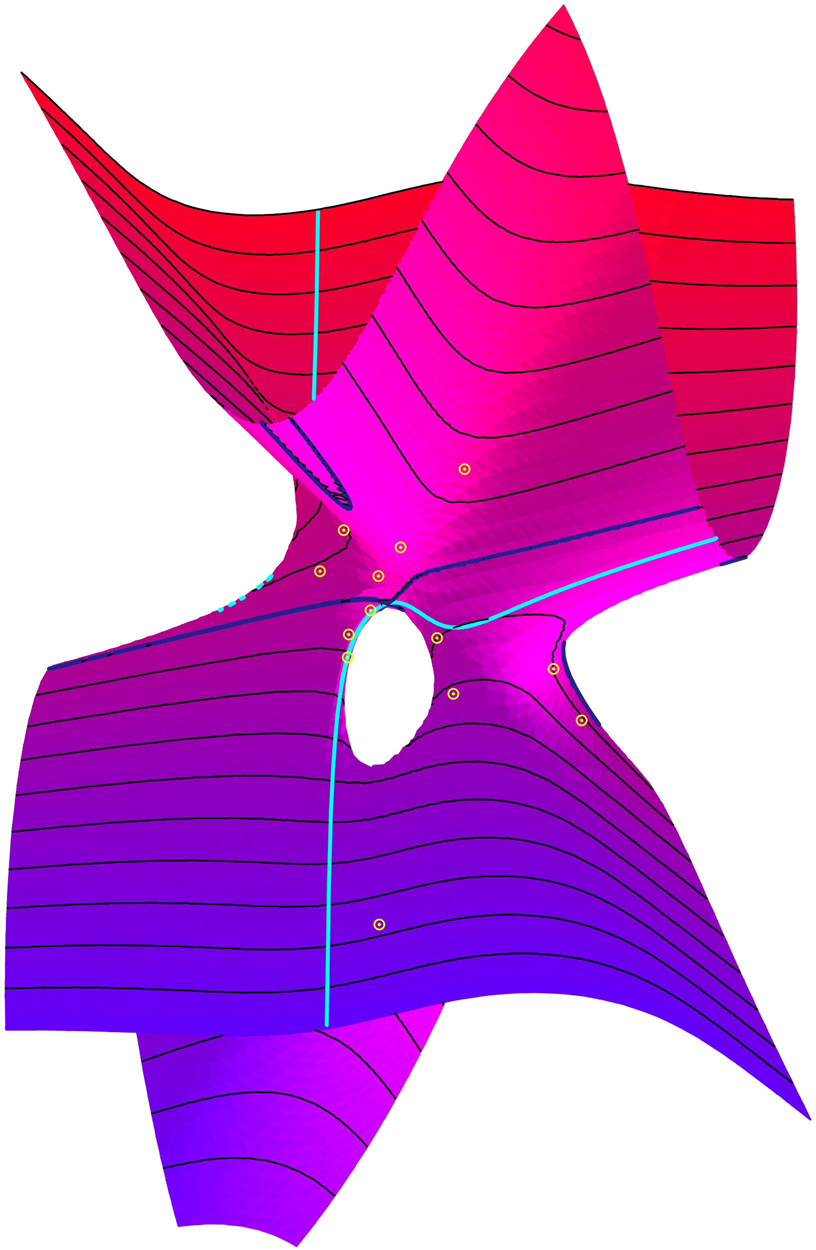
