Density of rational points on a family of Del Pezzo surfaces of degree one

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Surface: smooth, projective, geometrically integral scheme over a field, of dimension 2.

Surface X is Del Pezzo if anticanonical divisor $-K_X$ is ample.

Degree of Del Pezzo surface X is intersection number $d = K_X \cdot K_X$.

Examples:

- Intersection of two quadrics in $\mathbb{P}^4$ ($-K_S$ very ample; $d = 4$).
- A smooth cubic surface in $\mathbb{P}^3$ ($-K_S$ very ample; $d = 3$).
- Smooth double cover of $\mathbb{P}^2$, ramified over a quartic ($d = 2$). (Anticanonical map is the projection to $\mathbb{P}^2$).

Fact. A Del Pezzo surface over a separably closed field is birationally equivalent with $\mathbb{P}^2$.

Theorem (Segre, Manin, Kollár, Pieropan, Salgado–Testa–Várilly-Alvarado, Festi-vL).
Let S be a Del Pezzo surface of degree $d \geq 2$ over a field k. Suppose $P \in S(k)$ is a rational point. If $d = 2$ and k is infinite, then suppose, furthermore, that P does not lie on four exceptional curves, nor on the ramification locus of the anticanonical map. Then S is unirational over k.

Theorem (Kollár, Mella)
Let S be a Del Pezzo surface of degree $d = 1$ over a field k of characteristic not equal to 2. If S admits a conic bundle structure, then S is unirational.

Remark. When these are minimal, Picard number $\rho(S) = 2$.

Question 1. Is there a DP1 with $\rho = 1$ that is unirational?

Question 2. Is there a DP1 with $\rho = 1$ that is **not** unirational?

Philosophy. Geometry governs arithmetic.
Easier geometry (over algebraic closure) $\Rightarrow$ more rational points.

Conjecture. (follows from a conjecture by Colliot-Thélène).
If S is DP1 over a number field k, then $S(k)$ is Zariski dense in S.

Ultimate goal. Prove this.

Immediate goal. Prove this under some assumptions.

- Special families
- Existence of a special rational point

Every Del Pezzo surface S/k of degree $d = 1$ is isomorphic to a smooth sextic in $\mathbb{P}(2, 3, 1, 1)$, with coordinates x, y, z, w, given by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in k[z, w]$ homogeneous of degree i. (And vice versa.)

Linear system $|-K_S|$ induces rational map $S \dashrightarrow \mathbb{P}^1(z, w)$.

Unique base point $O = [1 : 1 : 0 : 0]$.

Curves in $|-K_S|$ are fibers of $\mathrm{Bl}_{\mathcal{O}}(S) \to \mathbb{P}^1$ (anticanonical fiber). Almost all are elliptic fibers, all are geometrically integral.

Theorem (Várilly-Alvarado). Let A, B be nonzero integers, and let S be the Del Pezzo surface of degree 1 over $\mathbb{Q}$ given by

$$y^2 = x^3 + Az^6 + Bw^6.$$

Assume that Tate–Shafarevich groups of elliptic curves over $\mathbb{Q}$ with j-invariant 0 are finite. If $3AB$ is not a square, or if A and B are relatively prime and $9 \nmid AB$, then $S(\mathbb{Q})$ is Zariski dense in S.

Question. What about $y^2 = x^3 + 243z^6 + 16w^6$?

Theorem (Salgado–vL). Let $S \subset \mathbb{P}(2,3,1,1)$ be a Del Pezzo surface of degree 1 over a number field k. Suppose F is a smooth anticanonical fiber with a point $P \in F(k)$ of order > 2 that does not lie on six exceptional curves. Set $U = \mathbb{P}(2,3,1,1) \setminus Z(z,w)$ and let $\mathcal{C}$ denote the curve of those sections of the projection $U \to \mathbb{P}^1(z,w)$ that meet S at the point P with multiplicity at least 5. If $\#\mathcal{C}(k) = \infty$, then $S(k)$ is Zariski dense.

Remark. If the order of P is 3, then $\#\mathcal{C}(k) = \infty$ for free. For Várilly-Alvarado's example

$$y^2 = x^3 + 243z^6 + 16w^6$$

there is a 3-torsion point $[0:4:0:1]$, but it lies on nine exceptional curves...

The coefficients of the curve $\mathcal{C}$ associated to the point $[-63:14:1:5]$ are too large to find points...

