KK-theory in geometry and physics

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The *index* of a linear map $L : V \to W$ between finite dimensional vector spaces:

$$\mathrm{Ind}L := \dim\ \ker L - \dim\ \mathrm{coker} L = \dim\ \ker L - \dim\ \ker L^*.$$

Dimension theorem $\Rightarrow \mathrm{Ind}L = \dim V - \dim W$.

Infinite dimensional spaces:

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad S(e_i) = e_{i+1} \quad \mathrm{Ind}S = -1.$$

Let H, K be Hilbert spaces.

A closed operator $D : \text{Dom } D \subset H \to K$ with densely defined adjoint is *Fredholm* if

- D has closed range,
- $\ker D$ and $\ker D^*$ are finite dimensional.

The index

$$\text{Ind}D := \dim \ \ker D - \dim \ \ker D^*$$

of Fredholm operators is well defined.

Theorem

The index is invariant under homotopies of Fredholm operators.

Hardy space on the circle:

$$H^2 = H^2(S^1) := \left\{ f \in L^2(S^1) : f = \sum_{n\geq 0} a_n e^{2\pi i n x} \right\} \simeq \ell^2(\mathbb{N}).$$

$P : L^2(S^1) \to H^2(S^1)$ orthogonal projection.

Toeplitz operator with symbol $f \in C(S^1)$:

$$T_f : H^2 \to H^2, \quad \phi \mapsto P(f)\phi.$$

Theorem (Gohberg–Krein)

For $f : S^1 \to \mathbb{C}^\times$ the operator $T_f : H^2 \to H^2$ is Fredholm and

$$\operatorname{Ind} T_f = -w(f),$$

with $w(f)$ the winding number of f. If f is C^1 then $w(f) = \int_{S^1} \frac{f'(z)}{f(z)} dz$.

M compact n-dimensional manifold, $S_\pm \to M$, smooth vector bundles.

$D_+ : C^\infty(M, S_+) \to C^\infty(M, S_-)$ first order elliptic differential operator.

$D_+ = a_0(x) + \sum_{i=1}^n a_i(x)\partial_i$ locally.

D is of *Dirac type* if:

- D^2 is a generalized Laplacian,
- for $f \in C^\infty(M)$ we have $[D, f] = c(df)$.

Here $c(df)$ denotes *Clifford mutliplication* by df.

M Riemannian manifold

$$S_+ := \bigwedge^{\mathrm{ev}} T^*M, \quad S_- := \bigwedge^{\mathrm{odd}} T^*M,$$

$$D_+ = d + d^* : C^\infty(M, \bigwedge^{\mathrm{ev}} T^*M) \to C^\infty\left(M, \bigwedge^{\mathrm{odd}} T^*M\right)$$

$$c(df) = \epsilon(df) - \iota(df^\sharp)$$

$E \to M$ another vector bundle.

$\nabla : C^\infty(M, E) \to C^\infty(M, E) \otimes \Omega^1(M)$ connection.

$\nabla(\psi f) = \nabla(\psi)f + \psi \otimes df \quad$ for $f \in C^\infty(M), \psi \in C^\infty(M, E)$.

$$\begin{aligned} 1 \otimes_\nabla D_+ : C^\infty(M, E \otimes S_+) &\to C^\infty(M, E \otimes S_-) \\ e \otimes \psi &\mapsto e \otimes D_+\psi + \nabla(e)\psi \end{aligned}$$

Proposition

The operator $1 \otimes_\nabla D_+ : C^\infty(M, E \otimes S_+) \to C^\infty(M, E \otimes S_-)$ *is of Dirac type. It extends to a densely defined closed Fredholm operator* $L^2(M, E \otimes S_+) \to L^2(M, E \otimes S_-)$.