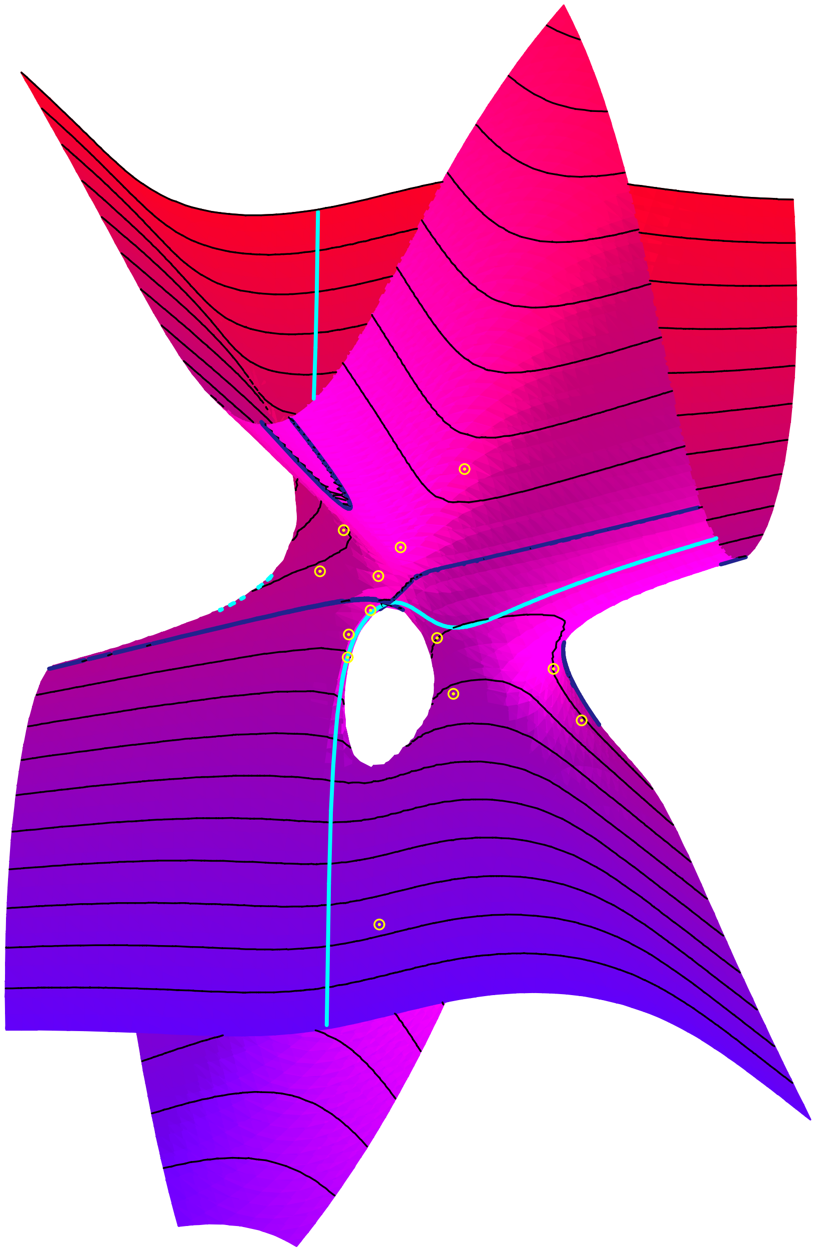


