# UNIRATIONALITY OF DEL PEZZO SURFACES OF DEGREE TWO OVER FINITE FIELDS

DINO FESTI, RONALD VAN LUIJK

Abstract. We prove that every del Pezzo surface of degree two over a finite field is unirational, building on the work of Manin and an extension by Salgado, Testa, and Várilly-Alvarado, who had proved this for all but three surfaces. Over general fields of characteristic not equal to two, we state sufficient conditions for a del Pezzo surface of degree two to be unirational.

## 1. Introduction

A *del Pezzo surface* is a smooth, projective, geometrically integral variety $X$ of which the anticanonical divisor $-K_X$ is ample. We define the *degree* of a del Pezzo surface $X$ as the self intersection number of $K_X$, that is, $\deg X = K_X^2$. If $k$ is an algebraically closed field, then every del Pezzo surface of degree $d$ over $k$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (with $d = 8$), or to $\mathbb{P}^2$ blown up in $9 - d$ points in general position.

Over arbitrary fields, the situation is more complicated and del Pezzo surfaces need not be birationally equivalent with $\mathbb{P}^2$. We therefore look at the weaker notion of unirationality. We say that a variety $X$ of dimension $n$ over a field $k$ is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over $k$. We prove the following theorem.

**Theorem 1.1.** *Every del Pezzo surface of degree 2 over a finite field is unirational.*

The analog for higher degree holds over any field. Works of B. Segre, Yu. Manin, J. Kollár, M. Pieropan, and A. Knecht prove that every del Pezzo surface of degree $d \geq 3$, defined over any field $k$, is unirational, provided that the set $X(k)$ of rational points is non-empty. For references, see [Seg43, Seg51] for $k = \mathbb{Q}$ and $d = 3$, see [Man86, Theorem 29.4 and 30.1] for $d \geq 3$ with the extra assumption for $d \in \{3, 4\}$ that $k$ has enough elements. See [Kol02, Theorem 1.1] for $d = 3$ and a general ground field. The earliest reference we could find for $d = 4$ and a general ground field is [Pie12, Proposition 5.19]. Independently, for $d = 4$, [Kne13, Theorem 2.1] covers all finite fields. Since all del Pezzo surfaces over finite fields have a rational point (see [Man86, Corollary 27.1.1]), this implies that every del Pezzo surface of degree at least 3 over a finite field is unirational.

Most of the work to prove Theorem 1.1 was already done. Building on work by Manin (see [Man86, Theorem 29.4]), C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational, except possibly for three isomorphism classes of surfaces (see [STVA14, Theorem 1]). In Section 3, we will present the three difficult surfaces and show that these are also unirational, thus proving Theorem 1.1.

Before that, in Section 2, we will recall the basics about del Pezzo surfaces of degree 2, including the fact that the linear system associated to the anti-canonical divisor induces a finite morphism to $\mathbb{P}^2$ of degree 2. We call this morphism the *anti-canonical morphism* associated to $X$. This allows us to state the second main theorem.

**Theorem 1.2.** *Suppose $k$ is a field of characteristic not equal to 2. Let $X$ be a del Pezzo surface of degree 2 over $k$, and let $\pi\colon X \to \mathbb{P}^2$ be its anti-canonical morphism. Assume that $X$ has a $k$-rational point, say $P$. Let $C \subset \mathbb{P}^2$ be a geometrically integral curve over $k$ of degree $d \geq 2$ and suppose that $\pi(P)$ is a point of multiplicity $d - 1$ on $C$. Suppose, moreover, that $C$ intersects the branch locus $B$ of the morphism $\pi$ with even multiplicity everywhere. Then the following statements hold.*

*(1) If $\pi(P)$ is not contained in $B$, then $X$ is unirational.*

(2) *If $\pi(P)$ is contained in $B$, and it is an ordinary singular point on $C$ and we have $d \in \{3,4\}$, then there exists a field extension $\ell$ of $k$ of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field $\ell$, the surface $X_\ell$ is unirational.*

The main tool for the proof of both theorems is Lemma 3.2 (that is, [STVA14, Theorem 17]), which states that, outside characteristic 2, a del Pezzo surface of degree 2 is unirational if it contains a rational curve. We prove Theorem 1.2 in Section 4 by showing that, under the hypotheses of Theorem 1.2, the pull-back of the curve $C$ to $X$ contains a rational component. Manin's original construction, and the generalisation by Salgado, Testa, and Várilly-Alvarado, produces a rational curve that corresponds to case (1) of Theorem 1.2, with $4-d$ equal to the number of exceptional curves that $P$ lies on.