Theorem (Atiyah–Singer)

Let M be a compact manifold, $D_+ : C^\infty(M, S_+) \to C^\infty(M, S_-)$ a Dirac type operator and $E \to M$ a vector bundle. For any connection $\nabla : C^\infty(M, E) \to C^\infty(M, E) \otimes \Omega^1(M)$ we have

$$\mathrm{Ind}(1 \otimes_\nabla D_+) = \int_M \mathrm{ch}(E)\widehat{\mathrm{A}}(S) = \langle E, D_+ \rangle \in \mathbb{Z},$$

where $\mathrm{ch}(E)$ and $\widehat{\mathrm{A}}(S)$ are characteristic classes of E and $S = S_+ \oplus S_-$ respectively.

- $D = d + d^*$ acting in $L^2(M, \bigwedge^* T^*M)$;

$$\mathrm{Ind}(d + d^*) = \mathrm{Eul}(M) = (2\pi)^{-n/2} \int_M \mathrm{Pf}(-R),$$

  where R is the Riemannian curvature;

- For M a surface and r_M the scalar curvature we recover

$$\mathrm{Eul}(M) = \frac{1}{2\pi} \int_M r_M dx$$

  the classical Gauss–Bonnet theorem.

A vector bundle E pairs with an elliptic operator D to give an integer:

$$\langle E, D\rangle := \mathrm{Ind}(1 \otimes_{\nabla} D) \in \mathbb{Z}.$$

This integer is a topological invariant computed by geometric methods.

Vector bundles generate the K-theory group $K^0(M)$.

Elliptic operators generate the K-homology group $K_0(M)$.

Index pairing $K^0(M) \times K_0(M) \to \mathbb{Z}, \quad (E, D) \mapsto \mathrm{Ind}(1 \otimes_\nabla D)$

The Chern character

$$\mathrm{Ch} : K^0(M) \to H^{ev}(M, \mathbb{R}) := \bigoplus_i H^{2i}(M, \mathbb{R})$$

is a rational isomorphism onto cohomology.

The classical index theorems can be viewed as a calculation of the index pairing:

$$\begin{array}{ccccc} K^0(M) & \times & K_0(M) & \longrightarrow & \mathbb{Z} \\ \downarrow{\scriptstyle Ch} & & \downarrow{\scriptstyle Ch} & & \downarrow \\ H^{ev}(M,\mathbb{R}) & \times & H_{ev}(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \end{array}$$

C^*-algebra: closed $*$-subalgebra of $B(H)$.

For a locally compact Hausdorff space X, $C_0(X)$ is a C^*-*algebra*.

Theorem (Gelfand–Naimark)

Any commutative C^*-algebra is $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

Theorem (Serre–Swan)

For a compact Hausdorff space X and a locally trivial complex vector bundle $E \to X$, the module of sections $\Gamma(X, E)$ is a finitely generated projective $C(X)$-module. Conversely every finitely generated projective $C(X)$-module arises in this way.

- The *Toeplitz algebra* $\mathcal{T} \subset B(\ell^2(\mathbb{N}))$ is the C^*-algebra generated by the shift $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$;
- *Group C^*-algebras*: for a discrete group Γ, $C_r^*(\Gamma) \subset B(\ell^2(\Gamma))$ is the closure of $\mathbb{C}[\Gamma]$ acting by convolution;
- *Crossed products*: if Γ acts on X by homeomorphisms we form $C_0(X) \rtimes \Gamma \subset B(L^2(X \times \Gamma))$;

Theorem

Suppose that Γ acts freely and properly on the locally compact Hausdorff space X. Then the C^*-algebras $C_0(X/\Gamma)$ and $C_0(X) \rtimes \Gamma$ are Morita equivalent.

$K_0(A) := \{[P] - [Q] : P, Q$ finitely generated projective modules$\}$.

Definition

A *spectral triple* $(\mathcal{A}, H, D)$ for A consists of

- a Hilbert space $H = H_+ \oplus H_-$ such that A is represented on $H_\pm$,
- $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ for a closed operator $D_+ : H_+ \to H_-$, $D_- := D_+^*$ and such that $a(1 + D^2)^{-1}$ is compact a operator,
- a dense subalgebra $\mathcal{A} \subset A$ such that for $a \in \mathcal{A}$ the commutators $[D_+, a]$ are bounded.