K3 surfaces with Picard number one and infinitely many rational points

February 16, 2010
Tokyo

Ronald van Luijk



Computing Picard groups of surfaces

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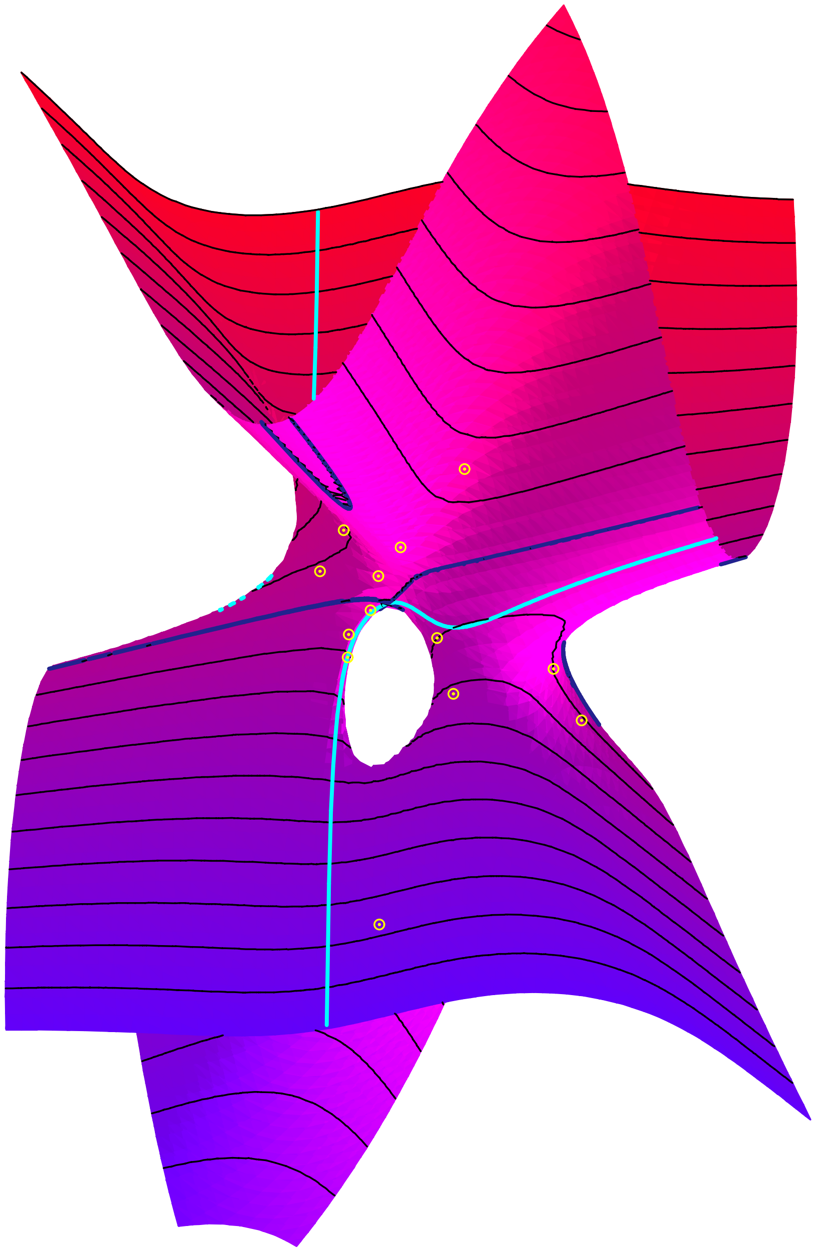

Computing Picard groups of surfaces (char.-p methods)

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0) Some definitions

Surface:
smooth, projective, geometrically integral, dimension 2 over a field.

K3 surface : a surface X with $\dim H^1(X, \mathcal{O}_X) = 0$
and trivial canonical sheaf.

Examples:

- A smooth quartic surface in $\mathbb{P}^3$.
- **Kummer surface**: minimal nonsingular model of $A/[-1]$, with A an abelian surface.

1) Advertisement for arithmetic

Example [Noam Elkies].

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

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Rational points are (Zariski) dense on surface

$$\mathbb{P}^3 \supset X : x^4 + y^4 + z^4 = t^4.$$

Example. Let $X \subset \mathbb{P}^3$ be given by

$$x^4 + 2y^4 = z^4 + 4w^4.$$

Question 1 (Swinnerton-Dyer, 2002)**.**
Does X have more than two rational points?

Example. Let $X \subset \mathbb{P}^3$ be given by

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Question 1 (Swinnerton-Dyer, 2002)**.**
Does X have more than two rational points?

Answer (Elsenhans, Jahnel, 2004):

$$1484801^4 + 2 \cdot 1203120^4 \;=\; 1169407^4 + 4 \cdot 1157520^4$$

Question 2 (open). *Does there exist a K3 surface X over a number field k such that the set $X(k)$ of k-rational points on X is neither empty nor dense?*

Néron-Severi group $\mathsf{NS}(X)$ of a surface X over a field k:
group of divisor classes modulo algebraic equivalence.

