1 Global Analysis Lecture 12.12.2017 (Substitute Lecturer: Thorben Kastenholz)

Have to do a slight detour, before we can continue doing geometry.

Definition 1.1. *Let V denote a vector field on some manifold M. Let J denote an interval in $\mathbb{R}$. We call a smooth curve $\gamma\colon J \to M$ an integral curve of V if $\gamma'(t) = V(\gamma(t))$. Furthermore if $0 \in J$ we call the point $\gamma(0)$ the starting point of γ.*

Examples:

- $V = \partial_x$ in $\mathbb{R}^2$. Integral curves: $\gamma(t) = (a + t, b)$
- $W = x\partial_y - y\partial_x =$ "ix" Integral curves: $\gamma(t) = (a\cos(t) - b\sin(t), a\sin(t) + b\cos(t))$

Lemma 1.2 (Translation Lemma)**.** *Let V be a smooth vector field on a smooth manifold M, let $J \subset \mathbb{R}$ be an open interval, and let $\gamma\colon J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, let $J + a$ denote the interval J shifted by a.*

Then the curve $\tilde{\gamma}\colon J + a \to M$ defined by $\tilde{\gamma}(t) = \gamma(t - a)$ is an integral curve of V.

Proof. Exercise □

Assume that every point $p \in M$ has a unique integral curve $\theta^p\colon \mathbb{R} \to M$ that starts at this point. Define $\theta_t\colon M \to M$ via $\theta_t(p) = \theta^p(t)$. If we set $q = \theta^p(s)$, the translation lemma implies that $t \mapsto \theta^p(t + s)$ is an integral curve starting at q. Using the assumption about uniqueness of integral curves we get $\theta^q(t) = \theta^p(t + s)$.

This translates to

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$$

Together with the property $\theta_0(p) = p$ we get that θ defines an action of the additive group $\mathbb{R}$ on M.

Conversely we define a global flow / one-parameter group action on M to be a left action of $\mathbb{R}$ on M. We call this action smooth if the corresponding map $\theta\colon M \times \mathbb{R} \to M$ is smooth.

In this case we define $\theta_t\colon M \to M$ by $\theta_t(p) = \theta(t, p)$. θ is a homeomorphism/diffeomorphism.

Furthermore define $\theta^p\colon \mathbb{R} \to M$ to be the parametrized orbit of p under this action i.e $\theta^p(t) = \theta(t, p)$.

We define the infinitesimal generator of such a smooth θ to be the vector field $V(p) = \partial_t\theta^p(0)$.

Proposition 1.3. *Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow. The infinitesimal generator V of θ is a smooth vector field on M, and each curve θ^p is an integral curve of V.*

Proof. There is a smooth global vector field ∂_t on $\mathbb{R} \times M$ and by definition $V = (\theta_*)|_{\{0\}\times M}(\partial_t)$.

Furthermore note that $\theta_*(\partial_t) = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0}\theta(t + t_0, p) = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0}\theta(t, \theta_{t_0}(p)) = V(\theta^p(t_0))$. □

Examples:

- For $V = \partial_x$ the global flow is given by $\theta_t(x, y) = (x + t, y)$
- For $W = x\partial_y - y\partial_x$ the global flow is given by

$$\theta_t(x, y) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

2 The Fundamental Theorem on Flows

We have seen that every global flow gives us a vector field. Want to know wether the converse holds.

It certainly doesn't: Consider V as before defined on $\mathbb{R}^2 \setminus \{0\}$.

Definition 2.1. *We call an open subset $\mathcal{D}$ of $\mathbb{R} \times M$ with the property, that every $\mathcal{D}^p = \{t \in \mathbb{R} \,|\, (t, p) \in \mathcal{D}\}$ is an open interval containing 0 a flow domain. A (smooth) flow on M is a (smooth) map $\theta\colon \mathcal{D} \to M$, that satisfies:*

- $\theta(0,p) = p$
- *$s \in \mathcal{D}^p$ and $t \in \mathcal{D}^{\theta(s,p)}$ such that $s+t \in \mathcal{D}^p$, then $\theta(t,\theta(s,p)) = \theta(t+s,p)$.*

For a smooth flow we can define analogously the infinitesimal generator.