$K^0(A) := \{[(A, H, D)] :$ homotopy classes of spectral triples$\}$.

For a manifold M, $(L^2(\wedge^* T^* M), d + d^*) \in K^*(C(M))$ is a spectral triple.

$[D] \in K^0(A)$ pairs with $[P] \in K_0(A)$ by choosing a connection

$$\nabla : P \to P \otimes_{\mathcal{A}} \Omega^1_D, \quad \Omega^1_D := \left\{ \sum a_i [D, b_i] : a_i, b_i \in \mathcal{A} \right\}$$

forming a connection operator

$$1 \otimes_\nabla D : P \otimes_A H \to P \otimes_A H$$

and taking its index:

$$K^0(A) \times K_0(A) \to \mathbb{Z}, \qquad (D, P) \mapsto \mathrm{Ind}(1 \otimes_\nabla D)$$

For a pair of C^*-algebras Kasparov constructed graded abelian group $KK_*(A, B)$ such that

- $KK_*(\mathbb{C}, A) \simeq K_*(A)$, the K-theory of A;
- $KK_*(A, \mathbb{C}) \simeq K^*(A)$, the K-homology of A;
- associative, bilinear product
  $KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C)$;
- recovers index pairing via
  $KK_0(\mathbb{C}, A) \times KK_0(A, \mathbb{C}) \to KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$;
- $x \in KK_0(A, B)$ defines a map $K_0(A) \to K_0(B)$;
- $KK_0(B, B)$ is a ring and $K^*(B), K_*(B)$ are modules over this ring.

The cycles for $KK_0(A, B)$ are given by pairs (X, S) consisting of

- A bimodule ${}_AX_B$;
- an operator $S : \mathrm{Dom}S \to {}_AX_B$ satisfying some analytic constraints, in particular S is B-linear but not A-linear.

Theorem (Mesland–Rennie 2015)

Up to equivalence, the Kasparov product for (X, S) and (Y, T) is given by

$$(X, S) \otimes_B (Y, T) := (X \otimes_B Y, S \otimes 1 + 1 \otimes_\nabla T),$$

where $\nabla : \mathcal{X} \to X \otimes \Omega^1_T$ is a densely defined connection.

A self-adjoint operator $D : \mathcal{D}(D) \to E$ in a Hilbert module E is *regular* if $D \pm i : \mathcal{D}(D) \to E$ are surjective

$S : \mathcal{D}(S) \to E$ and $T : \mathcal{D}(T) \to E$ self-adjoint regular operators

$\mathcal{F}(S,T) := \{x \in \mathcal{D}(S) \cap \mathcal{D}(T) : Sx \in \mathcal{D}(T), \quad Tx \in \mathcal{D}(S)\} \subset E$

$[S,T] := ST + TS : \mathcal{F}(S,T) \to E$

Theorem (Lesch–Mesland 2018)

Suppose that

- $(S+i)^{-1}\mathcal{D}(T) \subset \mathcal{F}(S,T)$
- $\forall x \in \mathcal{F}(S,T) \quad \langle [S,T]x, [S,T]x\rangle \leq C(\langle x,x\rangle + \langle Sx,Sx\rangle + \langle Tx,Tx\rangle).$

Then $S+T : \mathcal{D}(S) \cap \mathcal{D}(T) \to E$ *and* $S^2+T^2 : \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \to E$ *are self-adjoint and regular.*

- Factorisation of Dirac operators on G-principal bundles, for example the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$;
- Hecke operators and K-theory of arithmetic groups and manifolds;
- KK-factorisation and bulk-boundary correspondence for topological insulators.

