Finiteness theorems for K3 surfaces over arbitrary fields

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Definition. A K3 surface is a nice (smooth, projective, geometrically integral) surface X over a field k with $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Examples:

- smooth quartic surface in $\mathbb{P}^3$;
- smooth double cover of $\mathbb{P}^2$ ramified along a sextic curve;
- smooth complete intersection of quadric and cubic in $\mathbb{P}^4$;
- Kummer surface: desingularization of abelian surface modulo -1.

Picard lattice with intersection pairing.

- $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathrm{Num}(X)$.
- Rank $\rho(X)$ at most 22.
- Hodge Index Theorem: signature $(1, \rho - 1)$.
- For divisor D, get $\chi(\mathcal{O}_X(D)) = \frac{1}{2}D^2 + 2$, so $\mathrm{Pic}(X)$ is even.
- For divisor class D with $D^2 \geqslant -2$, get $D \gg 0$ or $-D \gg 0$.
- For a nice curve $C \subset X$, get $2p_a(C) - 2 = C^2$, so $C^2 \geqslant -2$.

Cones

Let X be a projective K3 surface over an algebraically closed field k.

- Inside $(\operatorname{Pic} X)_{\mathbb{R}}$ we have the cone

$$\{\alpha \in (\operatorname{Pic} X)_{\mathbb{R}} \mid \alpha^2 > 0\}$$

  consisting of two components; the one that contains all the ample divisor classes is the positive cone, denoted $\mathcal{C}_X$.

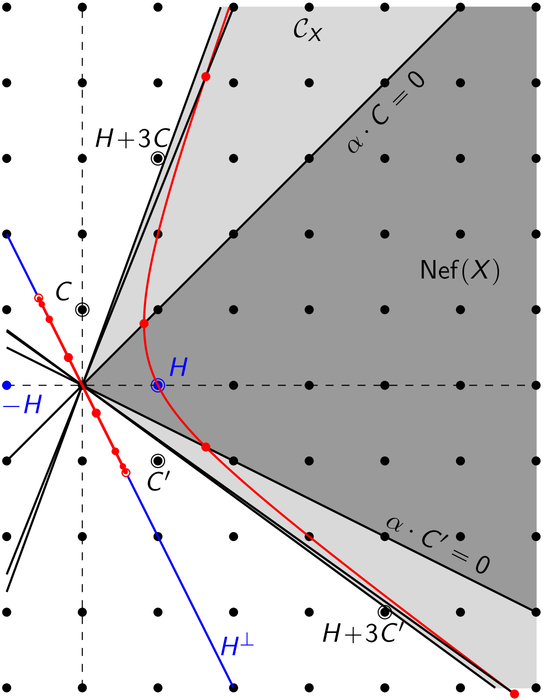
- There is also the nef cone

$$\operatorname{Nef}(X) = \{\alpha \in (\operatorname{Pic} X)_{\mathbb{R}} \mid \alpha \cdot C \geqslant 0 \text{ for all curves } C \subset X\}.$$

- The ample cone $\operatorname{Amp}(X) \subset \mathcal{C}_X$ is $\mathbb{R}$-generated by all ample classes. Nakai–Moishezon–Kleiman:

$$\operatorname{Nef}(X)^\circ = \operatorname{Amp}(X) \subset \overline{\operatorname{Amp}(X)} = \operatorname{Nef}(X).$$

- $\operatorname{Amp}(X) = \mathcal{C}_X \cap \{\, \alpha \in (\operatorname{Pic} X)_{\mathbb{R}} \mid \alpha \cdot C > 0 \text{ for all (-2)-curves } C \subset X\,\}.$



Example.

Conic $C \subset X \subset \mathbb{P}^3_{\mathbb{C}}$ quartic

Hyperplane section H.

Basis (H, C) for Pic(X).

$$\mathrm{Pic}(X)_{\mathbb{R}} \cong \left(\mathbb{R}^2, \begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}\right)$$

$$(aH + bC)^2 = 4a^2 + 4ab - 2b^2$$

(-2)-**class** $D = aH + bC$

$D^2 = -2 \Leftrightarrow$

$$(b - a) + a\sqrt{3} = \pm(2 + \sqrt{3})^n$$

D effective iff $H \cdot D > 0$.