Linear equivalence implies algebraic equivalence, so quotient map

$$\operatorname{Pic} X \to \mathsf{NS}(X).$$

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Group $\mathsf{NS}(X)$ is finitely generated.
The **Picard number** of X is $\rho(X) = \operatorname{rank} \mathsf{NS}(X)$.
The **geometric Picard number** of X is $\rho(\overline{X})$ with $\overline{X} = X \times_k \overline{k}$.

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K3 surface: (linear = algebraic = numerical) equivalence,
$\operatorname{Pic} X \cong \mathsf{NS}(X)$ is torsion-free and $1 \leq \rho(X) \leq \rho(\overline{X}) \leq B_2 = 22$.

Elkies'	$x^4 + y^4 + z^4 = t^4$	has	$\rho = 4$.
Swinnerton-Dyer's	$x^4 + 2y^4 = z^4 + 4w^4$	has	$\rho = 1$.

Elkies'	$x^4 + y^4 + z^4 = t^4$	has	$\rho = 4$.
Swinnerton-Dyer's	$x^4 + 2y^4 = z^4 + 4w^4$	has	$\rho = 1$.

Vague idea:

The higher the Picard number of X, the "easier" it is for X to have lots of rational points.

Let X be a K3 surface over a number field k. If there exists a finite field extension k'/k such that $X(k')$ is Zariski dense in X, then we say that rational points on X are *potentially dense*.

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Theorem[F. Bogomolov – Y. Tschinkel]
Let X be a K3 surface over a number field.
(a) If $\mathrm{Aut}\,\overline{X}$ is infinite or $\overline{X}$ has an elliptic fibration, then rational points on X are potentially dense.
(b) If $\rho(\overline{X}) \geq 2$, then in most cases rational points on X are potentially dense.

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Question 3. *Is there a K3 surface X over a number field with $\rho(\overline{X}) = 1$ on which the rational points are potentially dense?*

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Question 3. *Is there a K3 surface X over a number field with $\rho(\overline{X}) = 1$ on which the rational points are potentially dense?*
Question 4. *Is there a K3 surface X over a number field with $\rho(\overline{X}) = 1$ on which the rational points are* **not** *potentially dense?*

2) The main problem

Question 5 (Swinnerton-Dyer). *Is there a K3 surface over a number field with Picard number* 1 *on which there are infinitely many rational points?*

We will see that they do exist, even with the *geometric* Picard number equal to 1. We can also take the ground field to be $\mathbb{Q}$.

2) The main problem

Question 5. *Is there a K3 surface over a number field with Picard number* 1 *on which there are infinitely many rational points?*

1) infinitely many rational points

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Theorem[P. Deligne, 1973]
A general quartic surface in $\mathbb{P}^3$ has geometric Picard number 1.

2) The main problem

Question 5. *Is there a K3 surface over a number field with Picard number 1 on which there are infinitely many rational points?*

1) infinitely many rational points
2) geometric Picard number 1 (hardest, despite:)

Theorem[P. Deligne, 1973]
A general quartic surface in $\mathbb{P}^3$ has geometric Picard number 1.

Quartic surfaces in $\mathbb{P}^3$ are parametrized by $\mathbb{P}^{34}$. "General" means "up to a countable union of proper closed subsets of $\mathbb{P}^{34}$".

A priori this could exclude all quartic surfaces defined over $\overline{\mathbb{Q}}$!

What was known?

Theorem[T. Terasoma, 1985; J. Ellenberg, 2004]
K3 surfaces over $\mathbb{Q}$ with geometric Picard number 1 exist.

Theorem[T. Shioda]
For every prime $m \geq 5$ the surface in $\mathbb{P}^3$ given by

$$w^m + xy^{m-1} + yz^{m-1} + zx^{m-1} = 0$$

has geometric Picard number 1.

Theorem[vL] The quartic surface in $\mathbb{P}^3(x,y,z,w)$ given by

$$wf = 3pq - 2zg$$

with $f \in \mathbb{Z}[x,y,z,w]$ and $g,p,q \in \mathbb{Z}[x,y,z]$ equal to

$$\begin{aligned} f &= x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 \\ &\quad + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \\ g &= xy^2 + xyz - xz^2 - yz^2 + z^3, \\ p &= z^2 + xy + yz, \\ q &= z^2 + xy, \end{aligned}$$

has geometric Picard number 1 and infinitely many rational points.