Proposition 2.2. *If $\theta\colon \mathcal{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve θ^p is an integral curve of V.*

Proof. The proof of the global flow situation carries over almost verbatim, so it is left as an easy exercise. □

Theorem 2.3 (Fundamental Theorem on Flows)**.** *Let V be a smooth vector field on a smooth manifold M. There is a unique maximal smooth flow $\theta\colon \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:*

(a) For each $p \in M$, the curve $\theta^p\colon \mathcal{D}^p \to M$ is the unique maximal integral curve of V starting at p.

(b) If $s \in \mathcal{D}^p$, then $\mathcal{D}^{\theta(s,p)}$ is the interval $\mathcal{D}^p - s$.

(c) For each $t \in \mathbb{R}$, the set $M_t = \{p \in M\colon (t,p) \in \mathcal{D}\}$ is open in M, and $\theta_t\colon M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

(d) For each $(t,p) \in \mathcal{D}, (\theta_t)_* V(p) = V(\theta_t(p))$.

We will call the unique flow mentioned above the flow generated by V.

To prove the theorem, we need the following theorem about ODE's, which we will prove later. (This is also called the flowbox theorem)

Theorem 2.4 (ODE Existence, Uniqueness and Smooothness)**.** *Let $U \subset \mathbb{R}^n$ be open, and let $V: U \to \mathbb{R}^n$ be a smooth map. For $t_0 \in \mathbb{R}$ and $x \in U$, consider the following initial value problem:*

$$(\gamma^i)'(t) = V^i(\gamma(t)), \gamma^i(t_0) = x^i$$

(a) EXISTENCE: For any $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists an open interval J_0 containing t_0 and an open set $U_0 \subset U$ containing x_0 such that for each $x \in U_0$, there is a smooth curve $\gamma\colon J_0 \to U$ that solves the initial value problem.

(b) UNIQUENESS: Any two differentiable solutions to the initial value problem agree on their common domain.

(c) SMOOTHNESS: Let t_0, x_0, J_0 and U_0 be as in (a), and define a map $\theta\colon J_0 \times U_0 \to U$ by letting $\theta(t,x) = \gamma(t)$, where $\gamma\colon J_0 \to U$ is the unique solution to the initial value problem with initial condition x. Then θ is smooth.

Proof of the fundamental theorem on flows. Let γ and $\tilde{\gamma}$ denote two integral curves of V defined on the same interval J such that $\gamma(t_0) = \tilde{\gamma}(t_0)$. Then by our ODE Theorem we get that the set on which γ and $\tilde{\gamma}$ agree is open, but on the other side it is closed by definition. Therefore the two curves agree on the whole domain. Let $\mathcal{D}^p$ denote the union of all open intervals $J \subset \mathbb{R}$ containing 0 on which an integral curve starting at p is defined. We define $\theta^p(t) = \gamma(t)$, where γ is an integral curve through p of V. Our previous considerations give, that this is well defined and furthermore unique.

Define $\mathcal{D}$ as the union of $\mathcal{D}^p \times \{p\}$ and $\theta(t,p) = \theta^p(t)$. By definition θ fulfills condition (a) of the theorem. To prove (b) fix any $p \in M$ and $s \in \mathcal{D}^p$ and write $q = \theta(s,p)$. The curve $\gamma\colon \mathcal{D}^p - s \to M$ defined by $\gamma(t) = \theta^p(t+s)$ satisfies $\gamma(0) = q$ and the translation lemma shows that this is an integral curve, which agress with θ^q. This gives us that θ is actually a flow.

By maximality of θ^q, the domain of γ cannot be larger than $\mathcal{D}^q$, which means that $\mathcal{D}^p - s \subset \mathcal{D}^q$. Since $0 \in \mathcal{D}^p$ this implies, that $-s \in \mathcal{D}^q$, and the group laws give, that $\theta^q(-s) = p$. Analogously $\mathcal{D}^q + s \subset \mathcal{D}^p$, which is the same as $\mathcal{D}^q \subset \mathcal{D}^p - s$. This proves (b).

The open sets of (a) in the ODE Theorem give immediatly, that $\mathcal{D}$ is an open subset of $\mathbb{R} \times M$. Furthermore part (c) of the ODE Theorem implies, that θ is smooth on these sets. But local smoothness implies smoothness, and therefore θ is a smooth map.

Furthermore since $\mathcal{D}$ is open M_t is open as well. Part (b) gives us, that $p \in M_t$ implies, that $\theta_t(p) \in M_{-t}$ and the group laws immediately imply, that θ_t and θ_{-t} are mutual inverses.