Residual conic $C' = H - C \gg 0$.

Every $\phi \in \mathrm{Aut}(X)$ fixes:

- $\mathcal{C}_X$ and Nef(X)
- class of H (so ϕ is linear)
- plane containing C and C'
- set $C \cap C'$

Aut(X) is finite!

Automorphisms and cones in general

Study $\operatorname{Aut} X$ by relating it to the group $O(\operatorname{Pic} X)$ of isometries of $\operatorname{Pic} X$.

- Any $\phi \in \operatorname{Aut} X$ induces an isometry of $\operatorname{Pic} X$ fixing $\mathcal{C}_X$ and $\operatorname{Nef}(X)$.
- More isometries of $\operatorname{Pic} X$: reflection s_δ in $\delta^\perp$ for any (-2)-class δ.

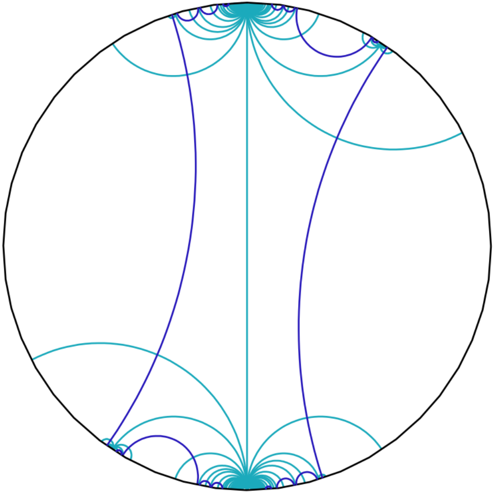
$$s_\delta(x) = x + (x \cdot \delta)\delta, \text{ where } \delta^2 = -2.$$

  These fix $\mathcal{C}_X$, but never fix $\operatorname{Nef}(X)$, since s_δ interchanges the two half-spaces $\{(\alpha \cdot \delta) > 0\}$ and $\{(\alpha \cdot \delta) < 0\}$.
- Weyl group $W(\operatorname{Pic} X) := \langle s_\delta \rangle_\delta$. From reflection groups theory:

  > $\operatorname{Nef}(X) \cap \mathcal{C}_X$ is a locally polyhedral fundamental domain for the action of $W(\operatorname{Pic} X)$ on $\mathcal{C}_X$.

  (Where $\operatorname{Nef}(X)$ meets the boundary of $\mathcal{C}_X$, it need not be locally polyhedral.)

Example of hyperbolic picture for rank 3 [by A. Baragar]



Dark blue: $Nef(X)$. Light blue: tiling by $O(Pic(X))$

Theorem (Pjateckiĭ-Šapiro–Šafarevič, Sterk, Lieblich–Maulik)

Let X be a K3 surfaces over $k = \overline{k}$ with $\operatorname{char} k \neq 2$.

1. $\operatorname{Nef}(X) \cap \mathcal{C}_X$ *is a locally polyhedral fundamental domain for the action of* $W(\operatorname{Pic} X)$ *on* $\mathcal{C}_X$.
2. *The homomorphism*

$$\operatorname{Aut} X \to \operatorname{O}(\operatorname{Pic} X)/W(\operatorname{Pic} X)$$

   has finite kernel and cokernel.
3. *The group* $\operatorname{Aut} X$ *is finitely generated.*
4. *The action of* $\operatorname{Aut} X$ *on real convex hull* $\operatorname{Nef}^e(X)$ *of* $\operatorname{Nef}(X) \cap \operatorname{Pic}(X)$ *admits a rational polyhedral fundamental domain.*
5. *For any d, there are only finitely many orbits under* $\operatorname{Aut} X$ *of classes of irreducible curves of self-intersection $2d$.*

All five have analogues over non-algebraically closed fields, but first two need adjustment. The last three then follow just as before.

Let X be a K3 surface over any field k and set $\bar{X} = X \times_k \bar{k}$.

