SPECTRAL TRIPLES ON NONCOMMUTATIVE BOUNDARIES

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Abstract. We discuss several examples of spectral triples for C^*-algebras that arise as "noncommutative boundaries" of commutative dynamical systems. The examples concern Cuntz-Krieger algebras and crossed products of Kleinian groups by their limit sets. The common theme for these spectral triples is that they carry non-trivial K-homological content, they are not finitely summable, and their operators formally resemble the logarithm of the Laplacian on $\mathbb{R}^n$. The latter admits a representation as a singular integral operator, the expression for which makes sense in the context of metric measure spaces. These notes are based on a talk of the same title that was delivered at the workshop "Spectral approaches to fractal geometry" at Chalmers University, Gothenburg, Sweden on September 16th, 2018.

Contents

1. Fractional calculus on $\mathbb{R}^n$

For $0 < \alpha < \frac{n}{2}$, the fractional powers of the positive Laplacian $\Delta := -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$ admit the following expressions as a singular integral operators:

$$\Delta^{\alpha} f(x) = c_{\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{\|x - y\|^{n+2\alpha}} dy$$

$$\Delta^{-\alpha} f(x) = c_{-\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{\|x - y\|^{n-2\alpha}} dy. \tag{1}$$

Here $c_{\alpha} > 0$ is a constant depending only on α. The integral transform (1) is referred to as the *Riesz potential*. We are interested in such operators in the context of compact metric measure spaces.

1.1. Definition. Let (X, ρ) be a compact metric space. A measure μ on X is called *Ahlfors d-regular* if there exist constants c_1, c_2 such that for all $r > 0$ and all $x \in X$ we have

$$c_1 r^d \leqslant \mu(B(x, r)) \leqslant c_2 r^d.$$

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We have the following result for Riesz potentials of Ahlfors regular measures.

1.2. Theorem ([14]). *Let (X, ρ) be a compact metric space and μ an Ahlfors d-regular measure. For $0 < \alpha < d$ the map*

$$I_\alpha f(x) := \int_X \frac{f(y)}{\rho(x,y)^{d-\alpha}} d\mu(y),$$

defines a compact operator in $L^2(X, \mu)$.

The operator

$$f \mapsto \int_X \frac{f(x) - f(y)}{\rho(x,y)^d} d\mu(y), \tag{2}$$

can be thought of as an analogue of the logarithm of a suitbale Laplacian in $L^2(X, \mu)$. Due to the boundedness of the Riesz potential

$$\left\| \int_X \frac{f(x) - f(y)}{\rho(x,y)^d} d\mu(y) \right\| \leqslant \|f\|_{\text{Lip}} \left\| \int_X \frac{f(y)}{\rho(x,y)^{d-1}} d\mu(y) \right\|,$$

we see that the operator (2) is defined on Lipschitz functions and thus in particular has a dense domain in $L^2(X, \mu)$.

2. Dirac operators on metric measure spaces with dynamics

We consider a compact metric space (X, ρ) with an Ahlfors regular measure μ and we furthermore assume that a semigroup G acts on X by continuous homeomorphisms. We do not require this action metirc or measure preserving. Our goal is to construct spectral triples on $C(X) \rtimes G$ for a suitable notion of semigroup crossed product. We focus on two particular examples.

2.1. Example. Cuntz-Krieger algebras. Fix a positive integer N and consider the compact space

$$\Omega_N := \{1, \cdots, N\}^{\mathbb{N}} = \{x_1 x_2 x_3 \cdots : x_i \in \{1, \cdots N\}\}.$$

The *cylinder sets*

$$C_{x_1 \cdots x_k} := \{x_1 \cdots x_k x_{k+1} \cdots\} \subset \Omega_N$$

generate the topology on Ω_N. The metric on X is given by $d(x, y) = e^{-\min\{i : x_i \neq y_i\}}$ and the measure is determined by $\mu(C_{x_1 \cdots x_k}) = N^{-k}$. The is a surjective local homeomorphism, the shift map $\sigma : \Omega_N \to \Omega_N$ defined by

$$\sigma(x_1 x_2 \cdots) := \sigma(x_2 \cdots).$$

The map σ does not preserve the metric nor the measure on Ω_N.