For the three difficult surfaces one can use case (2) of Theorem 1.2 (see Remark 4.1). Here we benefit from the fact that if $k$ is a finite field, then any curve that becomes birationally equivalent with $\mathbb{P}^1$ over an extension of $k$, already is birationally equivalent with $\mathbb{P}^1$ over $k$ itself.

For interesting examples and more details about the proof of Theorem 1.2, Manin's construction, as wel as a generalisation of Theorem 1.2, we refer the reader to an extended version of this paper [FvL14].

The authors would like to thank Bjorn Poonen, Damiano Testa and Anthony Várilly-Alvarado for useful conversations.

## 2. Del Pezzo surfaces of degree two

The statements in this section are well known and we will use them freely. Let $X$ be a del Pezzo surface of degree 2 over a field $k$ with canonical divisor $K_X$. The Riemann-Roch spaces $\mathcal{L}(-K_X)$ and $\mathcal{L}(-2K_X)$ have dimension 3 and 7, respectively. Let $x,y,z$ be generators of $\mathcal{L}(-K_X)$ and choose an element $w \in \mathcal{L}(-2K_X)$ that is not contained in the image of the natural map $\mathrm{Sym}^2\,\mathcal{L}(-K_X) \to \mathcal{L}(-2K_X)$. Then $X$ embeds into the weighted projective space $\mathbb{P} = \mathbb{P}(1,1,1,2)$ with coordinates $x,y,z$, and $w$. We will identify $X$ with its image in $\mathbb{P}$, which is a smooth surface of degree 4. Conversely, every smooth surface of degree 4 in $\mathbb{P}$ is a del Pezzo surface of degree 2. There are homogeneous polynomials $f,g \in k[x,y,z]$ of degrees 2 and 4, respectively, such that $X \subset \mathbb{P}$ is given by

$$w^2 + fw = g. \tag{1}$$

If the characteristic of $k$ is not 2, then after completing the square on the left-hand side, we may assume $f = 0$. For more details and proofs of these facts, see [Kol96, Section III.3, Theorem III.3.5] and [Man86, Section IV.24].

The restriction to $X$ of the 2-uple embedding $\mathbb{P} \to \mathbb{P}^6$ corresponds to the complete linear system $|-2K_X|$. Every hyperplane section of $X \subset \mathbb{P}$ is linearly equivalent with $-K_X$. The projection $\mathbb{P} \dashrightarrow \mathbb{P}^2$ onto the first three coordinates restricts to a finite, separable morphism $\pi_X\colon X \to \mathbb{P}^2$ of degree 2, which corresponds to the complete linear system $|-K_X|$. This is the anti-canonical morphism mentioned in the introduction.

The morphism $\pi_X$ is ramified above the branch locus $B_X \subset \mathbb{P}^2$ given by $f^2 + 4g = 0$. If the characteristic of $k$ is not 2, then $B_X$ is a smooth curve. We denote the ramification locus $\pi^{-1}(B_X)$ of $\pi_X$ by $R_X$. As for every double cover, the morphism $\pi_X$ induces an involution $\iota_X\colon X \to X$ that sends a point $P \in X$ to the unique second point in the fiber $\pi_X^{-1}(\pi_X(P))$, or to $P$ itself if $\pi_X$ is ramified at $P$. If $X$ is clear from the context, then we sometimes leave out the subscripts and write $\pi, \iota, B$, and $R$ for $\pi_X, \iota_X, B_X$, and $R_X$, respectively.

## 3. Proof of the first main theorem

Set $k_1 = k_2 = \mathbb{F}_3$ and $k_3 = \mathbb{F}_9$. Let $\gamma \in k_3$ denote an element satisfying $\gamma^2 = \gamma + 1$. Note that $\gamma$ is not a square in $k_3$. For $i \in \{1,2,3\}$, we define the surface $X_i$ in $\mathbb{P} = \mathbb{P}(1,1,1,2)$ with coordinates $x, y, z, w$ over $k_i$ by

$$\begin{aligned} X_1 : \ -w^2 &= (x^2+y^2)^2 + y^3z - yz^3, \\ X_2 : \ -w^2 &= x^4 + y^3z - yz^3, \\ X_3 : \ \gamma w^2 &= x^4 + y^4 + z^4. \end{aligned}$$

These surfaces are smooth, so they are del Pezzo surfaces of degree 2. C. Salgado, D. Testa, and A. Várilly-Alvarado proved the following result.

