Hecke operators, KK-theory and arithmetic groups

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K-theory, Hecke algebras and representation theory
marking Roger Plymen's 75th birthday
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Geometry and number theory

Bianchi groups and Bianchi manifolds:

- Let K be an imaginary quadratic field, $K = \mathbb{Q}(\sqrt{-d})$ for some square free positive integer d
- $O_K \subset K$ its ring of integers
- a torsion free finite index subgroup $\Gamma \subset PSL_2(O_K)$
- $\Gamma \subset PSL_2(\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3) = G$
- $M = \mathbb{H}^3/\Gamma$ noncompact hyperbolic manifold
- $H^*(M, \mathbb{Z}) = H^*(\Gamma, \mathbb{Z})$
- $H_*(M, \mathbb{Z}) = H_*(\Gamma, \mathbb{Z})$

Hecke operators on (co)homology

Generel setting: Γ discrete subgroup of a Lie group G

- For $g \in G$, write $\Gamma_g := \Gamma \cap g\Gamma g^{-1} \subset \Gamma$
- $C_G(\Gamma) = \{g \in G : [\Gamma : \Gamma_g], [\Gamma : \Gamma_{g^{-1}}] < \infty\}$

Hecke operator

$$\begin{array}{ccc} H^*(\Gamma, \mathbb{Z}) & \xrightarrow{T_g} & H^*(\Gamma, \mathbb{Z}) \\ \text{res} \downarrow & & \uparrow \text{cores} \\ H^*(\Gamma_g, \mathbb{Z}) & \xrightarrow{\mathrm{Ad} g} & H^*(\Gamma_{g^{-1}}, \mathbb{Z}) \end{array} \tag{1}$$

The double coset algebra

T_g depends only on the double coset $\Gamma g \Gamma$

$$\mathbb{Z}[\Gamma, G] = \left\{ \sum_{g \in C_G(\Gamma)} z_g [\Gamma g \Gamma] : z_g \in \mathbb{Z}, z_g \neq 0 \text{ for finitely many } g \right\}$$

with *Shimura product*:

$$[\Gamma g^{-1} \Gamma] \cdot [\Gamma h^{-1} \Gamma] := \sum_{k=1}^{K} m_k [\Gamma g_{i(k)} h_{j(k)} \Gamma].$$

$g_{i(k)}, h_{j(k)}$ coset representatives for $\Gamma_{g^{-1}}, \Gamma_{h^{-1}} \subset \Gamma$ such that

$$\Gamma g^{-1} \Gamma h^{-1} \Gamma = \bigsqcup_{k=1}^{K} \Gamma g_{i(k)} h_{j(k)} \Gamma$$

$H^*(\Gamma, \mathbb{Z})$ is a module over $\mathbb{Z}[\Gamma, G]$ via the Hecke action.

Geometric picture

Γ torsion free, $\quad \mathbb{H} := K \setminus G$ symmetric space

For $g \in C_G(\Gamma)$, set $M_g := \mathbb{H}/\Gamma_g$ and $M_{g^{-1}} := \mathbb{H}/\Gamma_{g^{-1}}$

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\ \downarrow & & \downarrow \\ M_g & \xrightarrow{g} & M_{g^{-1}} \\ {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle \pi_{g^{-1}}} \\ M & & M \end{array} \tag{2}$$

Hecke correspondence $M \xleftarrow{\pi_{g^{-1}} \circ g} M_g \xrightarrow{\pi_g} M$

$$T_g := (\pi_{g^{-1}} \circ g)^* \circ \pi_{g!} : H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{Z})$$

Modular forms

Classically, for a finite index torsion-free subgroup $\Gamma \subset PSL_2(\mathbb{Z})$, there is a *Hecke equivariant* isomorphism

$$S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \oplus \mathrm{Eis}(\Gamma) \to H^1(\Gamma, \mathbb{C}),$$

where $S_2(\Gamma)$ is the space of weight 2 holomorphic cusp forms:

$$f : \mathbb{H} \to \mathbb{C}, \quad (f \circ \gamma)(z) = (cz + d)^2 f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Such f defines a cocycle $c : \Gamma \to \mathbb{C}$ via

$$c(\gamma) := \int_w^{w\gamma} f(z)dz, \quad w \in \mathbb{H}$$

and the Hecke operators on $S_2(\Gamma)$ are defined explicitly.