Theorem The quartic surface S in $\mathbb{P}^3(x, y, z, w)$ given by

$$wf = 3pq - 2zg$$

has geometric Picard number 1 and infinitely many rational points.

Infinitely many rational points:

The curve $C = S \cap (H_w\colon w = 0)$, has infinitely many rational points.

The plane H_w is tangent to S at $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$.

Therefore, $g(C) \leq 1$, so consistent with Faltings' Theorem.

3) Bounding the Picard number from above

Let X be a (smooth, projective, geometrically integral) surface over $\mathbb{Q}$ and let $\mathcal{X}$ be an integral model with good reduction at the prime p.

From étale cohomology we get injections

$$\mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_l \hookrightarrow \mathrm{NS}(\mathcal{X}_{\overline{\mathbb{F}}_p}) \otimes \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1)).$$

The second injection respects Frobenius.

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The second injection respects Frobenius.

Corollary The rank $\rho(X_{\overline{\mathbb{Q}}})$ is bounded from above by the number of eigenvalues λ of Frobenius acting on $H^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$ for which λ is a root of unity.

$$\mathsf{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_l \hookrightarrow \mathsf{NS}(\mathcal{X}_{\overline{\mathbb{F}}_p}) \otimes \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1)).$$

We can compute the characteristic polynomial f of Frobenius by computing traces of its powers through the Lefschetz formula

$$\#\mathcal{X}(\mathbb{F}_{p^n}) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}(\textbf{Frob}^n \textbf{ on } H^i_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)).$$

Note the difference between $\mathbb{Q}_l$ and the twist $\mathbb{Q}_l(1)$.

Knowing the traces, the characteristic polynomial f follows from simple linear algebra and scaling of the roots by a factor p.

Problem!

The degree of f is B_2, so even (22) for K3 surfaces.

Lemma Let f be a polynomial with real coefficients and even degree, such that all its roots have complex absolute value 1. Then the number of roots of f that are roots of unity is even.

Proof. All the real roots of f are roots of unity. The remaining roots come in conjugate pairs, either both being a root of unity or both not being a root of unity. Therefore, the number of roots that are *not* roots of unity is even (independent of the parity of the degree).

4) A trick!

The intersection pairing gives the Néron-Severi group the structure of a *lattice*. The injection

$$\mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_l \hookrightarrow \mathrm{NS}(\mathcal{X}_{\overline{\mathbb{F}}_p}) \otimes \mathbb{Q}_l$$

of $\mathbb{Q}_l$-vector spaces respects the inner product.

Lemma If Λ' is a sublattice of finite index of Λ, then we have

$$\operatorname{disc} \Lambda' = [\Lambda : \Lambda']^2 \operatorname{disc} \Lambda.$$

This implies that $\operatorname{disc} \Lambda$ and $\operatorname{disc} \Lambda'$ have the same image in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

Sketch of proof

We find finite-index sublattices M_2 and M_3 of the Néron-Severi groups over $\overline{\mathbb{F}}_2$ and $\overline{\mathbb{F}}_3$ respectively. Both will have rank 2, which already shows that the rank of $\mathrm{NS}(S_{\overline{\mathbb{Q}}})$ is at most 2. We get the following diagram

$$\begin{array}{ccccc} \mathrm{NS}(S_{\overline{\mathbb{Q}}}) & \subset & \mathrm{NS}(S_{\overline{\mathbb{F}}_2}) & \supset & M_2 \\ \| & & & & \\ \mathrm{NS}(S_{\overline{\mathbb{Q}}}) & \subset & \mathrm{NS}(S_{\overline{\mathbb{F}}_3}) & \supset & M_3 \end{array}$$

Example chosen so that the images of $\operatorname{disc} M_2$ and $\operatorname{disc} M_3$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ are different, so $\mathrm{NS}(S_{\overline{\mathbb{Q}}})$ has rank at most 1.