**Theorem 3.1.** *Let $X$ be a del Pezzo surface of degree 2 over a finite field. If $X$ is not isomorphic to $X_1, X_2$, and $X_3$, then $X$ is unirational.*

*Proof.* See [STVA14, Theorem 1]. □

We will use the following lemma to prove the complementary statement, namely that $X_1, X_2$, and $X_3$ are unirational as well.

**Lemma 3.2.** *Let $X$ be a del Pezzo surface of degree 2 over a field $k$. Suppose that $\rho\colon \mathbb{P}^1 \to X$ is a nonconstant morphism; if the characteristic of $k$ is 2 and the image of $\rho$ is contained in the ramification divisor $R_X$, then assume also that the field $k$ is perfect. Then $X$ is unirational.*

*Proof.* See [STVA14, Theorem 17]. □

For $i \in \{1, 2, 3\}$, we define a morphism $\rho_i\colon \mathbb{P}^1 \to X_i$ by extending the map $\mathbb{A}^1(t) \to X_i$ given by

$$t \mapsto (x_i(t) : y_i(t) : z_i(t) : w_i(t)),$$

where

$$\begin{aligned} x_1(t) &= t^2(t^2-1), & x_2(t) &= t(t^2+1)(t^4-1), & x_3(t) &= (t^4+1)(t^2-\gamma^3), \\ y_1(t) &= t^2(t^2-1)^2, & y_2(t) &= -t^4, & y_3(t) &= (t^4-1)(t^2+\gamma^3), \\ z_1(t) &= t^8-t^2+1, & z_2(t) &= t^8+1, & z_3(t) &= (t^4+\gamma^2)(t^2-\gamma), \\ w_1(t) &= t(t^2-1)(t^4+1)(t^8+1), & w_2(t) &= t^2(t^2+1)(t^{10}-1), & w_3(t) &= \gamma^2 t(t^8-1)(t^2+\gamma). \end{aligned}$$

It is easy to check for each $i$ that the morphism $\rho_i$ is well defined, that is, the polynomials $x_i, y_i, z_i$, and $w_i$ satisfy the equation of $X_i$, and that $\rho_i$ is non-constant.

**Theorem 3.3.** *The del Pezzo surfaces $X_1, X_2$, and $X_3$ are unirational.*

*Proof.* By Lemma 3.2, the existence of $\rho_1, \rho_2$, and $\rho_3$ implies that $X_1, X_2$, and $X_3$ are unirational. □

*Proof of Theorem 1.1.* This follows from Theorems 3.1 and 3.3. □

## 4. Proof of the second main theorem

If $C$ is a plane curve with an ordinary singularity $Q$ and $\tilde{C}$ is the normalisation of $C$, then we can think of the points of $\tilde{C}$ above $Q$ as corresponding with the branches of $C$ through $Q$. The intersection multiplicity of $C$ with another plane curve $B$ at $Q$ is then the sum of the intersection multiplicities of $B$ with all the branches of $C$ through $Q$. This point of view is used in the following proof. For more technical details about this approach, see the extended version of this paper [FvL14].

*Proof of Theorem 1.2.* Let $\iota\colon X \to X$ denote the involution associated to the double cover $\pi$. Set $Q = \pi(P)$. Projection away from the point $Q \in C \subset \mathbb{P}^2$ yields a birational map $C \dashrightarrow \mathbb{P}^1$ whose inverse $\vartheta\colon \mathbb{P}^1 \to C$ can be identified with the normalisation map of $C$. The map $\vartheta$ restricts to an isomorphism $\mathbb{P}^1 \setminus \vartheta^{-1}(Q) \to C \setminus \{Q\}$, and $C$ is smooth away from $Q$. Let $D = \pi^{-1}(C)$ be the inverse image of $C$ under $\pi$, and let $\tilde{D}$ be its normalisation. Then $\pi$ induces a double cover $\tilde{\pi}\colon \tilde{D} \to \mathbb{P}^1$.