We construct a groupoid out of this non-invertible dynamics as follows. Let

$$G_N := \{(x, n, y) \in \Omega_N \times \mathbb{Z} \times \Omega_N : \exists k \in \mathbb{N} \quad \sigma^{k+n}(x) = \sigma^k(y)\}.$$

The source and range maps $r, s : G_N \to \Omega_N$ are given by

$$r(x, n, y) = x, \quad s(x, n, y) = y,$$

whereas composition and inversion are defined by

$$(x, n, y)^{-1} = (y, -n, x), \quad (x, n, y) \circ (y, m, z) := (x, n + m, z).$$

The algebraic groupoid G_N comes with two further maps $c : G_N \to \mathbb{Z}$ and $\kappa : G_N \to \mathbb{N}$ determined by

$$c(x, n, y) := n, \quad \kappa(x, n, y) := \min\{k : \sigma^{n+k}(x) = \sigma^k(y).\}$$

There is a compatible topology on G_N which is determined by requiring the above maps r, s, c and κ to be continuous. The groupoid G_N admits a Haar system consisting of counting measures, which allows to extend the measure μ to all of G_N. The groupoid C^*-algebra $C^*(G_N)$ is isomorphic to the Cuntz algebra O_N. The latter algebra can be defined as the universal unital C^*-algebra generated by isometries $S_1, \cdots, S_N$ satisfying the relation

$$\sum_{i=1}^{N} S_i S_i^* = 1.$$

The isomorphism $O_N \simeq C^*(G_N)$ is implemented by the map that sends the generator S_i to characteristic function of the clopen set

$$G_N^i := (x, 1, y) \in G_N : x \in C_i, \quad \sigma(x) = y\}.$$

Details can be found in **[13]**. The algebra O_N is a simple, nuclear and purely infinite C^*-algebra.

2.2. Example. The second example of interest concerns boundary actions of Kleinian groups. Denote by $\mathbf{H}^{n+1}$ the hyperbolic space of dimension $n+1$. We realize this space as the open unit ball in $\mathbb{R}^{n+1}$ with the Riemannian metric $\frac{dx^2}{(1-\|x\|^2)^2}$ and we denote $G := \mathrm{Isom}(\mathbf{H}^{n+1})$. The geodesic boundary of $\mathbf{H}^{n+1}$ is the unit sphere S^n and the action of G extends to an action on S^n, but this action is not isometric for the Euclidean metric on S^n. Let $\Gamma \subset G$ be a discrete group. We are interested in the crossed product C^*-algebra $C(S^n) \rtimes \Gamma$. In case the group Γ is *non-elementary*, the C^*-algebra $C(S^n) \rtimes \Gamma$ is simple, nuclear and purely infinite.

Of special interest are the cases $n = 1, 2$. For $n = 1$, $\mathbf{H}^2$ can be identified with the complex upper halfplane and its boundary with $\mathbb{P}^1(\mathbb{R})$. The isometry group $\mathrm{Isom}(\mathbf{H}^2) \simeq PSL(2, \mathbb{R})$ and the action of G on the (full) complex plane is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}, \tag{3}$$

and preserves the upper and lower half planes as well as the real line. The action (3) extends to an action of $PSL(2, \mathbb{C})$. The latter group can be identified with $\mathrm{Isom}(\mathbf{H}^3)$ and $\mathbb{P}^1(\mathbb{C})$ with its boundary.

2.3. Definition. Let A be a unital C^*-algebra. A *spectral triple* $(\mathscr{A}, H, D)$ for A consists of

(1) a representation $A \to B(H)$;
(2) a self-adjoint operator $D : \mathrm{Dom}\, D \to H$ for which $(D \pm i)^{-1} \in \mathbb{K}(H)$ (the conpact operators on H);
(3) a dense subalgebra $\mathscr{A} \subset A$ such that for all $a \in \mathscr{A}$ we have $a : \mathrm{Dom}\, D \to \mathrm{Dom}\, D$ and $[D, a]$ extends to a bounded operator.