The example was $wf = 3pq - 2zg$.

The reduction S_3 of S at 3 is given by $wf = zg$, so it contains the line $L\colon w = z = 0$. By the adjunction formula

$$L \cdot (L + K_{S_3}) = 2g(L) - 2 = -2,$$

with canonical divisor $K_{S_3} = 0$, we find $L^2 = -2$.

Let M_3 be the lattice generated by the hyperplane section H and L. With respect to $\{H, L\}$ the inner product on M_3 is given by

$$\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}.$$

With respect to $\{H, L\}$ the inner product on M_3 is given by

$$\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}.$$

We get $\operatorname{disc} M_3 = -9$. By counting points as described before we find that the characteristic polynomial of Frobenius acting on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l(1))$ factors over $\mathbb{Q}$ as

$$\begin{aligned}(x-1)^2(x^{20} &+ \tfrac{1}{3}x^{19} - x^{18} + \tfrac{1}{3}x^{17} + 2x^{16} - 2x^{14} + \tfrac{1}{3}x^{13} \\ &+ 2x^{12} - \tfrac{1}{3}x^{11} - \tfrac{7}{3}x^{10} - \tfrac{1}{3}x^{9} + 2x^{8} + \tfrac{1}{3}x^{7} - 2x^{6} \\ &+ 2x^{4} + \tfrac{1}{3}x^{3} - x^{2} + \tfrac{1}{3}x + 1).\end{aligned}$$

As the second factor is not integral, we find that exactly two of its roots are roots of unity. We conclude that M_3 has finite index in $\mathrm{NS}(S_{\overline{\mathbb{F}}_3})$.

The example is still $wf = 3pq - 2zg$.

The reduction S_2 is given by $wf = pq$, for some quadratic forms p and q. It therefore contains a conic C given by $w = p = 0$. By the adjunction formula

$$C \cdot (C + K_{S_2}) = 2g(C) - 2 = -2,$$

we find $C^2 = -2$. Let M_2 be the lattice generated by the hyperplane section H and C. With respect to $\{H, C\}$ the inner product on M_3 is given by

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}.$$

With respect to $\{H, C\}$ the inner product on M_2 is given by

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}.$$

We get $\operatorname{disc} M_2 = -12$. By counting points as described before we find that the characteristic polynomial of Frobenius acting on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_2}, \mathbb{Q}_l(1))$ factors over $\mathbb{Q}$ as

$$(x-1)^2(x^{20} + \tfrac{1}{2}x^{19} - \tfrac{1}{2}x^{18} + \tfrac{1}{2}x^{16} + \tfrac{1}{2}x^{14} + \tfrac{1}{2}x^{11} + x^{10} + \tfrac{1}{2}x^9 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x + 1).$$

The last factor is not integral, so M_2 has finite index in $\mathrm{NS}(S_{\overline{\mathbb{F}}_2})$.

$$
\begin{array}{ccccc}
\mathsf{NS}(S_{\overline{\mathbb{Q}}}) & \subset & \mathsf{NS}(S_{\overline{\mathbb{F}}_2}) & \supset & M_2 \\
|| & & & & \\
\mathsf{NS}(S_{\overline{\mathbb{Q}}}) & \subset & \mathsf{NS}(S_{\overline{\mathbb{F}}_3}) & \supset & M_3
\end{array}
$$

As $\operatorname{disc} M_3 = -9$ and $\operatorname{disc} M_2 = -12$ do not have the same image in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, we have proven that $\mathsf{NS}(S_{\overline{\mathbb{Q}}})$ has rank 1. By the adjunction formula the lattice is even, so it is generated by H.

$$
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This trick works if $\rho = \rho(\overline{X})$ is odd and primes p_1, p_2 are such that

$$\rho(X_{\overline{\mathbb{F}}_{p_i}}) = \rho + 1$$

and the images of $\operatorname{disc} \mathrm{NS}\, X_{\overline{\mathbb{F}}_{p_i}}$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ are different.