Fuchsian groups and surfaces:

- $\mathbb{H}$ complex upper half plane and $G := PSL_2(\mathbb{R}) = \mathrm{Isom}(\mathbb{H})$
- a torsion free discrete index subgroup $\Gamma \subset G$
- $M = \mathbb{H}/\Gamma$ noncompact hyperbolic surface
- $H^*(M, \mathbb{Z}) = H^*(\Gamma, \mathbb{Z})$

A *weight 2 modular form* for Γ is a holomorphic function

$$f : \mathbb{H} \to \mathbb{C}, \quad (f \circ \gamma)(z) = (cz + d)^2 f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$\omega_f := f(z)dz$ defines a holomorphic differential form on $M := \mathbb{H}/\Gamma$

$C_G(\Gamma) = \{g \in G : [\Gamma : \Gamma_g], [\Gamma : \Gamma_{g^{-1}}] < \infty\} \quad \Gamma_g := \Gamma \cap g\Gamma g^{-1}$

The *Hecke operators* on modular forms are defined explicitly:

$$T_g(f)(z) := \sum_{i=1}^{d} (c_i z + d_i)^{-2} f(g_i z), \qquad \Gamma g \Gamma = \bigsqcup_{i=1}^{d} g_i \Gamma,$$

for any $g \in C_G(\Gamma)$ and encode deep arithmetic information.

- $M := \mathbb{H}/\Gamma$
- $M_g := \mathbb{H}/\Gamma_g$ with pair of covering maps

$$\begin{tikzcd} & M_g \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ M & & M \end{tikzcd}$$

- Hecke operator $T_g := \pi_* \circ \varphi_! : H_*(M, \mathbb{Z}) \to H_*(M, \mathbb{Z})$
- Similarly $T_g : H^*(\Gamma, \mathbb{Z}) \to H^*(\Gamma, \mathbb{Z})$
- After Connes-Skandalis $[M \leftarrow M_g \to M] \in KK_0(C_0(M), C_0(M))$.

For a finite index torsion-free subgroup $\Gamma \subset PSL_2(\mathbb{Z})$, there is a *Hecke equivariant* isomorphism

$$S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \oplus \mathrm{Eis}(\Gamma) \to H^1(\Gamma, \mathbb{C}),$$

where $S_2(\Gamma)$ denotes the space of weight 2 cusp forms.

The free abelian group $\mathbb{Z}[\Gamma, G]$ on the double cosets $\Gamma g \Gamma$ with $g \in C_G(\Gamma)$ can be made into a ring under the *Shimura product*.
$H^1(\Gamma, \mathbb{Z})$ is a module over this ring under the Hecke action.

Theorem (Mesland–Sengun 2017)

For any $C_G(\Gamma)$-C^*-algebra B there is a ring homomorphism

$$\mathbb{Z}[\Gamma, G] \mapsto KK_0(B \rtimes \Gamma, B \rtimes \Gamma), \quad \Gamma g \Gamma \mapsto T_g.$$

If Γ acts freely and properly on X with quotient $M = X/\Gamma$ this map coincides with the map

$$\mathbb{Z}[\Gamma, G] \mapsto KK_0(C_0(M), C_0(M)), \quad \Gamma g \Gamma \mapsto [M \leftarrow M_g \rightarrow M].$$

Theorem (Mesland–Sengun 2017)

The Chern character $\mathrm{Ch} : K^0(M) \to H^{ev}(M)$ is a $\mathbb{Z}[\Gamma, G]$ module map.

To model a disordered solid material we conisder *Delone sets* $\mathcal{L} \subset \mathbb{R}^d$.

An (r, R)-Delone set is *uniformly discrete*: $\forall x \in \mathbb{R}^d : |B(x, r) \cap \mathcal{L}| \leq 1$,

and *relatively dense*: $\forall x \in \mathbb{R}^d |B(x, R) \cap \mathcal{L})| \geq 1$.

Consider

- $\Omega := \{\mathcal{L} + x : x \in \mathbb{R}^d\}$, the *hull* of $\mathcal{L}$
- $\Omega_0 := \{\omega \in \Omega : 0 \in \omega\}$ the *transversal* of $\mathcal{L}$
- $\mathcal{G} := \{(\omega, x) : x \in \omega\} \subset \Omega_0 \times \mathbb{R}^d$ can be made into an *étale groupoid*
- A 2-cocycle $\sigma : \mathcal{G} \times \mathcal{G} \to S^1$ (eg from magnetic field)

Observables: the twisted groupoid C^*-algebra $C^*(\mathcal{G}, \sigma)$.