Theorem (Bulthuis-vL). Let S be a DP1 over a number field k. Suppose F is a smooth anticanonical fiber with a point $P \in F(k)$ of finite order $n > 1$. Then the linear system

$$|-nK_S - nP| = \{ D \in |-nK_S| \ : \ \mu_P(D) \geq n \}$$

induces a fibration $\varphi\colon \mathrm{Bl}_P(S) \to \mathbb{P}^1$ of curves of genus 1.

Moreover, the set $S(k)$ is Zariski dense if and only if there is a $Q \in S(k) \setminus \{P\}$ such that the fiber $G_Q = \varphi^{-1}(\varphi(Q))$ is smooth, and on the elliptic curve (G_Q, Q), the sum of the points above P has infinite order.

Theorem (Desjardins–Winter).
Let k be a field of characteristic 0 with $a, b, c \in k$ and $ac \neq 0$.
Let $S \subset \mathbb{P}(2,3,1,1) \supset \mathbb{A}^3(x, y, t)$ be given by

$$y^2 = x^3 + at^6 + bt^3 + c.$$

If S contains a k-rational point (x_0, y_0, t_0) with $t_0 \neq 0, \infty$ that is non-torsion on its fiber, then $S(k)$ is Zariski dense in S.

If rational points are dense on some fiber F, then also on S.

Theorem (Desjardins–Winter + Colliot-Thélène).
If k is of finite type over $\mathbb{Q}$, then the converse holds as well.

Note: $at^6 + bt^3 + c = a_6(f(t))$ for $a_6 = au^2 + bu + c$ and $f = t^3$.

Theorem (Nijgh, vL)
Let k be a field, and $a_1, a_2, a_3, a_4, a_6 \in k[u]$ and $f = \sum f_i t^i \in k[t]$ polynomials with $3 \deg(a_i) \leq i$ and $\deg(f) = 3$. Let $S \subset \mathbb{A}^3(x, y, t)$ be the surface given by

$$y^2 + a_1(f(t))xy + a_3(f(t))y = x^3 + a_2(f(t))x^2 + a_4(f(t))x + a_6(f(t)).$$

Suppose that S is smooth, $\deg(4a_6 + a_3^2) = 2$, and $\operatorname{char} k = 0$.

If there is a $t_0 \in k$ such that

1. $3t_0 f_3 \neq -f_2$,
2. $f - f(t_0)$ is separable,
3. the curve $F \subset S$ given by $t = t_0$ is smooth, and
4. the set $F(k)$ of k-rational points on F is infinite,

then the set $S(k)$ of k-rational points is Zariski dense in S.

If k is of finite type over $\mathbb{Q}$, then the converse holds as well.

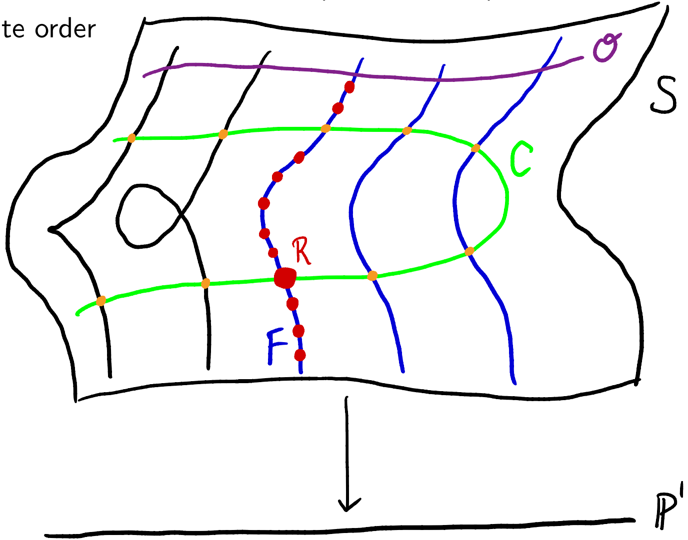
If rational points are dense on some fiber F, then also on S.

Standard attack:

Find multisection with

(1) infinitely many rational points (so genus ≤ 1)

(2) infinite order



Given $R \in F$, construct a curve $C_R \subset S$.