For $\Gamma = SL(2,\mathbb{Z})$ and the element

$$g := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{Q}),$$

we have

$$\Gamma \cap g^{-1}\Gamma g = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod p \right\}.$$

The operator T_g arising from this element corresponds to the Hecke operator T_p on cusp forms under the aforementioned isomorphism.

Motivation

In the study of higher dimensional modular forms, the cohomology of Γ as a *Hecke module* plays a pivotal rôle.

The absence of a complex structure in dimension 3 creates the need for new tools to study the Hecke modules

$$H^1(\Gamma, \mathbb{Z}), \quad H^1(M, \mathbb{Z}).$$

There is a third picture in which modular forms appear as distributions on the boundary $\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$.

We propose the use of Fredholm operators and analytic K-homology to study the action of Γ on $\mathbb{H}, \overline{\mathbb{H}}$ and $\partial\mathbb{H}$.

(Generalised) (co)homology for C^*-algebras

KK-theory for C^*-algebras provides a unifying framework to study Hecke modules from all these perspectives at once.

To a pair of separable C^*-algebras, Kasparov associated a graded abelian group $KK_*(A, B)$ in such a way that:

- $KK_*(A, \mathbb{C}) \simeq K^*(A)$, the K-homology of A
- $KK_*(\mathbb{C}, A) \simeq K_*(A)$, the K-theory of A
- there is an associative, bilinear product $KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C)$.

Geometric picture of KK-theory

Prototype: $(C(M), L^2(\wedge^* M), d + d^*) \in KK_0(C(M), \mathbb{C})$

$KK_0(A, B)$ are given by *unbounded Fredholm modules* (X, D):

- A Hilbert C^*-bimodule ${}_AX_B$
- a self-adjoint regular operator $D : \mathrm{Dom}\, D \to {}_AX_B$ which is A-Fredholm
- Kasparov product:

$$(X, S) \otimes_B (Y, T) := (X \otimes_B Y, S \otimes 1 + 1 \otimes_\nabla T),$$

where $\nabla : \mathcal{X} \to X \otimes \Omega^1_T$ is a densely defined *connection*.

Lemma

Let $g \in C_G(\Gamma)$ and $M \xleftarrow{\pi_{g^{-1}} \circ g} M_g \xrightarrow{\pi_g} M$ the associated Hecke correspondence. The C^*-algebra $C_0(M_g)$ can be made into a C^*-$C_0(M)$-bimodule, whose left action is by compact operators.

The above bimodule is denoted T_g^M and defines an element $[T_g^M] \in KK_0(C_0(M), C_0(M))$.

Hecke operators in KK-theory

Lemma

Let B be a $C_G(\Gamma)$-C^*-algebra and $g \in C_G(\Gamma)$. The space $C_c(\Gamma g^{-1}\Gamma, B)$ admits a completion into a $B \rtimes_r \Gamma$-bimodule T_g^Γ whose left action is by compact operators.

Thus for $g \in C_G(\Gamma)$ we obtain a $B \rtimes_r \Gamma$-bimodule T_g^Γ and an element $[T_g^\Gamma] \in KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$.

The Shimura product in KK-theory

Theorem (Sengun-M.)

For any $C_G(\Gamma)$-C^*-algebra B, the map

$$\mathbb{Z}[\Gamma, G] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma), \quad [\Gamma g \Gamma] \mapsto T^{\Gamma}_{g^{-1}},$$

is a ring homomorphism.

Morita compatibilty

Γ torsion free, $\quad \mathbb{H} := K \setminus G, \quad M := \mathbb{H}/\Gamma.$

Morita equivalence bimodule X for $(C_0(\mathbb{H}) \rtimes \Gamma, C_0(M))$.

Proposition (Sengun, M.)

There is a unitary isomorphism of $(C_0(\mathbb{H}) \rtimes \Gamma, C_0(M))$-bimodules

$$T_g^{\Gamma} \otimes_{C_0(\mathbb{H}) \rtimes \Gamma} X \xrightarrow{\sim} X \otimes_{C_0(M)} T_g^{M}.$$

In particular

$$[T_g^{\Gamma}] \otimes [X] = [X] \otimes [T_g^{M}] \in KK_0(C_0(\mathbb{H}) \rtimes \Gamma, C_0(M)).$$

Morita compatibilty

Corollary

Under the composition

$$\mathbb{Z}[\Gamma, G] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma) \xrightarrow{\sim} KK_0(C_0(M), C_0(M)),$$

the Shimura product $[\Gamma g^{-1}\Gamma] \cdot [\Gamma h^{-1}\Gamma]$ *maps to the composition of correspondences* $[M \xleftarrow{\tau_g} M_{g\,\pi_g} \times_{\tau_h} M_h \xrightarrow{\pi_h} M]$.

