commensurability class.

We address the more general question that asks whether a 3-manifold is commensurable to a fibered knot complement in any $\mathbb{Z}/2\mathbb{Z}$-homology sphere. Calegari and Dunfield [4] found sufficient conditions [4, Thm. 6.1] under which certain hyperbolic knot complements are not commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. One consequence [4, Thm. 7.1] of their work is that the nonfibered two-bridge knots $K(p,q)$ with $0 < p < 40$ have complements that are not commensurable with a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.

Hoste and Shanahan [15] extended these results to the nonfibered twist knots (the knots $J(2,n)$ with $n \neq 0, \pm 1, \pm 2$) and the knots $J(3,2n)$, for $-33 < n < 0$. Our explicit defining equations allow us to use Calegari and Dunfield's results to prove the following, conjectured by Hoste and Shanahan [15, Conj. 1].

Theorem 7.2. *Let k, l be integers for which the knot $J(k,l)$ has a nonfibered complement M in $\mathbb{S}^3$. Then $J(k,l)$ is hyperbolic and M is not commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.*

Before beginning the proof, we use the Alexander polynomial to identify the fibered $J(k,l)$ knots. Since these knots are two-bridge knots, which are alternating, they are fibered if and only if their Alexander polynomial is monic, i.e., with leading coefficient ± 1 [3, Prop. 13.26]. Without loss of generality we may assume that l is even. We can easily compute the Alexander polynomials.

Lemma 7.3. *For all nonzero integers k and $l = 2n$, the Alexander polynomial $\Delta_{k,l}(t)$ of the knot $J(k,l)$ is*

(1) $nmt^2 + (1 - 2nm)t + nm$
if $k = 2m$,
(2) $mt^{2n} + (1 + 2m)(-t^{2n-1} + \cdots - t) + m$
if $k = 2m + 1$ and $l > 0$, and
(3) $(m + 1)t^{-2n} + (1 + 2m)(-t^{-2n-1} + \cdots - t) + (m + 1)$
if $k = 2m + 1$ and $l < 0$.

It follows that $J(k,2n)$ is fibered only for the unknot $J(0,l) = J(k,0)$, the figure-eight $J(2,-2) = J(-2,2)$, the trefoil $J(2,2) = J(-2,-2)$, the knots $J(3,2n) = J(-3,-2n)$ for any $n > 0$ and $J(1,2n) = J(-1,-2n)$ for any n.

Proof of Theorem 7.2. First note that $J(k,l)$ is hyperbolic, as the only nonhyperbolic knots of the form $J(k',l')$ are the torus knots $J(\pm 1, 2n)$, the unknot and the trefoil $J(2,2) = J(-2,-2)$ (see [11, Thm. 1]).

Also note that $M = \mathbb{S}^3 \setminus J(k,l)$ is not arithmetic, as the only arithmetic knot is the fibered figure-eight knot; for the definition of arithmetic and the proof of this fact, see [24] and [18, Section 9.4]. We will show that M is in fact *generic*, which for a 1-cusped hyperbolic 3-manifold means that it is not arithmetic and its commensurator orbifold has a flexible cusp, i.e., a cusp that is not rigid. See [23, section 2.1] for an explanation of the latter condition, which for any hyperbolic complement M' of a nonarithmetic knot is equivalent with the fact that M' has no hidden symmetries (isometries of a finite cover of M' that are not the lift of an isometry of M') by [21, Prop. 9.1]. Reid and Walsh show that the complement of no hyperbolic two-bridge knot other than the figure-eight has hidden symmetries [23, Thm. 3.1]. We conclude that M is indeed generic.

A representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ is called *integral* if for all $\gamma \in \pi_1(M)$ the trace $\mathrm{tr}(\overline{\rho(\gamma)})$ of a lift $\overline{\rho(\gamma)} \in \mathrm{SL}_2(\mathbb{C})$ of $\rho(\gamma)$ is an algebraic integer. Calegari and Dunfield [4, Thm. 6.1] prove that if M' is a generic hyperbolic knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere and if $Y_0\big(\pi_1(M')\big)$ contains the character of a nonintegral reducible representation, then M' is not commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. From the above discussion, it suffices to show that the component of $C(k,l)$ (see §3) corresponding to $Y_0(k,l)$ contains the character of a nonintegral reducible representation. Without loss of generality we will assume that $l = 2n$ is even. We use the notation from §3. A representation is reducible exactly when $r = 2$. The points $(r, y) \in C(k, l)$ with $r = 2$ satisfy $F(y) = 0$ with

$$F(y) = f_n(t)\Big(\Phi_{-k}(2)\Phi_{k-1}(2)(y-2) - 1\Big) + f_{n-1}(t)$$

and

$$t = \Phi_{-k}(2)\Psi_k(2)(y-2) + 2. \tag{23}$$

By Theorem 6.1 the curve $C(k, l)$ is irreducible unless $k = l$. By Lemma 3.3 we have $\Phi_{2j}(2) = f_j(2) = j$, $\Phi_{2j+1}(2) = 1$, $\Psi_{2j}(2) = 0$, and $\Psi_{2j+1}(2) = 1$ for all integers j.

First, consider the case where k is odd, say $k = 2m + 1$. Then $k \neq l$, so $C(k, l)$ is irreducible. Here, $t = y$. Then

$$\begin{aligned} F(t) &= f_n(t)(m(t-2) - 1) + f_{n-1}(t) \\ &= mf_{n+1}(t) - kf_n(t) + (m+1)f_{n-1}(t), \end{aligned}$$

where we used $tf_n(t) = f_{n-1}(t) + f_{n+1}(t)$ in the last inequality. For any integer j, the constant terms of f_{2j} and f_{2j+1} are 0 and $(-1)^j$ respectively. Therefore, the constant term of $F(t)$ is ± 1 if n is even and $\pm k$ if n is odd. The leading term is m if l is positive and $m + 1$ if l is negative. As $k = 2m + 1$, we conclude that in all cases the leading term and constant term are relatively prime. Therefore, $F(t)$ has a nonintegral root exactly when the leading term is not ± 1. The leading term is 1 only when $m = 1$ ($k = 3$) and $l > 0$ or when $m = 0$ ($k = 1$) and $l < 0$. It is -1 only when $m = -1$ ($k = -1$) and $l > 0$ or when $m = -2$ ($k = -3$) and $l < 0$. All cases correspond to fibered knots by Lemma 7.3, so we conclude that $F(t)$ does have a nonintegral root y_0 corresponding to a nonintegral point $(2, y_0)$ on $C(k, l)$ and thus on $Y_0(k, l)$.

Now it suffices to assume k is even, say $k = 2m$. From (23) we get $t = 2$. By Lemma 3.9 there is a unique point $P = (2, 2 - 1/mn)$ on $C(k, l)$ with $r = 2$. If $k \neq l$ then $C(k, l)$ is irreducible, so P corresponds to a nonintegral point on $Y_0(k, l)$. If $k = l$, then the birational morphism to the new model $D(k, k)$ (see Proposition 4.4) sends P to $(2, 2)$, which lies on the component corresponding to $Y_0(k, k)$ by Proposition 4.6. We conclude that P is a nonintegral point on $Y_0(k, k)$ in this case as well. □

Proof of Theorem 1.3. Given that every manifold is commensurable with itself, this follows immediately from Theorem 7.2. □

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