In Connes' proposal for noncommutative geometry **[3]**, spectral triples, and more generally unbounded KK-cycles, are objects that encode both topological and geometric information. The Dirac operator on a spin manifold recovers both the Riemannian metric and the fundamental class in K-homology. Moreover, the *square* of a Dirac operator is directly related to the Laplacian on the manifold. In noncommutative situations, it is thus of interest to construct spectral triples that carry both spectral and K-homological information.

Denote by $\mathscr{L}^n(H)$ the n-th Schatten ideal in $B(H)$. A spectral triple is called *finitely summable* if $(D \pm i)^{-1} \in \mathscr{L}^n(H)$ for some n. Finite summability can be viewed as a notion of finite dimensionality for spectral triples. It plays an important role in the analysis of index formulas in noncommutative geometry. Also recall that a *continuous trace* on a C^*-algebra A is a continuous map $\tau : A \to \mathbb{C}$ such that fro all $a, b \in A$ it holds that $\tau(ab) = \tau(ba)$. The following result relates summability of spetral triples to traces on its C^*-algebra.

2.4. Theorem (Connes **[2]**). *Let $(\mathscr{A}, H, D)$ be a finitely summable spectral triple for A such that the representation $A \to B(H)$ is injective. The there exists a nonzero continuous trace on A.*

For our purposes, we observe that a simple and purely infinite C^*-algebra does not admit any continuous trace. There we conclude that such C^*-algebras do not admit any finitely summable spectral triples. This holds in particular for the C^*-algebras O_N and $C(S^n) \rtimes \Gamma$ discussed above.

3. Hilbert modules and KK-theory

Kasparov's KK-theory is a bivariant homology theory that associates to a pair of C^*-algebras (A, B) a $\mathbb{Z}/2$-graded abelian group $KK_*(A, B)$. Elements in a KK-group are given by a generalization of spectral triples.

3.1. Definition. A *Hilbert module* over a C^*-algebra B is a right B-module X_B that comes equipped with an inner product

$$X_B \times X_B \to B, \quad (x, y) \mapsto \langle x, y \rangle.$$

The inner product is required to satisfy the properties

$$\langle x, y \rangle^* = \langle y, x \rangle, \quad \langle x, yb \rangle = \langle x, y \rangle b,$$

for all $x, y \in X_B$ and $b \in B$. Moreover we require *positivity* $\langle x, x \rangle \geqslant 0$ (positivity in the C^*-algebra B) and *nondegeneracy*, $\langle x, x \rangle = 0$ only if $x = 0$. Lastly we require that X_B is complete in the norm $\|x\|^2 := \|\langle x, x \rangle\|_B$.

In case the C^*-algebra $B = \mathbb{C}$, the notion reduces to that of a complex Hilbert space. If $B = C(X)$ with X a compact Hausdorff space, then a Hilbert C^*-module over B is the same as a continuous field of Hilbert spaces over X. Operator theory on Hilbert modules is developed analogously to the Hilbert space case, while taking into account several subtleties. An operator $T : X_B \to X_B$ is *adjointable* if there exists an operator $T^* : X_B \to X_B$ such that for all $x, y \in X_B$ it holds that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

Adjointable operators are automatically right B-linear and bounded. The set $\operatorname{End}^*_B(X)$ of all adjointable operators is a C^*-algebra in the operator norm. A pair of elements $x, y \in X_B$ induces an operator $|x\rangle\langle y| : X_B \to X_B$ via

$$|x\rangle\langle y| : (z) := x\langle y, z \rangle.$$

These operators are adjointable with $(|x\rangle\langle y|)^* = |y\rangle\langle x|$. The *algebra of compact endomorphisms* is the closed linear span of the operators $|x\rangle\langle y|$ and is denoted $\mathbb{K}(X)$.

3.2. Definition. Let (A, B) be a pair of (seperable) C^*-algebras with A unital. A *Kasparov module* for (A, B) is a tuple (X_B, F) consisting of

(1) a right Hilbert B-module X_B and a $*$-homomorphism $A \to \text{End}^*_B(X)$;
(2) an operator $F \in \text{End}^*_B(X_B)$ with $F - F^*, 1 - F^2$ and $[F, a] \in \mathbb{K}(X)$.