Let $S \in \mathbb{P}^1$ be a point and set $T = \vartheta(S) \in C$. The curve $B$ is given locally around $T$ by the vanishing of a rational function on $\mathbb{P}^2$ that is regular at $T$. We let $h$ denote the image of such a function in the local ring $\mathcal{O}_{C,T}$ of $T$ in $C$.

If $T \neq Q$, then $T$ is a smooth point of $C$, so the ring $\mathcal{O}_{C,T} = \mathcal{O}_{\mathbb{P}^1,S}$ is a discrete valuation ring. In this case, the valuation of $h$ equals the intersection multiplicity of $B$ and $C$ at $T$, which is even. Since the characteristic of $k$ is not 2, this implies that adjoining a square root of $h$ to $\mathcal{O}_{C,T}$ yields an unramified extension, so the morphism $\tilde{\pi} \colon \tilde{D} \to \mathbb{P}^1$ is not ramified above $S$ when $T \neq Q$.

Suppose that $Q$ is not contained in $B$. Then for $T = Q$, the element $h$ is a unit in the local ring $\mathcal{O}_{C,T}$, and therefore also in the ring extension $\mathcal{O}_{\mathbb{P}^1,S}$. Hence, as before, since the characteristic of $k$ is not 2, this implies that the morphism $\tilde{\pi}$ is not ramified above $S$. This means that $\tilde{\pi}$ is unramified. Since $\mathbb{P}^1_{\overline{k}}$ has no nontrivial unramified covers, this means that the curve $\tilde{D}$, and hence the curve $D \subset X$, splits into two components over some quadratic extension $\ell$ of $k$. Exactly one of the components of $D$ contains the rational point $P$ and the other component contains $\iota(P)$. This implies that the Galois group $\mathrm{Gal}(\ell/k)$ sends each component to itself, so these components are defined over $k$. Each maps isomorphically to $C$, so $X$ contains a curve that is birationally equivalent to $\mathbb{P}^1$ and therefore $X$ is unirational by Lemma 3.2. This proves (1).

Suppose that $Q$ is contained in $B$ and that it is an ordinary singular point on $C$. Then $\vartheta^{-1}(Q)$ consists of exactly $d-1$ points over $\overline{k}$, each corresponding to the tangent direction of one of the $d-1$ branches of $C$ at $Q$. At most one of tangent directions is tangent to $B$, so at least $d-2$ of the branches intersect $B$ with multiplicity 1. The total intersection multiplicity of $B$ and $C$ at $Q$ is even. If $d$ is odd, then the contribution $(d-2)\cdot 1$ of the $d-2$ branches with intersection multiplicity 1 is odd, so the last branch intersects $B$ with odd multiplicity as well; hence all $d-1$ branches intersect $B$ with odd multiplicity, which implies that $\tilde{\pi} \colon \tilde{D} \to \mathbb{P}^1$ is ramified above all $d-1$ points above $Q$. If $d$ is even, then the contribution of the $d-2$ branches of $C$ that intersect $B$ with multiplicity 1 is even as well, so the last branch intersects $B$ with even multiplicity; as before, this means that $\tilde{\pi}$ is not ramified above the point in $\vartheta^{-1}(Q) \subset \mathbb{P}^1$ that corresponds to this last branch, so $\tilde{\pi}$ is ramified above exactly $d-2$ of the $d-1$ points above $Q$. For $d \in \{3,4\}$, these two cases ($d$ odd or even) imply that the map $\tilde{\pi} \colon \tilde{D} \to \mathbb{P}^1$ is ramified at exactly two points, so $\tilde{D}$ is a geometrically integral curve of genus 0 by the theorem of Riemann-Hurwitz. Indeed, this implies that there is a field extension $\ell$ of $k$ of degree at most 2 for which $\tilde{D}_\ell$, and thus $D_\ell = \pi^{-1}(C_\ell)$, is birationally equivalent with $\mathbb{P}^1_\ell$. For each such field, the surface $X_\ell$ is unirational by Lemma 3.2. This proves (2). □