The isomorphism $K^*(C_0(\mathbb{H}) \rtimes \Gamma) \xrightarrow{\otimes[X]} K^*(C_0(M))$ is a $\mathbb{Z}[\Gamma, G]$-module map.

Chern character

Theorem (M. -Sengun)

Let $M := \mathbb{H}/\Gamma$ *be a locally symmetric space. The Chern character*

$$\mathrm{Ch} : K^*(M) \to \bigoplus_i H^{*+i}_{DR}(M),$$

is a homomorphism of $\mathbb{Z}[\Gamma, G]$*-modules.*

The descent theorem

Theorem (Sengun-M.)

Let A and B be $C_G(\Gamma)$-C^*-algebras, $x \in KK_n^{C_G(\Gamma)}(A, B)$ and

$$j_\Gamma : KK_n^\Gamma(A, B) \to KK_n(A \rtimes_r \Gamma, B \rtimes_r \Gamma),$$

the descent map. For any separable C^*-algebra C the induced maps

$$j_\Gamma(x) \otimes : KK_i(B \rtimes \Gamma, C) \to KK_{i+n}(A \rtimes \Gamma, C),$$

$$\otimes j_\Gamma(x) : KK_i(C, A \rtimes \Gamma, C) \to KK_{i+n}(C, B \rtimes \Gamma),$$

are homomorphisms of $\mathbb{Z}[\Gamma, G]$-modules..

Extensions and Hecke equivariant exact sequences

Corollary

Let

$$0 \to B \to E \to A \to 0,$$

be a $C_G(\Gamma)$-equivariant and Γ-exact extension of $C_G(\Gamma)$-C^*-algebras. Then the induced exact sequences in KK-theory are Hecke equivariant.

Borel-Serre compactification

The Borel-Serre compactification $(\overline{M}, \partial\overline{M})$ is constructed via a partial $C_G(\Gamma)$-compactification $\overline{\mathbb{H}}^{BS}$ of $\mathbb{H}$.

The C^*-extension

$$0 \to C_0(\mathbb{H}) \to C(\overline{\mathbb{H}}^{BS}) \to C_0(\partial\overline{\mathbb{H}}^{BS}) \to 0$$

is $C_G(\Gamma)$-equivariant.

Any torsion free Γ acts freely and properly on $\overline{\mathbb{H}}^{BS}$ and $\overline{\mathbb{H}}^{BS}/\Gamma = \overline{M}$.

$$\cdots K^i(\partial\overline{M}) \to K^i(\overline{M}) \to K^i(M) \to K^{i-1}(\partial\overline{M}) \to \cdots,$$

is a Hecke equivariant exact sequence in topological K-theory.

Geodesic compactification

The extension

$$0 \to C_0(\mathbb{H}) \to C(\overline{\mathbb{H}}) \to C(\partial\mathbb{H}) \to 0,$$

is G-equivariant.

$C_0(\mathbb{H}) \rtimes \Gamma \sim C_0(M)$ Morita equivalence

$C(\overline{\mathbb{H}}) \rtimes \Gamma \sim C_r^*(\Gamma)$ KK-equivalence (Meyer-Nest)

Gysin sequence

Theorem (Sengun-M.)

The exact sequence

$$\begin{array}{ccccc} K^0(C_0(M)) & \xrightarrow{\partial} & K^1(C(\partial\mathbb{H}) \rtimes \Gamma) & \longrightarrow & K^1(C_r^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K^0(C_r^*(\Gamma)) & \longleftarrow & K^0(C(\partial\mathbb{H}) \rtimes \Gamma) & \xleftarrow[\partial]{} & K^1(C_0(M)) \end{array}$$

is Hecke equivariant.

Back to dimension 3

Theorem (Sengun-M.)