- The positive, ample and nef cones of X are just the intersections with $(\operatorname{Pic} X)_{\mathbb{R}} \subset (\operatorname{Pic} \bar{X})_{\mathbb{R}}$ of the positive, ample and nef cones of $\bar{X}$.
- Suppose that X contains a pair of disjoint, conjugate (-2)-curves C_1, C_2. The class $[C_1 + C_2] \in \operatorname{Pic} X$ defines a wall of the ample cone that may not correspond to a (-2)-class defined over k.
- So we will need to replace the group $W(\operatorname{Pic} X)$.
- For $g \in \mathrm{O}(\operatorname{Pic} \bar{X})$ and $\delta \in \operatorname{Pic} \bar{X}$ with $\delta^2 = -2$, get

$$g s_\delta g^{-1} = s_{g\delta},$$

  so

  - $W(\operatorname{Pic} \bar{X})$ is normal in $\mathrm{O}(\operatorname{Pic} \bar{X})$,
  - $\operatorname{Aut}(\bar{k}/k)$ acts through $\mathrm{O}(\operatorname{Pic} \bar{X})$ on $W(\operatorname{Pic} \bar{X})$ by conjugation,
  - may define $R_X = W(\operatorname{Pic} \bar{X})^{\operatorname{Aut}(\bar{k}/k)}$,
  - R_X is normal in $\mathrm{O}(\operatorname{Pic} \bar{X})^{\operatorname{Aut}(\bar{k}/k)}$,
  - $\operatorname{Aut} X$ acts through $\mathrm{O}(\operatorname{Pic} \bar{X})^{\operatorname{Aut}(\bar{k}/k)}$ on R_X by conjugation.

Theorem (Bright, Logan, vL, 2018)

Let X be a K3 surface over any field of characteristic $\neq 2$.

1. *Then* $\mathrm{Nef}(X) \cap \mathcal{C}_X$ *is a fundamental domain for action of* R_X *on* $\mathcal{C}_X$.
2. ~~*The map* $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Pic}\, X)/R_X$ *has finite kernel and cokernel.*~~ *There is a natural map* $\mathrm{Aut}(X) \ltimes R_X \to \mathrm{O}(\mathrm{Pic}\, X)$ *with finite kernel and image of finite index.*

Proof. First replace $\bar{k}$ by k^s. Let $k \subset \bar{k}$ be separably closed.

- The Picard scheme $\mathbf{Pic}_{X/k}$ exists, with $\mathbf{Pic}_{X/k}(k) = \mathrm{Pic}\, X$ and $\mathbf{Pic}_{X/k}(\bar{k}) = \mathrm{Pic}\, \bar{X}$.
- $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ implies that $\mathbf{Pic}_{X/k}$ is étale over k, and therefore $\mathrm{Pic}\, X \to \mathrm{Pic}\, \bar{X}$ is an isomorphism.
- This also shows that all (-2)-curves on $\bar{X}$ are defined over k.
- Similarly, $\mathrm{H}^0(X, T_X) = 0$ shows that the automorphism scheme $\mathbf{Aut}_{X/k}$ is étale over k, and so $\mathrm{Aut}\, X \to \mathrm{Aut}\, \bar{X}$ is an isomorphism.

Acting on $\operatorname{Pic} X$

Set $\Gamma_k = \operatorname{Gal}(k^s/k)$.

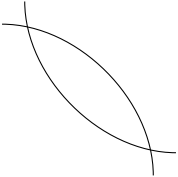
- We have $\operatorname{Pic} X \subset (\operatorname{Pic} X^s)^{\Gamma_k}$, but they need not be equal. (They are equal if X has a k-rational point).
- The quotient maps into $\operatorname{Br} k$, so is finite.
- It is clear that the action of $R_X = W(\operatorname{Pic} X^s)^{\Gamma_k}$ on $\operatorname{Pic} X^s$ preserves $(\operatorname{Pic} X^s)^{\Gamma_k}$, but not immediately obvious that it preserves $\operatorname{Pic} X$.
- Fortunately, we can see this from an explicit description of R_X.