Lastly we have to proof, that V is invariant under θ i.e. if $\theta(t_0, p) = q$, then $(\theta_{t_0})_*(V(p)) = V(q)$. For this we apply $(\theta_{t_0})_*(V(p))$ to a function f.

$$
\begin{aligned}
((\theta_{t_0})_*(V(p)))f = V_p(f \circ \theta_{t_0}) &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} f \circ \theta_{t_0} \circ \theta^p(t) \\
&= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} f(\theta_{t_0+t}(p)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} f(\theta^q(t)) \\
&= \left(\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \theta^q(t) \right) f = V(q)f
\end{aligned}
$$

The second equation uses, that θ^p is an integral curve for V, the third and fourth equation uses the group law and the last equation uses again, that θ^q is an integral curve for V. □

3 Proving the ODE Theorem

A little more abstract setting: We are given an open set $U \subset \mathbb{R}^n$ and a map $V : U \to \mathbb{R}^n$, which is Lipschitz continuous and for any $t_0 \in \mathbb{R}$ and any $x \in U$ we will consider the following ODE initial value problem:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}(\gamma^i)(t) &= V^i(\gamma(t)), \\
\gamma^i(t_0) &= x^i
\end{aligned}
$$

Note that this includes our previous considerations, because a smooth vector field $V : M \to TM$ is locally Lipschitz continuous and the following theorems can be applied locally to get the ODE Theorem as it was stated before.

Lemma 3.1 (Gronwall's Lemma)**.** *Suppose $J_0 \subset \mathbb{R}$ is an open interval containing t_0, and $u : J_0 \to \mathbb{R}^n$ is a differentiable map satisfying the following differential inequality for some non-negative constants A and B and all $t \in J_0$:*

$$|u'(t)| \leq A|u(t)| + B$$

Then the following inequality holds for all $t \in J_0$:

$$|u(t)| \leq e^{A|t-t_0|}|u(t_0)| + \frac{B}{A}(e^{A|t-t_0|} - 1)$$

Proof. We will show the inequality for the set J_0^+ of all $t \geq t_0$, because one can easily deduce it for $\tilde{t} \leq t_0$ by substituting $t_0 - \tilde{t} = t - t_0$.

If $|u(t)| > 0$ then $|u(t)|$ is a differentiable function of t and we see that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}|u(t)| = \frac{\mathrm{d}}{\mathrm{dt}}(u(t) \cdot u(t))^{0.5} &= \frac{1}{2}(u(t) \cdot u(t))^{-0.5}(2u(t) \cdot u'(t)) \\
&\leq \frac{1}{2}|u(t)|^{-1}(2|u(t)||u'(t)|) = |u'(t)| \leq A|u(t)| + B
\end{aligned}
$$

Define $g : J_0^+ \to \mathbb{R}$ by

$$g(t) = e^{A(t-t_0)}|u(t_0)| + \frac{B}{A}(e^{A(t-t_0)} - 1)$$

By definition $g(t_0) = |u(t_0)|$, $g(t) > 0$ for $t > t_0$ and g satisfies $g'(t) = Ag(t) + B$. Consider $f: J_0^+ \to \mathbb{R}$ defined by

$$f(t) = e^{-A(t-t_0)}(|u(t)| - g(t))$$

We see that $f(t) = 0$ and for $t \in J_0^+$ the claim of the lemma is equivalent to $f(t) \leq 0$. Suppose that $f(t) > 0$, this implies that $|u(t)| > 0$ and therefore that $|u(t)|$ is differentiable. Then

$$\begin{aligned} f'(t) &= -Ae^{-A(t-t_0)}(|u(t)| - g(t)) + e^{-A(t-t_0)}\left(\frac{\mathrm{d}}{\mathrm{dt}}|u(t)| - g'(t)\right) \\ &\leq -Ae^{-A(t-t_0)}(|u(t)| - g(t)) + e^{-A(t-t_0)}(A|u(t)| + B - Ag(t) - B) \\ &= 0 \end{aligned}$$

Suppose there exists some $t_1 \in J_0^+$ such that $f(t_1) > 0$. Define $\tau = \sup\{t \in [t_0, t_1]: f(t) \leq 0\}$. Then $f(\tau) = 0$ by continuity and $f(t) > 0$ for $t \in (\tau, t_1]$. Because f is increasing in this interval, but its derivative exists and is negative, the mean value theorem on this interval proves the claim. □

Theorem 3.2 (Existence and Uniqueness of ODE Solutions)**.** *Let $U \subset \mathbb{R}^n$ be an open set, and suppose $V: U \to \mathbb{R}^n$ is Lipschitz continuous. Let $(t_0, x_0) \in \mathbb{R} \times U$ be given. There exist an open interval J_0 containing t_0, an open set $U_0 \subset U$ containing x_0 and for for each $x \in U_0$ a C^1 curve $\gamma: J_0 \to U$ satisfying the initial value problem.*

Furthermore if we have two solutions to the initial value problem, then they agree on their common domains.