$R = R_0 = (x_0, y_0, t_0)$ $\quad t_0 \in k$

$\varphi(R) = (x_0, y_0, f(t_0))$

$f(t) - f(t_0) = f_3 (t-t_0)(t-t_1)(t-t_2)$

$R_1 = (x_0, y_0, t_1)$ $\quad t_1, t_2 \overset{?}{\in} k$

$R_2 = (x_0, y_0, t_2)$

φ étale around R_0, R_1, R_2

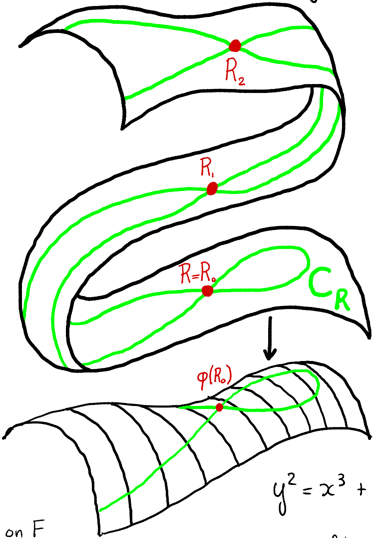
Tangent plane

$T_{S', \varphi(R)} : \alpha x + \beta y + \gamma u = 0$

$C_R = \underbrace{\varphi^{-1}(T_{S', \varphi(R)})}_{\alpha x + \beta y + \gamma f(t) = 0} \cap S$

γ nonzero if $2R \neq 0$ on F

$y^2 = x^3 + a_4(f(t))x + a_6(f(t))$



$S \subset \mathbb{A}^3(x,y,t)$

$(x,y,t) \mapsto (x,y,f(t))$ $\quad \varphi$

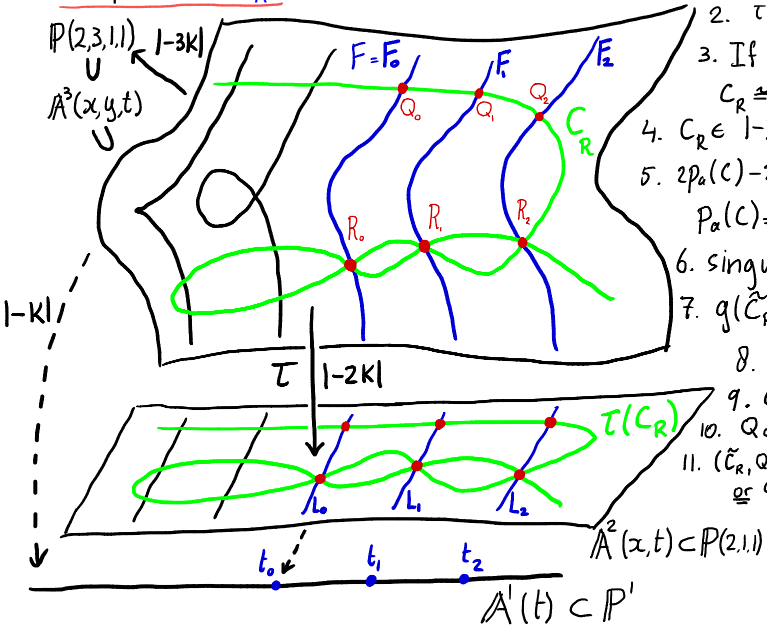
$S' \subset \mathbb{A}^3(x,y,u)$

$y^2 = x^3 + a_4(u)x + a_6(u)$

singular fiber of type IV^* at $u = \infty$

$t \longmapsto f(t)$ totally ramified above $u = \infty$

Properties of C_R



1. $C_{-R} = -C_R$
2. $\tau(C_{-R}) = \tau(C_R)$
3. If $2R \neq O$, then $C_R \cong \tau(C_R) \subset \mathbb{P}(2,1,1)$
4. $C_R \in |-3K|$
5. $2p_a(C) - 2 = C_R \cdot (C_R + K)$

   $p_a(C) = \frac{1}{2} \cdot 6K^2 + 1 = 4$
6. singular at R_0, R_1, R_2
7. $g(\tilde{C}_R) \leq 4 - 3 = 1$
8. $C_R \cdot F_i = 3$ Q_i
9. $C_R \cap F_i = \{R_i, -2R_i\}$
10. $Q_0 \in C_R(k)$
11. $(\tilde{C}_R, Q_0)$ is elliptic curve

    or $C_R \longleftrightarrow \mathbb{P}^1$ or

    C_R contains except. curve

A family of curves C_R.

Set $F^\circ = F \setminus F[2]$ and $A = \mathbb{A}^1(x) \setminus x(F[2])$.

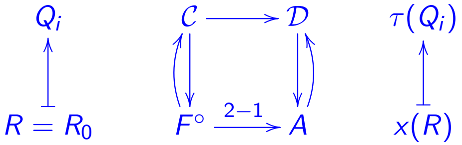
For $x \in A$ define $D_x = \tau(C_R) \subset \mathbb{P}(2,1,1)$ where $x(R) = x$.

Set $\mathcal{C} = \{\, (R,P) \,:\, R \in F^\circ,\ P \in C_R \,\} \subset F^\circ \times S$.