Description of R_X

Theorem (Hée; Lusztig; Geck–Iancu)

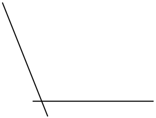
Let (W, T) be a Coxeter system. Let G be a group of permutations of T that induce automorphisms of W. Let F be the set of G-orbits $I \subset T$ for which W_I is finite, and for $I \in F$ let ℓ_I be the longest element of (W_I, I). Then $(W^G, \{\ell_I : I \in F\})$ is a Coxeter system.

In our situation, we have $W = W(\operatorname{Pic} X^s)$ and $T = \{s_\delta : \delta \text{ a } (-2)\text{-curve}\}$.

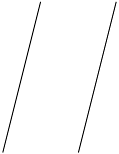


If two (-2)-curves have intersection number $\geqslant 2$, then the corresponding reflections generate an infinite dihedral group in W. So a Galois orbit containing two such curves will not lie in F, and will not contribute a generator to R_X.

Description of R_X

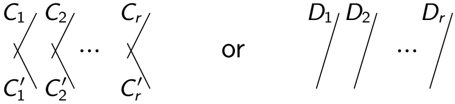


If an orbit consists of two (-2)-curves C, C' intersecting with multiplicity 1, then $s_C, s_{C'}$ generate a subgroup of W isomorphic to $S_3 = W(A_2)$; the longest element is the reflection $s_C s_{C'} s_C = s_{C'} s_C s_{C'}$: the Galois-invariant reflection $s_{[C]+[C']}$ associated to the (-2)-class $[C] + [C']$.

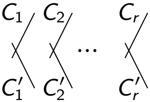


If an orbit consists of two disjoint (-2)-curves C, C', then $s_C, s_{C'}$ commute and generate a subgroup of W isomorphic to $C_2 \times C_2 = W(A_1 \times A_1)$; the Galois-invariant subgroup is generated by $s_C s_{C'}$, which is the reflection defined by the (-4)-class $[C] + [C']$.

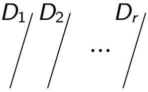
In general, the only orbits contributing to R_X are disjoint unions of these.



Fundamental domain for R_X



or



with

$$\ell_I = s_{C_1+C'_1} \circ \cdots \circ s_{C_r+C'_r} \colon \quad x \mapsto x + 2(x \cdot C_1)(C_1 + C'_1 + \cdots + C_r + C'_r),$$

$$\ell_I = s_{D_1} \circ \cdots \circ s_{D_r} \colon \quad x \mapsto x + (x \cdot D_1)(D_1 + \cdots + D_r)$$

So R_X does preserve (and therefore act on) $\operatorname{Pic} X \subset \operatorname{Pic} X^s$.
Proof that $\operatorname{Nef} X \cap \mathcal{C}_X$ is a fundamental domain for R_X acting on $\mathcal{C}_X$:

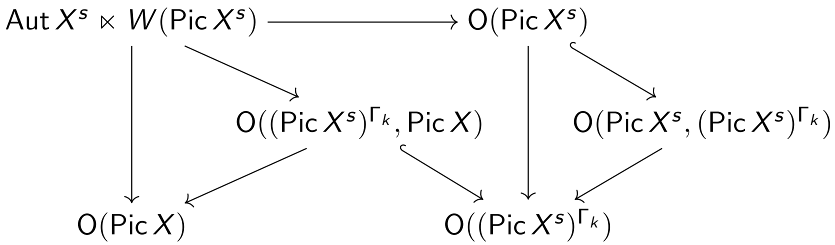
1. Case $\alpha \in \mathcal{C}_X$ has trivial stabiliser in $W(\operatorname{Pic} X^s)$.
   Then there is a unique $g \in W(\operatorname{Pic} X^s)$ with $g(\alpha) \in \operatorname{Nef} X^s$. Any $\sigma \in \Gamma_k$ preserves $\operatorname{Nef} X^s$, so $\sigma(g(\alpha)) = (\sigma g)(\sigma\alpha) = (\sigma g)(\alpha)$ also lies in $\operatorname{Nef} X^s$. By uniqueness, $\sigma g = g$ for all $\sigma \in \Gamma_k$, so g lies in R_X.
2. Case $\alpha \in \mathcal{C}_X$ has non-trivial stabiliser (i.e. lies on a wall).
   Write it as the limit of elements with trivial stabiliser.
3. To show that two translates of $(\operatorname{Nef} X \cap \mathcal{C}_X)$ intersect only in their boundaries, use $\partial(\operatorname{Nef} X) = \partial(\operatorname{Nef} X^s) \cap (\operatorname{Pic} X)_{\mathbb{R}}$.