The proof of this theorem relies on the Banach Fixed-Point Theorem, whose proof is left as an exercise.

Theorem 3.3 (Banach Fixed-Point Theorem)**.** *Let (X, d) be a non-empty complete metric space with a contraction mapping $T: X \to X$ (i.e a Lipschitz continuous map with Lipschitz constant less than 1). Then T admits a unique fixed point x^* in X.*

Proof of the Existence and Uniqueness of ODE Solutions. Suppose we have a solution γ to the initial value problem. Then the ODE fulfilled by γ implies, that γ is in C^1. Integrating the ODE and use of the fundamental theorem of calculus gives

$$\gamma^i = x^i + \int_{t_0}^{t} V^i(\gamma(s))ds$$

Conversely if a map γ fulfills the above equation, then the fundamental theorem of calculus implies that γ is a C^1 solution to the initial value problem.

Suppose J_0 is an open interval containing t_0. For any continuous curve $\gamma: J_0 \to U$ we define a new curve $I\gamma: J_0 \to \mathbb{R}^n$ by

$$I\gamma(t) = x + \int_{t_0}^{t} V(\gamma(s))ds$$

We are looking for a fixed point of $I\gamma$ in a suitable metric space of curves. V being Lipschitz continuous implies that there exists C such that $|V(x) - V(\tilde{x})| \leq C|x - \tilde{x}|$. Let M denote the supremum of $|V|$ on the compact set $\bar{B}_r(x_0)$, where r denotes some radius, such that the ball with radius r around x_0 is contained in U. Choose $\delta > 0$ and $\epsilon > 0$ small enough that

$$\delta < \frac{r}{2}, \epsilon < \min\left(\frac{r}{2M}, \frac{1}{C}\right)$$

and set $J_0 = (t_0 - \epsilon, t_0 + \epsilon)$ and $U_0 = B_\delta(x_0)$. For any $x \in U_0$ let $\mathcal{M}_x$ denote the set of all continuous curves $\gamma: J_0 \to \bar{B}_r(x_0)$ satisfying $\gamma(t_0) = x$. We define a metric on this space by

$$d(\gamma, \tilde{\gamma}) = \sup_{t \in J_0} |\gamma(t) - \tilde{\gamma}(t)|$$

Because every Cauchy sequence with respect to this metric is uniformly Cauchy we get that $\mathcal{M}_x$ is a complete metric space. We want to define $I: \mathcal{M}_x \to \mathcal{M}_x$ as considered above.

So we have to show that I is well-defined. That $I\gamma$ is continuous and $I\gamma(t_0) = x$ is seen immediately. To show that the image of $I\gamma$ is contained in $\bar{B}_r(x_0)$ we compute:

$$\begin{aligned} |I\gamma(t) - x_0| &= \left| x + \int_{t_0}^{t} V(\gamma(s))ds - x_0 \right| \\ &\le |x - x_0| + \int_{t_0}^{t} |V(\gamma(s))|ds \\ &< \delta + M\epsilon < r \end{aligned}$$

Next we show that I is a contraction. If $\gamma, \tilde{\gamma} \in \mathcal{M}_x$, then

$$\begin{aligned} d(I\gamma, I\tilde{\gamma}) &= \sup_{t \in J_0} \left| \int_{t_0}^{t} V(\gamma(s))ds - \int_{t_0}^{t} V(\tilde{\gamma}(s))ds \right| \\ &\le \sup_{t \in J_0} \int_{t_0}^{t} |V(\gamma(s)) - V(\tilde{\gamma}(s))|ds \\ &\le \sup_{t \in J_0} \int_{t_0}^{t} C\left|\gamma(s) - \tilde{\gamma(s)}\right|ds \le C\epsilon d(\gamma, \tilde{\gamma}) \end{aligned}$$

Because we have chose ϵ so that $C\epsilon < 1$ this shows that I is a contraction. The Banach Fixed-point Theorem implies, that I has a unique fixed-point. Uniqueness can also be seen using the following observation:

Suppose γ and $\tilde{\gamma}$ both solve the initial value problem for (t_0, x_0) then

$$\left| \frac{\mathrm{d}}{\mathrm{dt}}(\tilde{\gamma}(t) - \gamma(t)) \right| = |V(\tilde{\gamma}(t)) - V(\gamma(t))| \le C|\tilde{\gamma}(t) - \gamma(t)|.$$

Applying the Gronwall inequality implies

$$|\tilde{\gamma}(t) - \gamma(t)| \le e^{C|t-t_0|}|\tilde{\gamma}(t_0) - \gamma(t_0)|$$

Thus if both curves satisfy the same initial value condition, they agree. □

Theorem 3.4 (Smoothness of ODE Solutions)**.** *Suppose $U \subset \mathbb{R}^n$ is an open set and $V : U \to \mathbb{R}^n$ is Lipschitz continuous. Suppose also that $U_0 \subset U$ is an open set, $J_0 \subset \mathbb{R}$ is an open interval containing t_0, and $\theta : J_0 \times U_0 \to U$ is any map such that for each $x \in U_0$, $\gamma(t) = \theta(t, x)$ solves our initial value problem. If V is of class C^k for some $k \ge 0$, then so is θ.*

Proof. We will proof the theorem using induction on k. The hardest parts are the two cases $k = 0$ and $k = 1$. For continuity it suffices to show that for an arbitrary point $(t_1, x_1) \in J_0 \times U_0$ θ is continuous on some neighborhood of that point.

Let J_1 be a bounded open interval containing t_0 and t_1 and such that $\bar{J_1} \subset J_0$. Choose $r > 0$ such that $\bar{B}_{2r}(x_1) \subset U_0$ and let $U_1 = B_r(x_1)$. Let C be a Lipschitz constant for V and define constants M and T by

$$M = \sup_{\bar{U}_1} |V|, \qquad T = \sup_{\bar{J}_1} |t - t_0|$$

We will show that θ is continuous on $\bar{J}_1 \times \bar{U}_1$. Using our previous considerations for the uniqueness of solutions we get:

$$|\theta(t, \tilde{x}) - \theta(t, x)| \le e^{CT}|\tilde{x} - x|$$

Thus for each t, θ is Lipschitz continuous as a function of x. We have to show, that it is continuous in both variables. Let $(t, x), (\tilde{t}, \tilde{x}) \in \bar{J}_1 \times \bar{U}_1$ be arbitrary. Since the solutions to the initial value problem all solve

$$\theta^i(t, x) = x^i + \int_{t_0}^{t} V^i(\theta(s, x))ds$$

and therefore (assuming without loss of generality $\tilde{t} \geq t$)

$$\begin{aligned}
\left|\theta(\tilde{t},\tilde{x}) - \theta(t,x)\right| &\leq |\tilde{x}-x| + \left|\int_{t_0}^{\tilde{t}} V(\theta(s,\tilde{x}))ds - \int_{t_0}^{t} V(\theta(s,x))ds\right| \\
&\leq |\tilde{x}-x| + \int_{t_0}^{t} |V(\theta(s,\tilde{x}) - V(\theta(s,x))|ds \\
&\quad + \int_{t}^{\tilde{t}} |V(\theta(s,\tilde{x})|ds \\
&\leq |\tilde{x}-x| + C\int_{t_0}^{t} |\theta(s,\tilde{x}) - \theta(s,x)|ds + \int_{t}^{\tilde{t}} M ds \\
&\leq |\tilde{x}-x| + CTe^{CT}|\tilde{x}-x| + M\left|\tilde{t}-t\right|
\end{aligned}$$

This implies, that θ is continuous.

Next we address the $k = 1$ part, which is the hardest part of the proof. Suppose that V is of class C^1 and define $\bar{J}_1, \bar{U}_1$ as before. Expressed in terms of θ our initial value theorem becomes:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\theta^i(t,x) &= V^i(\theta(t,x)) \\
\theta^i(t_0,x) &= x^i
\end{aligned}$$

Since θ is continuous this implies, that the time derivative of θ^i exists and is continuous. So let us prove that $\partial_j\theta^i$ exists and is continuous.