The module is *even* if there is self-adjoint unitary $\gamma \in \text{End}^*_B(X)$ such that $\gamma a = a\gamma$ for all $a \in A$ and $D\gamma + \gamma D = 0$. Otherwise the module is *odd*.

The operator F in a Kasparov module is sometimes referred to as an A-Fredholm operator. By imposing a certain homotpy equivalence relation on the set of all Kasparov modules, one obtains the groups $KK_*(A, B)$ as groups of equivalence classes, see **[7]**. The KK-groups recover K-theory and K-homology of a C^*-algebra by setting A resp. B equal to the complex numbers $\mathbb{C}$. Moreover there is a bilinear and associative pairing

$$KK_*(A, B) \times KK_*(B, C) \to KK_*(A, C),$$

the *Kasparov product*. It generalises all cohomological product operations at once. An element $x \in KK(A, B)$ induces a map

$$[x] \otimes - : KK(B, \mathbb{C}) \to KK_*(A, \mathbb{C}),$$

via the Kasparov product.

We now proceed with the unbounded picture of Kasparov theory. The theory of unbounded self-adjoint operators extends to C^*-modules with minor modifications. A closed, densly defined symmetric operator $D : \text{Dom } D \to X_B$ is *self-adjoint and regular* if $D \pm i : \text{Dom } D \to X_B$ are surjective. In this case the resolvents $(D \pm i)^{-1} \in \text{End}^*_B(X)$ and have norm $\leqslant 1$. As in the case of Hilbert spaces, self-adjoint regular operators admit a nice functional calculus for functions $f \in C(\mathbb{R})$ (see **[8]**).

3.3. Definition ([1]). Let (A, B) be a pair of (seperable) C^*-algebras with A unital. An *unbounded Kasparov module* for (A, B) is a triple $(\mathscr{A}, X_B, D)$ consisting of

(1) a right Hilbert B-module X_B and a $*$-homomorphism $A \to \text{End}^*_B(X)$;
(2) a self-adjoint regular operator $D : \text{Dom } D \to X_B$ with $(D \pm i)^{-1} \in \mathbb{K}(X)$;
(3) a dense subalgebra $\mathscr{A} \subset A$ such that $a : \text{Dom } D \to \text{Dom } D$ and $[D, a]$ is bounded.

The module is *even* is there is self-adjoint unitary $\gamma \in \text{End}^*_B(X)$ such that $\gamma a = a\gamma$ for all $a \in A$ and $D\gamma + \gamma D = 0$. Otherwise the module is *odd*.

This should be compared to the definition of spectral triple: the only difference is that we work on a Hilbert B-module instead of a Hilbert space. The link between bounded and unbounded Kasparov modules is given by direct construction. For a self-adjoint regular operator D its *bounded transform* is the operator

$$F_D := D(1 + D^2)^{-\frac{1}{2}}.$$

The operator $(1 + D^2)^{-\frac{1}{2}}$ maps to module X_B bijectively onto the domain of D. As such F_D contains all information about D, see **[8]**. For an unbounded Kasparov module $(\mathscr{A}, X, D)$ the tuple (X, F_D) is an (A, B) Kasparov module. Therefore unbounded Kasparov modules, and in paritucular spectral triples, define elements in KK-theory.

4. Extensions and KK-theory

A short exact sequence

$$0 \to B \to E \xrightarrow{q} A \to 0, \tag{4}$$

of C^*-algebras, with B and ideal in A (which we assume to be unital) is referred to as an extension. The extension is called *semi-split* if there is a completely positive linear map $\ell : A \to E$ such that $q \circ \ell = \mathrm{id}_A$. An extension induces an exact sequence

$$\cdots K^{*-1}(B) \xrightarrow{\partial} K^*(A) \to K^*(E) \to K^*(B) \xrightarrow{\partial} \cdots,$$

in K-homology and more generally in KK-theory. The map ∂ is called the *boundary map* associated to the extension. Kasparov realised that a semisplit extension (4) defines an element $[\mathrm{ext}] \in KK_1(A, B)$, in such a way that the boundary map in a long exact sequence induced from (4) is given by the Kasparov product with the class $[\mathrm{ext}]$.