Set $\mathcal{D} = \{\, (x,P) \,:\, x \in A,\ P \in D_x \,\} \subset A \times \mathbb{P}(2,1,1)$.

The $4-1$ map
$$F^\circ \times S \longrightarrow A \times \mathbb{P}(2,1,1)$$
$$(R,P) \mapsto (x(R), \tau(P))$$

induces a Cartesian diagram.



The projection map $\mathcal{C} \to S$ is dominant, and $|F^\circ(k)| = \infty$.

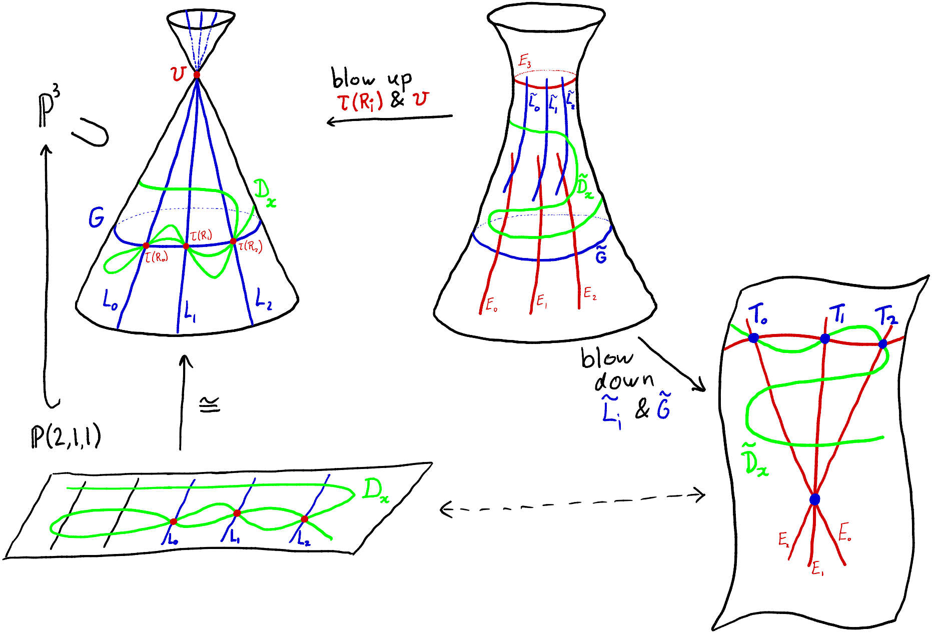
"Conclusion". Enough to show $\mathcal{D} \to A$ has section of infinite order.

Real conclusion.
Let η be the generic point of F, and take $R = R_0 = \eta$.
May assume generic fiber of $\mathcal{D} \to A$ is geometrically integral and singular only at the three points $\tau(R_i)$.

Let $\tilde{\mathcal{D}} \to A$ correspond to the normalisation.

Enough to show that on the elliptic surface $\tilde{\mathcal{D}} \to A$ with zero section $\tau(Q_0)$, the section $\tau(Q_1) + \tau(Q_2)$ has infinite order.

Claim.
The surface $\tilde{\mathcal{D}}$ embeds into $A \times \mathbb{P}^2$ as family of cubic curves.



$\mathbb{P}^3$
$\supset$
$\mathbb{P}(2,1,1)$
$\cong$
v
G
D_x
$\tau(R_0)$
$\tau(R_1)$
$\tau(R_2)$
L_0
L_1
L_2
D_x
L_0
L_1
L_2
blow up
$\tau(R_i)$ & v
E_3
$\tilde{L}_0$
$\tilde{L}_1$
$\tilde{L}_2$
$\tilde{D}_x$
$\tilde{G}$
E_0
E_1
E_2
blow down
$\tilde{L}_i$ & $\tilde{G}$
T_0
T_1
T_2
$\tilde{D}_x$
E_2
E_0
E_1

Take

$$a_1 = \epsilon, \ \ a_2 = \gamma, \ \ a_3 = \delta_1 u + \delta_0, \ \ a_4 = \beta_1 u + \beta_0, \ \ a_6 = \alpha_2 u^2 + \alpha_1 u + \alpha_0$$

$$F_0 : \ y^2 + a_1(f(t_0))xy + a_3(f(t_0))y = x^3 + a_2(f(t_0))x^2 + a_4(f(t_0))x + a_6(f(t_0))$$