*Remark* 4.1. Let the surfaces $X_1, X_2, X_3$ and the morphisms $\rho_1, \rho_2, \rho_3$ be as in the previous section. Take any $i \in \{1,2,3\}$. Set $A_i = \rho_i(\mathbb{P}^1)$ and $C_i = \pi_i(A_i)$, where $\pi_i = \pi_{X_i} \colon X_i \to \mathbb{P}^2$ is as described in the previous section. By Remark 2 of [STVA14], the surface $X_i$ is minimal, and the Picard group $\mathrm{Pic}\, X_i$ is generated by the class of the anticanonical divisor $-K_{X_i}$. The same remark states that the linear system $|-nK_{X_i}|$ does not contain a geometrically integral curve of geometric genus zero for $n \leq 3$ if $i \in \{1,2\}$, nor for $n \leq 2$ if $i = 3$. For $i \in \{1,2\}$, the curve $A_i$ has degree 8, so it is contained in the linear system $|-4K_{X_i}|$. The curve $A_3$ has degree 6, so it is contained in the linear system $|-3K_{X_i}|$. This means that the curve $A_i$ has minimal degree among all rational curves on $X_i$. The restriction of $\pi_i$ to $A_i$ is a double cover $A_i \to C_i$. The curve $C_i \subset \mathbb{P}^2$ has degree 4 for $i \in \{1,2\}$ and degree 3 for $i = 3$, and $C_i$ is given by the vanishing of $h_i$, with

$$\begin{aligned}
h_1 &= x^4 + xy^3 + y^4 - x^2yz - xy^2z, \\
h_2 &= x^4 - x^2y^2 - y^4 + x^2yz + yz^3, \\
h_3 &= x^2y + xy^2 + x^2z - xyz + y^2z - xz^2 - yz^2 - z^3.
\end{aligned}$$

For $i \in \{1,2\}$, the curve $C_i$ has an ordinary triple point $Q_i$, with $Q_1 = (0:0:1)$, $Q_2 = (0:1:1)$. The curve $C_3$ has an ordinary double point at $Q_3 = (1:1:1)$. For all $i$, the point $Q_i$ lies on the branch locus $B_i = B_{X_i}$.

Using the polynomial $h_i$, one can check that the curve $C_i$ intersects the branch locus $B_i$ with even multiplicity everywhere. In fact, had we *defined* $C_i$ by the vanishing of $h_i$, then one would easily check that $C_i$ satisfies the conditions of part (2) of Theorem 1.2. This gives an alternative proof of unirationality of $X_i$ without the need of the explicit morphism $\rho_i$; here we may use the fact that if $k$ is a finite field, then any curve that becomes birationally equivalent to $\mathbb{P}^1$ over an extension of $k$, already is birationally equivalent with $\mathbb{P}^1$ over $k$. Indeed, in practice we first found the curves $C_1$, $C_2$, and $C_3$, and then constructed the parametrisations $\rho_1, \rho_2, \rho_3$, which allow for the more direct proof that we gave of Theorem 3.3 in the previous section.

## References


[FvL14] Dino Festi and Ronald van Luijk. Unirationality of del Pezzo surfaces of degree two over finite fields (extended version). *preprint,* available at `arXiv:1408.0269`, 2014.

[Kne13] A. Knecht. Degree of unirationality for del Pezzo surfaces over finite fields. *preprint,* available at `http://www67.homepage.villanova.edu/amanda.knecht/UnirationalDegree.pdf`, 2013.

[Kol96] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[Kol02] J. Kollár. Unirationality of cubic hypersurfaces. *J. Inst. Math. Jussieu*, 1(3):467–476, 2002.

[Man86] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[Pie12] M. Pieropan. On the unirationality of del Pezzo surface over an arbitrary field. *Master thesis,* available at `http://www.algant.eu/documents/theses/pieropan.pdf`, 2012.

[Seg43] B. Segre. A note on arithmetical properties of cubic surfaces. *Journal of the London Mathematical Society*, 1(1):24–31, 1943.

[Seg51] B. Segre. On the rational solutions of homogeneous cubic equations in four variables. *Math. Notae*, 11:1–68, 1951.

[STVA14] C. Salgado, D. Testa, and A. Várilly-Alvarado. On the unirationality of del Pezzo surfaces of degree two. *Journal of the London Mathematical Society*, 90:121–139, 2014.