Proposition (Freed–Moore, Thiang, Bourne–Carey–Rennie, Kellendonk, Kubota)

Suppose that $h = h^* \in C_r^*(\mathcal{G}, \sigma)$ has a spectral gap. Then h determines a class in $K_0(C_r^*(\mathcal{G}, \sigma))$. If h has a chiral symmetry, then h determines a class in $K_1(C_r^*(\mathcal{G}, \sigma))$.

Numerical invariants arise via

$$K_*(C_r^*(\mathcal{G}, \sigma)) \times KK^*(C_r^*(\mathcal{G}, \sigma), C(\Omega_0)) \to K_0(C(\Omega_0)) \xrightarrow{\int} \mathbb{C}.$$

Here $\int$ is the integral associated to a translation invariant measure on Ω_0.

To obtain a class in $KK_d(C^*(\mathcal{G},\sigma), C(\Omega_0))$ we consider

- The coordinate functions $X_k : \mathcal{G} \to \mathbb{R}, \quad (\omega, x) \mapsto x_k$
- The spinor bundle S_d of $\mathbb{R}^d$
- $D := \sum_{k=1}^{d} \gamma_k X_k : C_c(\mathcal{G}, S_d) \to C_c(\mathcal{G}, S_d)$ extends to a KK-cycle (E, D)

The class $[(E,D)] \in KK_d(C^*(\mathcal{G},\sigma), C(\Omega_0))$ gives the map

$$K_d(C^*(\mathcal{G},\sigma)) \xrightarrow{\otimes[D]} K_0(C(\Omega_0))$$

Invariant measures on Ω_0 give maps $K_0(C(\Omega_0)) \to \mathbb{C}$ and invariants of $C^*(\mathcal{G},\sigma)$.

Resctriction to boundary of the material: dropping the coordinate x_d.

Consider

- The subgroupoid $\mathcal{H} := \{(\omega, x) \in \mathcal{G} : x_d = 0\} \subset \mathcal{G}$, and $C^*(\mathcal{H}, \sigma)$,
- The function $X_d : C_c(\mathcal{G}) \to C_c(\mathcal{G})$ extends to a KK-cycle (X, S).

The class $[(X, S)] \in KK_1(C^*(\mathcal{G}, \sigma), C^*(\mathcal{H}, \sigma))$ gives a map

$$K_d(C^*(\mathcal{G}, \sigma)) \xrightarrow{\otimes [S]} K_{d-1}(C^*(\mathcal{H}, \sigma)).$$

The boundary algebra $C^*(\mathcal{H}, \sigma)$ carries its own fundamental cycle (Y, T) built from $X_1, \cdots, X_{d-1}$ and the spinor bundle S_{d-1} on $\mathbb{R}^{d-1}$.

Theorem (Bourne–Mesland 2018)

There is a factorisation $D = S \otimes 1 + 1 \otimes_{\nabla} T$ on $X \otimes Y$ in unbounded KK-theory. Consequently there is a commutative diagram

$$\begin{array}{ccc} K_d(C^*(\mathcal{G},\sigma)) & \xrightarrow{\otimes X_d} & K_{d-1}(C^*(\mathcal{H},\sigma)) \\ \Big\downarrow{\scriptstyle \otimes D} & & \Big\downarrow{\scriptstyle \otimes T} \\ K_0(C(\Omega_0)) & = & K_0(C(\Omega_0)). \end{array}$$

Every K-theoretic invariant of the bulk algebra $C^*(\mathcal{G},\sigma)$ has a corresponding invariant in the edge algebra $C^*(\mathcal{H},\sigma)$. Their numerical invariants computed from the fundamental KK-class and an invariant measure on Ω_0 coincide.

- Vector bundles pair with Fredholm operartors to give integers,
- Index theorems relate K-theory to cohomology,
- KK-theory as an analytic tool in index theory,
- Explicit description of the Kasparov product after geometric examples,
- Hecke operators in KK-theory,
- Bulk-boundary correspondence and KK-factorisation.