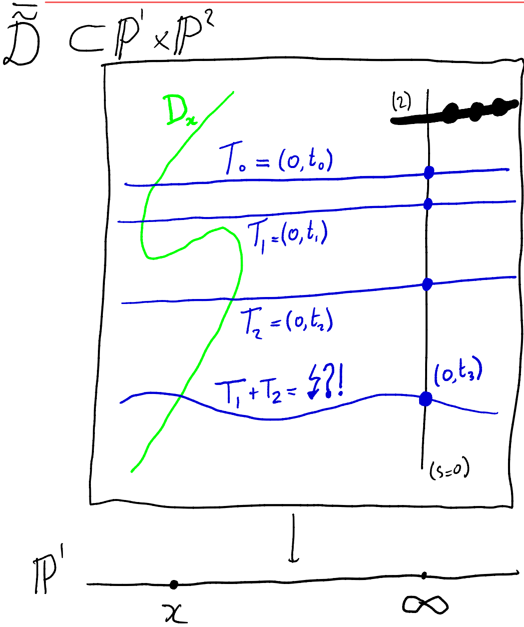
$$\begin{aligned}
\Phi_2 =& \Phi_2(F_0) = (\text{2-division polynomial of } F_0) = 4x^3 + \dots \\
\Phi_3 =& \Phi_3(F_0) = (\text{3-division polynomial of } F_0) = 3x^4 + \dots \\
\Psi_1 =& (-12\alpha_2 - 3\delta_1^2)x^2 + (-8\alpha_2\gamma - 2\alpha_2\epsilon^2 + 2\beta_1^2 + 2\beta_1\delta_1\epsilon - 2\gamma\delta_1^2)x \\
& + 2\alpha_1\beta_1 + \alpha_1\delta_1\epsilon - 4\alpha_2\beta_0 - 2\alpha_2\delta_0\epsilon - \beta_0\delta_1^2 + \beta_1\delta_0\delta_1 \\
\Psi_2 =& (-2\beta_1 - \delta_1\epsilon)x^3 + (-6\alpha_1 - 3\delta_0\delta_1)x^2 + (-4\alpha_1\gamma - \alpha_1\epsilon^2 + 2\beta_0\beta_1 + \beta_0\delta_1\epsilon + \beta_1\delta_0\epsilon - 2\gamma\delta_0\delta_1)x \\
& + 4\alpha_0\beta_1 + 2\alpha_0\delta_1\epsilon - 2\alpha_1\beta_0 - \alpha_1\delta_0\epsilon - \beta_0\delta_0\delta_1 + \beta_1\delta_0^2 \\
\Psi_3 =& (4\alpha_2 + \delta_1^2)x^3 + (4\alpha_2\gamma + \alpha_2\epsilon^2 - \beta_1^2 - \beta_1\delta_1\epsilon + \gamma\delta_1^2)x^2 + (-2\alpha_1\beta_1 - \alpha_1\delta_1\epsilon + 4\alpha_2\beta_0 \\
& + 2\alpha_2\delta_0\epsilon + \beta_0\delta_1^2 - \beta_1\delta_0\delta_1)x + 4\alpha_0\alpha_2 + \alpha_0\delta_1^2 - \alpha_1^2 - \alpha_1\delta_0\delta_1 + \alpha_2\delta_0^2
\end{aligned}$$

Then $\tilde{\mathcal{D}} \subset A(x) \times \mathbb{A}^2(s,t) \subset \mathbb{P}^1 \times \mathbb{P}^2$ is given by

$$\Psi_3 s^3 + (\Psi_1 f(t_0) + \Psi_2)s^2 + \Phi_3 \cdot s = \Phi_2 \cdot (f(t_0) - f(t)).$$

The point (section) $\tau(Q_i)$ corresponds with $\big(x, (0, t_i)\big)$.

The section $\tau(Q_1)+\tau(Q_2)$ has infinite order!



On minimal regular model, fiber at ∞ is of type I_0^*.

Addition on Néron model specialises on additive fiber to

$$(0,t_1)+(0,t_2)=(0,t_3)$$

with

$$t_3 - t_0 = (t_1 - t_0) + (t_2 - t_0).$$

Since $\mathrm{char}(k)=0$, the group $(k,+)$ has no torsion, so ∞ order if nonzero, i.e.,

$$3t_0 \neq t_0 + t_1 + t_2 = -f_2/f_3.$$

So section $T_1 + T_2$ has infinite order!

The converse

If k is of finite type over $\mathbb{Q}$ and $S(k)$ is dense, then there are infinitely many fibers F with infinitely many rational points.

proof

If k is a number field, then by Merel, the order of torsion points on elliptic curves over k is uniformly bounded, say by N. An extension by Colliot-Thélène states the same for k of finite type over $\mathbb{Q}$.

Hence there is a proper closed subset of S containing all k-points that have finite order on their fiber.

Thanks!