Descending finite kernel and image of finite index

Proposition (Bright, Logan, vL, 2018)

Let Λ be a lattice and $H \subset O(\Lambda)$ a subgroup such that $M = \Lambda^H$ is non-degenerate. Set $O(\Lambda, M) = \{g \in O(\Lambda) : g(M) = M\}$. Then:

1. *the natural map $O(\Lambda, M) \to O(M)$ has finite cokernel;*
2. *if $M^\perp$ is definite, then $O(\Lambda, M) \to O(M)$ has finite kernel, and the centraliser $Z_{O(\Lambda)} H$ has finite index in $O(\Lambda, M)$.*



Example I

- Over the complex numbers, whether $\operatorname{Aut} X$ is finite can be read off from $\operatorname{Pic} X$. This is not true over arbitrary fields.
- Let M, N be the block diagonal matrices

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix}, \qquad N = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & -2I_4 \end{array}\right).$$

- Over $\mathbb{C}$, a K3 surface having intersection matrix M would have infinite automorphism group, whereas a K3 surface having intersection matrix N would have finite automorphism group.
- Using elliptic surfaces, we construct a K3 surface X over $\mathbb{Q}$ such that $\operatorname{Pic} X$ has intersection matrix M, but $\operatorname{Pic} \bar{X}$ has intersection matrix N. So $\operatorname{Aut} \bar{X}$, and a fortiori $\operatorname{Aut} X$, is finite.

Example II

Example I was sort of cheating – with a finite automorphism group over $\mathbb{C}$.

- Let
$$M = \begin{pmatrix} 6 & 4 \\ 4 & -4 \end{pmatrix}, \quad N = \begin{pmatrix} 6 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$
- Any K3 surface over $\mathbb{C}$ with intersection lattice either M or N has infinite automorphism group.
- We constructed a K3 surface X over $\mathbb{Q}$, having intersection matrix M over $\mathbb{Q}$ and intersection matrix N over $\bar{\mathbb{Q}}$, such that $\operatorname{Aut} X$ is finite.
- We took X to be the intersection of a quadric and a cubic in $\mathbb{P}^4$, containing a pair of disjoint Galois-conjugate conics and having geometric Picard number 3.

Example II

$$M = \begin{pmatrix} 6 & 4 \\ 4 & -4 \end{pmatrix}, \quad N = \begin{pmatrix} 6 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$

- $\operatorname{Pic} X$ has intersection matrix M, and $\mathrm{O}(\operatorname{Pic} X)$ is easy to compute - it is related to the unit group of the field $\mathbb{Q}(\sqrt{10})$. In particular, it contains a copy of $\mathbb{Z}$ with finite index.
- A K3 surface over $\mathbb{C}$ having this Picard lattice would contain no (-2)-curves, so would have infinite automorphism group.
- However, X does contain a Galois-conjugate disjoint pair (C, C') of (-2)-curves, and in fact contains many.
- With H a hyperplane section, $6[H] - 3[C] - 4[C']$ is the class of another (-2)-curve D, disjoint from its conjugate D'.
- The two reflections in the (-4)-classes $[C] + [C']$ and $[D] + [D']$ generate an infinite dihedral subgroup of R_X, showing that R_X has finite index in $\mathrm{O}(\operatorname{Pic} X)$, and so $\operatorname{Aut} X$ is finite.

Example III

An example of actual arithmetic interest.

Theorem (Bright, Logan, vL, 2018)

Let k be a field of characteristic zero, let $c \in k^\times$ be such that $[k(\zeta_8, \sqrt[4]{c}) : k] = 16$, and let $X \subset \mathbb{P}^3_k$ be the surface

$$x^4 - y^4 = c(z^4 - w^4).$$

Then $\rho(X) = 6$ and $\rho(\bar{X}) = 20$ and $\operatorname{Aut} X$ is finite.

Proof is computational, but not straightforward!