Let Γ be a discrete torsion free noncocompact subgroup of $PSL_2(\mathbb{C})$. There are explicit $\mathbb{Z}[\Gamma, G]$-module isomorphisms

$$H^1(\Gamma,\mathbb{Z}) \to K^1(C_r^*(\Gamma)), \quad H_2(\overline{M},\partial\overline{M},\mathbb{Z}) \to K^0(C_0(M)).$$

$$H^1(\Gamma,\mathbb{Z}) \oplus H_2(\overline{M},\partial\overline{M},\mathbb{Z}) \to K^1(C(\partial\mathbb{H}) \rtimes \Gamma).$$

Under these isomorphisms, the cohomology pairing $H_* \times H^* \to \mathbb{Z}$ corresponds to the index pairing $K_* \times K^* \to \mathbb{Z}$.

Here $\overline{M}$ denotes the Borel-Serre compactification.

All these isomorphisms are constructed via explicit spectral triples.

Geometric K-homology of M

The isomorphisms are obtained by constructing explicit unbounded Fredholm modules associated to a cohomology class.

The K-homology hexagon simplifies to

$$0 \to K^0(C_0(M)) \to K^1(C(\partial\mathbb{H}) \rtimes \Gamma) \to K^1(C_r^*(\Gamma)) \to 0.$$

Elements of $H_2(\overline{M}, \partial\overline{M})$ are given by properly embedded surfaces $(N, \partial N) \subset (\overline{M}, \partial\overline{M})$.

This endowes $\mathring{N} \subset M$ with a complete Riemannian metric and a self-adjoint Dirac operator

$$(C_0(M), L^2(\mathring{N}, \mathcal{S}), D_{\mathring{N}}) \in K^0(C_0(M))$$

Cocycle $c : \Gamma \to \mathbb{Z}$, kernel $\Gamma_c \subset \Gamma$

Hodge DeRham operator $D_{HR} = d + d^*$

Distance function $\rho(x) = d_{\mathbb{H}}(0, x)$

$$(C_r^*(\Gamma_c), L^2(\wedge^*\mathbb{H}), \gamma_{\mathbb{H}} := D_{HR} + \rho^s \hat{c}(d\rho))$$

Unbounded Fredholm module

$$(C_r^*(\Gamma), C_r^*(\Gamma) \otimes_{C_r^*(\Gamma_c)} L^2(\wedge^*\mathbb{H}), c \otimes \sigma + 1 \otimes \gamma_{\mathbb{H}})$$

Extends to $C(\partial\mathbb{H}) \rtimes \Gamma$.

Baum-Connes assembly map

For $\Gamma \subset G$ discrete, torsion free and cocompact.

Baum-Connes assembly map $\mu : K_{\Gamma}^{*}(\underline{E}\Gamma) \to K_{*}(C_r^{*}(\Gamma))$

Is it a Hecke module map?

Work in progress

$\underline{E}\Gamma = K \setminus G = \mathbb{H}$ classifying space for proper actions

$K_{\Gamma}^{*}(\underline{E}\Gamma) \simeq K_{*}(M)$ with $M = \mathbb{H}/\Gamma$.

The assembly map is the composition

$$K_{*}(M) \xrightarrow{PD} K^{*}(M) \xrightarrow{\otimes\alpha} K_{*}(C_{r}^{*}(\Gamma))$$

with PD the Poincaré duality map and $\alpha \in KK_{n}(C(M), C_{r}^{*}(\Gamma))$ the image of the Dirac element in $KK_{n}^{G}(\mathbb{C}, \mathbb{C})$ after descent and Morita equivalence.

Work in progress

Proposition

Let $M := \mathbb{H}/\Gamma$ be a compact locally symmetric space. Then the Poincaré duality map

$$PD^{-1} : K^*(M) \to K_*(M)$$

is Hecke equivariant.

Corollary

Let $\Gamma \subset G$ be cocompact and torsion free. Then the Baum-Connes assembly map

$$\mu : K_*^{\Gamma}(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma)),$$

is Hecke equivariant.

Outlook

- Generalize to general discrete groups
- Potential applications to the case $\Gamma = SL(3, \mathbb{Z})$ and other arithmetic groups

Summary

- Arithmetic groups and manifolds, cohomology and modular forms
- Hecke operators in KK-theory
- Morita equivalence and Chern character are compatible with Hecke operators
- All classical Hecke modules arise from KK-theory
- Hecke equivariant exact sequences from (noncommutative) $C_G(\Gamma)$ compactifications of locally symmetric spaces
- Hecke equivariance of the Baum-Connes assembly map