5) Extension by R. Kloosterman

Conjecture[Artin–Tate for K3] Let $X/\overline{\mathbb{F}}_q$ be a K3 surface. Let f be the characteristic polynomial of Frobenius acting on $H^i_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. Let ρ and Δ denote the rank and the discriminant of $\mathrm{NS}(X)$. Let $\mathrm{Br}\,X$ denote the Brauer group of X. Then

$$\lim_{T \to q} \frac{f(T)}{(T-q)^\rho} = -q^{21-\rho} \cdot \#\,\mathrm{Br}\,X \cdot \Delta.$$

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$$\lim_{T \to q} \frac{f(T)}{(T-q)^{\rho}} = -q^{21-\rho} \cdot \# \mathrm{Br}\, X \cdot \Delta.$$

Facts:

Tate conjecture $\Rightarrow$ Artin–Tate

$\mathrm{Br}\, X$ finite $\Rightarrow$ $\# \mathrm{Br}\, X$ is square (Liu–Lorenzini–Raynaud)

Conclusion: Artin conjecture gives $\Delta \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ without explicit generators of $\mathrm{disc}\, \mathrm{NS}(S_{\overline{\mathbb{F}}_q})$.

Application

Theorem[R. Kloosterman, 2005]
The elliptic K3 surface $\pi : X \to \mathbb{P}^1$ over $\overline{\mathbb{Q}}$ with Weierstrass equation

$$y^2 = x^3 + 2(t^8 + 14t^4 + 1)x + 4t^2(t^8 + 6t^4 + 1)$$

has $\rho(X) = 17$ and Mordell-Weil rank 15.

5) Extension by A.-S. Elsenhans and J. Jahnel

You do not necessarily need

$$\rho(X_{\overline{\mathbb{F}}_{p_i}}) = \rho + 1.$$

Example[A.-S. Elsenhans and J. Jahnel]
Let $S\colon w^2 = f_6(x,y,z)$ be a K3 surface of degree 2 over $\mathbb{Q}$. Assume the congruences

$$f_6 = y^6 + x^4y^2 - 2x^2y^4 + 2x^5z + 3xz^5 + z^6 \pmod 5$$

and

$$f_6 = 2x^6 + x^4y^2 + 2x^3y^2z + x^2y^2z^2 + x^2yz^3 + 2x^2z^4 + xy^4z \\ + xy^3z^2 + xy^2z^3 + 2xz^5 + 2y^6 + y^4z^2 + y^3z^3 \pmod 3.$$

Then $\mathbf{S}$ has geometric Picard rank $\mathbf{1}$.

5) Extension by A.-S. Elsenhans and J. Jahnel

Let L denote the pull-back of a line in $\mathbb{P}^2$.

The characteristic polynomial of Frobenius acting on the space

$$(\mathrm{NS}\, S_{\overline{\mathbb{F}}_3} \otimes \mathbb{Q}_l)/\langle L \rangle$$

equals

$$(t-1)(t^2+t+1).$$

There are only finitely many Galois-invariant subspaces of $\mathrm{NS}\, S_{\overline{\mathbb{F}}_3} \otimes \mathbb{Q}_l$ containing L. Their dimensions are $1, 2, 3, 4$.

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The characteristic polynomial of Frobenius acting on the space

$$(\mathrm{NS}\, S_{\overline{\mathbb{F}}_5} \otimes \mathbb{Q}_l)/\langle L \rangle$$

equals $(t-1)\Phi_5(t)\Phi_{15}(t)$, where Φ_n denotes the n-th cyclotomic polynomial. There are only finitely many Galois-invariant subspaces of $\mathrm{NS}\, S_{\overline{\mathbb{F}}_5} \otimes \mathbb{Q}_l$ containing L. Their dimensions are $1, 2, 5, 6, 9, 10, 13, 14$.

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Only common dimensions are 1 and 2. Compare discriminants up to squares of the subspaces of dimension 2.

6) Generators for the Néron-Severi group (Schütt–Shioda)

Assume: $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{NS}(X_{\overline{\mathbb{F}}_p})$ torsion free.