4.1. Theorem (Kasparov **[7]**). *Suppose that the extension* (4) *is semi-split. Then it defines a class in* $KK_1(A, B)$.

The idea here is to form the tensor product $A \otimes B$ and define a positive definite inner product

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_\ell := b_1^* \ell(a_1^* a_2) b_2.$$

The completion in the associated norm gives a C^*-module $A \otimes_\ell B$. The operator

$$P(a \otimes b) := 1 \otimes \ell(b),$$

defines an *almost projection*in the module $A \otimes_\ell B$ in the sense that $P^2 - P, P^* - P \in \mathbb{K}(A \otimes_\ell B)$. Moreover A acts on this module from the left by multiplication and the commutators $[P, a] \in \mathbb{K}(A \otimes_\ell B)$. The tupel $(A \otimes_\ell B, 2P - 1)$ is the sought for Kasparov module.

4.2. Example (See **[10, 12]**). Denote by $\overline{\mathbf{H}^{n+1}}$ denote the closed ball in $\mathbb{R}^{n+1}$. The extension of (commutative) C^*-algebras

$$0 \to C_0(\mathbf{H}^{n+1}) \to C(\overline{\mathbf{H}^{n+1}}) \to C(S^n) \to 0,$$

is compatible with the G-action of each of them. For a discrete group $\Gamma \subset G$, each action is amenable and gives a an extension of crossed products

$$0 \to C_0(\mathbf{H}^{n+1}) \rtimes \Gamma \to C(\overline{\mathbf{H}^{n+1}}) \rtimes \Gamma \to C(S^n) \rtimes \Gamma \to 0. \tag{5}$$

The extension is semisplit and admits a splitting coming from harmonic analysis. Let

$$P : S^n \times \mathbf{H}^{n+1} \to \mathbb{R}, \quad P(\xi, x) := \frac{1 - \|x\|^2}{\|\xi - x\|^2},$$

be the *Poisson kernel* which satisfies $P(g(\xi), g(x)) = |g'(\xi)|^{-1} P(\xi, x)$ for all $g \in G$. Denote by ν_0 the Lebesgue measure on S^n. For each $x \in \mathbf{H}^{n+1}$ the stabilizer $G_x \subset G$ is a compact group. Since also S^N is a compact space there exists a unique G_x-invariant probability measure ν_x on S^n. It holds that

$$d\nu_x \xi = P(\xi, x)^n d\nu_0 \xi,$$

and for $f \in C(S^n)$ the function

$$x \mapsto \ell(f)(x) = \int_{S^n} f(\xi) d\nu_x \xi,$$

dfines an element of $C(\overline{\mathbf{H}^{n+1}})$, is hyperbolically harmonic in $\mathbf{H}^{n+1}$ and for every $\eta \in S^n$ it holds that

$$\lim_{x\to\eta}\int_{S^n} f(\xi)d\nu_x\xi = f(\eta).$$

Since we also have

$$\ell(f)(g(x)) = \int_{S^n} f(\xi)d\nu_{gx}\xi = \int_{S^n} f(g\xi)d\nu_{gx}g\xi = \int_{S^n} f(g\xi)d\nu_x\xi = \ell(f\circ g)(x),$$

the splitting is G-equivariant and thus also induces a splitting of (5). We write

$$T_1\mathbf{H}^{n+1} := S^n \times \mathbf{H}^{n+1},$$

as every pair (ξ, x) can be identified with a unique unit tangent vector at x. The construction of the module $C(S^n)\otimes_\ell C_0(\mathbf{H}^{n+1})$ yields the space $L^2(T_1\mathbf{H}^{n+1},\nu_x)$ of sections of the continuous fields of Hilbert spaces over $\mathbf{H}^{n+1}$ whose fiber at $x\in\mathbf{H}^{n+1}$ is $L^2(S^n,\nu_x)$. The inner product is given by

$$\langle\Phi,\Psi\rangle(x) := \int_{S^n}\overline{\Phi(\xi,x)}\Psi(\xi,x)d\nu_x\xi,$$

and the almost projection P is an actual projection namely

$$P\Phi(x,\eta) := \int_{S^n}\Phi(x,\xi)d\nu_x\xi,$$

the pointwise expectation operator. We have thus found that the tuple $(L^2(T_1\mathbf{H}^{n+1},\nu_x), 2P-1)$ represents the extension (5).