Goal: Show $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}})$ is primitive.

6) Generators for the Néron-Severi group (Schütt–Shioda)

Assume: $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{NS}(X_{\overline{\mathbb{F}}_p})$ torsion free.
Goal: Show $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}})$ is primitive.

If not primitive, then there is $g \in G$ and a prime

$$r \,|\, \mathrm{disc}\, G = [\hat{G} : G]^2 \cdot \mathrm{disc}\, \hat{G}$$

with $\hat{G} = (G \otimes \mathbb{Q}) \cap \mathrm{NS}(X_{\overline{\mathbb{Q}}})$ and

$$0 \neq \overline{g} \in G \otimes \mathbb{F}_r \quad \text{and} \quad 0 = \overline{g} \in \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r,$$

so

$$G \otimes \mathbb{F}_r \to \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r$$

is not injective.

Assume: $G \subset \mathsf{NS}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \mathsf{NS}(X_{\overline{\mathbb{F}}_p})$ torsion free.

Goal: Show $G \subset \mathsf{NS}(X_{\overline{\mathbb{Q}}})$ is primitive.

If not primitive, then $\exists r : G \otimes \mathbb{F}_r \to \mathsf{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r$ is not injective.

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Goal: Show $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}})$ is primitive.

If not primitive, then $\exists r \colon G \otimes \mathbb{F}_r \to \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r$ is not injective.

For such r and every $H \subset \mathrm{NS}(X_{\overline{\mathbb{F}}_p})$ the composition

$$G \otimes \mathbb{F}_r \to \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r \to \mathrm{Hom}(H \otimes \mathbb{F}_r, \mathbb{F}_r)$$

induced by

$$\mathrm{NS}(X_{\overline{\mathbb{Q}}}) \to \mathrm{Hom}(H, \mathbb{Z}), \qquad D \mapsto (x \mapsto D \cdot x)$$

is not injective.

Assume: $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{NS}(X_{\overline{\mathbb{F}}_p})$ torsion free.

Goal: Show $G \subset \mathrm{NS}(X_{\overline{\mathbb{Q}}})$ is primitive.

If not primitive, then $\exists r:$ for every $H \subset \mathrm{NS}(X_{\overline{\mathbb{F}}_p})$ the composition

$$G \otimes \mathbb{F}_r \to \mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r \to \mathrm{Hom}(H \otimes \mathbb{F}_r, \mathbb{F}_r)$$

sending D to $(x \mapsto D \cdot x)$ is not injective.

Assume: $G \subset \mathsf{NS}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \mathsf{NS}(X_{\overline{\mathbb{F}}_p})$ torsion free.

Goal: Show $G \subset \mathsf{NS}(X_{\overline{\mathbb{Q}}})$ is primitive.

If not primitive, then $\exists r:$ for every $H \subset \mathsf{NS}(X_{\overline{\mathbb{F}}_p})$ the composition

$$G \otimes \mathbb{F}_r \to \mathsf{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r \to \mathsf{Hom}(H \otimes \mathbb{F}_r, \mathbb{F}_r)$$

sending D to $(x \mapsto D \cdot x)$ is not injective.

Sufficient: Find for each prime r with $r^2 | \operatorname{disc} G$ an $H \subset \mathsf{NS}(X_{\overline{\mathbb{F}}_p})$ with

$$G \otimes \mathbb{F}_r \to \mathsf{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{F}_r \to \mathsf{Hom}(H \otimes \mathbb{F}_r, \mathbb{F}_r)$$

injective (just linear algebra).

Theorem[Mizukami ($m = 4$), Schütt–Shioda–vL ($m \leq 100$)]
For any integer $1 \leq m \leq 100$ the Néron-Severi group of the Fermat surface $S_m \subset \mathbb{P}^3$ over $\mathbb{C}$ given by

$$x^m + y^m + z^m + w^m = 0$$

is generated by the lines on S_m if and only if $m \leq 4$ or $(m, 6) = 1$.