4.3. Example (see **[11]**)**.** The algebra O_N also appears as the quotient in an extension. Consider the vector space $\mathbb{C}^N$ with its standard inner product and define *Fock space* of $\mathbb{C}^N$ as

$$F_N := \bigoplus_{k=0}^{\infty} F_N^k, \quad F_N^k := (\mathbb{C}^N)^{\otimes k}, \quad (\mathbb{C}^N)^{\otimes 0} := \mathbb{C}.$$

We view F_N as a graded Hilbert space whose degree k part is F_N^k. For $v\in\mathbb{C}^N$ define a bounded linear operator

$$T_v : F_N \to F_N, \quad T_v(w_1\otimes w_k) := v\otimes w_1\otimes w_k.$$

Its adjoint acts as $T_v^*(w_1\otimes\cdots\otimes w_k) = \langle v, w_1\rangle w_2\otimes\cdots\otimes w_k$ for $k\geqslant 1$ and as 0 for $k=0$. The *Toeplitz algebra* $T_N \subset B(F_N)$ is the C^*-algebra generated by T_v, $v\in\mathbb{C}^N$. Let e_i be the standard orthonormal basis of $\mathbb{C}^N$ and write $T_i := T_{e_i}$. We then have $T_i^*T_i = 1$ and $T_iT_i^* = 1 - p_0$, where p_0 is the projection onto the degree 0 part of $F(\mathbb{C}^N)$. It then follows that T_N contains $\mathbb{K}(F_N)$ and by simplicity of O_N it follows that $O_N = T_N/\mathbb{K}(F_N)$. Note that $\mathbb{K}(F_N)$ is Morita equivalent to $\mathbb{C}$.

4.4. Proposition. *Under the Morita equivalence isomorphism $KK_1(O_n,\mathbb{K}(F_N))\simeq KK_1(O_N,\mathbb{C})$, the class of the extension*

$$0\to\mathbb{K}(F_N)\to T_N\to O_N\to 0, \tag{6}$$

generates the group $K^1(O_n) = KK_1(O_N,\mathbb{C})\simeq\mathbb{Z}/(N-1)\mathbb{Z}$.

Consider the set $X_N := \{(x, n, \sigma^n(x)) : x \in \Omega_N\} \subset G_N$ and $X_N^\mu = X_N \cap r^{-1}(C_\mu)$. The subspace $L^2(X_N, \mu)$ is isomorphic to F_N, via a unitary isomorphism $u : F_N \to L^2(X_N, \mu)$ mapping $e_w = e_{w_1} \otimes \cdots \otimes e_{w_{|w|}}$ to the normalised indicator function of X_N^μ. Now take $P_N \in B(L^2(X, \mu))$ to be the orthogonal projection onto $L^2(X_N)$. Then

$$T_i = u^* P_N S_i P_N u = u^* S_i u,$$

which gives a completely positive splitting of the extension (6). The tuple $(L^2(G_N, \mu), 2P_N - 1)$ is a Kasparov module and its class represents the extension (6) after Morita equivalence.

5. Spectral triples

5.1. Kleinian groups. The action of a Kleinian group Γ on the sphere S^n does not admit an invariant metric. For computational purposes, we do not consider the round metric on S^n, but rather the *chordal distance* between boundary points ξ, η, defined to be $\|\xi - \eta\|$ using the ambient Euclidean metric on the ball. As with the field of harmonic measures, we consider a family of metrics on S^n paramtetrized by $x \in \mathbf{H}^{n+1}$. We introduce the metrics

$$d_x(\xi, \eta) := P(\xi, x)^{1/2} P(\eta, x)^{1/2} \|\xi - \eta\|,$$

which satisfy the invariance condition $d_{gx}(g\xi, g\eta) = d_x(\xi, \eta)$. The metrics d_x are mutually Lipschtiz equivalent, but the constant of equivalence C_x determined by

$$C_x^{-1} d_x(\xi, \eta) \leqslant d_0(\xi, \eta) \leqslant C_x d(\xi, \eta),$$

becomes large as x approaches the boundary. We denote by $\mathrm{Lip}_x(S^n)$ the Lipschitz functions on S^n for the metrix d_x. We now consider the operator

$$\Delta\Psi(\xi, x) := \int_{S^n} \frac{\Psi(\xi, x) - \Psi(\eta, x)}{d_x(\xi, \eta)^n} d\nu_x \eta.$$

We view Δ as a densely defined operator in $L^2(T_1(\mathbf{H}^{n+1}), \nu_x)$. Denoting the hyperbolic distance in $\mathbf{H}^{n+1}$ by $\rho(x, y)$ we define the function $\rho(x) = \rho_0(x) = \rho(0, x)$.

5.1. Theorem ([9]). *The operator Δ is self-adjoint and regular and $\Delta + \rho$ is positive. Moreover $P\Delta = \Delta P = 0$ and the operator*

$$S := (2P - 1)(\Delta + \rho) = -\Delta + \rho(2P - 1),$$

is self-adjoint, regular and G-invariant in the module $L^2(T_1(\mathbf{H}^{n+1}), \nu_x)$. The triple

$$(\mathrm{Lip}_0(S^n), L^2(T_1(\mathbf{H}^{n+1}), \nu_x), S),$$

is an unbounded representative for the extension (4) *in the group $KK_1^G(A, B)$.*

Now let $\not{D}$ be the hyperbolic Dirac operator on $\mathbf{H}^{n+1}$, acting in the spinor bundle $\not{S}$ with grading operator σ and μ the hyperbolic volume element. To obtain a spectral triple for $C(S^n) \rtimes \Gamma$ we couple the operator S above to the Dirac operator $\not{D}$ and obtain the following theorem.

5.2. Theorem. *For any discrete subgroup $\Gamma \subset G$ the data*

$$(\mathrm{Lip}_0(S^n) \rtimes^{\mathrm{alg}} \Gamma, L^2(T_1(\mathbf{H}^{n+1}), \not{S}, \nu_x \otimes \mu), S \otimes \sigma + \not{D}),$$

defines a spectral triple. Its class is the image of the Dirac class $[\not{D}] \in K^*(C_0(\mathbf{H}^{n+1}) \rtimes \Gamma)$ under the boundary map

$$\partial : K^*(C_0(\mathbf{H}^{n+1}) \rtimes \Gamma) \to K^{*+1}(C(S^n) \rtimes \Gamma),$$

induced by the extension (5).

The boundary map ∂ is known to be injective in odd dimensions as well as for noncocompact groups Γ.

5.2. The algebra O_N. For the Cuntz algebra O_N, we consider the boundary map

$$\partial : K^0(\mathbb{K}(F_N)) \to K^1(O_N),$$

associated to the Toeplitz extension (6), which is well-known to be surjective. After Morita equivalence this is a map $K^0(\mathbb{C}) \to K^1(O_N)$ and $K^0(\mathbb{C}) \simeq \mathbb{Z}$, so the image of the generator of this group generates $K^1(O_N)$. To construct a spectral triple for O_N generating $K^1(O_N)$ we proceed analogously to the Kleinian group case. Recall that an *extended metric* on a space is a metric that allows pairs of points ot be at infinite distance. Also recall that the Hausdorff dimension of Ω_N and hence of G_N is $\log N$.

5.3. Theorem ([5, 6]). *There exists an extended metric ρ_N on the groupoid G_N such that the operator*

$$\Delta_N f(\xi) := \frac{1}{(1-N^{-1})}\int_{G_N} \frac{f(\xi)-f(\eta)}{\rho(\xi,\eta)^{\log N}} d\mu_N(\eta),$$

is densely defined and self-adjoint. Then $(\mathrm{Lip}(G_N, \rho_N), L^2(G_N, \mu), -\Delta_N + 2(P_N - 1)|c|)$, *with $c(x, n, y) = n$, is a spectral triple on O_N that represents the class of the extension* (6) *and thus generates $K^1(O_N)$.*

This spectral triple has the remarkable property that it is not finitely summable but the operator $F_D = D|D|^{-1}$ is finitely summable.

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