We define the differential quotient $(\Delta_h)^i_j \colon \bar{J}_1 \times \bar{U}_1 \to \mathbb{R}$ by

$$(\Delta_h)^i_j(t,x) = \frac{\theta^i(t,x+he_j) - \theta^i(t,x)}{h}$$

By definition $\partial_j\theta^i(t,x) = \lim_{h\to 0}(\Delta_h)^i_j)(t,x)$ if the limit exists. We will show that $(\Delta_h)^i_j)$ converges uniformly as $h \to 0$, which implies, that the limit exists and is continuous as the uniform limit of continuous functions.

We write every index of the differential quotient in a matrix to get $\Delta_h \colon \bar{J}_1 \times \bar{U}_1 \to \mathrm{Mat}(n\times n, \mathbb{R})$. Our previous considerations showed that $\left|(\Delta_h)^i_j(t,x)\right| \leq e^{CT}$ for every index. This implies that $|\Delta_h(t,x)| \leq ne^{CT}$.

We can Taylor approximate V for all $t \in \bar{J}_1, y \in \bar{U}_1$ and $v \in B_r(0)$ to get

$$V^i(y+v) - V^i(y) = v^k\frac{\partial V^i}{\partial y^k}(y) + v^k\int_0^1 \left(\frac{\partial V^i}{\partial y^k}(y+sv) - \frac{\partial V^i}{\partial y^k}(y)\right) ds$$

We will write $G^i_k(y,v)$ for the integral part of this equation. Thus we get:

$$V^i(y+v) = V^i(y) + v^k\frac{\partial V^i}{\partial y^k}(y) + v^k G^i_k(y,v)$$

where G^i_k is continuous and zero whenever $v = 0$. Since $G^i_k(y,v)$ is defined on a compact set, it is uniformly continuous i.e for every $\epsilon > 0$ there exists $\delta > 0$ such that the matrix valued function G satisfies

$$|G(y,v)| < \epsilon \text{ for all } y \in \bar{U}_1 \text{ and all } |v| < \delta$$

Since $\theta(t_0,x) = x$ we have that Δ_h satisfies the following initial condition: $(\Delta_h)^i_j(t_0,x) = \delta^i_j$. We want to compute the time derivative of (Δ_h). Denote $\theta(t,x)$ by y and define $v = (v^1,\ldots,v^n)$ as

$$v^k = \theta^k(t,x+he_j) - \theta^k(t,x) = h(\Delta_h)^k_j(t,x)$$

This gives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}(\Delta_h)^i_j(t,x) &= \frac{1}{h}\left(\frac{\mathrm{d}}{\mathrm{dt}}\,\theta^i(t,x+he_j) - \frac{\mathrm{d}}{\mathrm{dt}}\,\theta^i(t,x)\right)\\ &= \frac{1}{h}(V^i(\theta(t,x+he_j)) - V^i(\theta(t,x)))\\ &= \frac{1}{h}\left(v^k\frac{\partial V^i}{\partial y^k}(\theta(t,x)) + v^k G^i_k(y,v)\right)\\ &= \left(\frac{\partial V^i}{\partial y^k}(\theta(t,x)) + G^i_k(y,v)\right)(\Delta_h)^k_j(t,x)\end{aligned}$$

Thus for any nonzero $h, \tilde{h} \in \bar{B}_r(0)$

$$\begin{aligned}&\frac{d}{dt}\left((\Delta_h)^i_j(t,x) - (\Delta_{\tilde{h}})^i_j(t,x)\right)\\ =&\frac{\partial V^i}{\partial y^k}(\theta(t,x))\left((\Delta_h)^k_j(x) - (\Delta_{\tilde{h}})^k_j(t,x)\right)\\ &+G^i_k(y,v)(\Delta_h)^k_j(t,x) - G^i_k(y,\tilde{v})(\Delta_{\tilde{h}})^k_j(t,x)\end{aligned}$$

Here $\tilde{v}$ is defined analogously to v with $\tilde{h}$ instead of h. Now chose $\delta \le r$ such that G is uniformly bounded by some $\epsilon > 0$. Let E denote the supremum of $|DV|$ on $\bar{U}_1$. By specifying $|h|$ and $|\tilde{h}|$ to be smaller than $\delta e^{-CT}/n$ we get that v and $\tilde{v}$ are both bounded by δ. We compute

$$\left|\frac{\mathrm{d}}{\mathrm{dt}}(\Delta_h(t,x) - \Delta_{\tilde{h}}(t,x))\right| \le E\big|\Delta_h(t,x) - \Delta_{\tilde{h}}(t,x)\big| + 2\epsilon n e^{CT}$$

Because $\Delta_h(t_0,x) - \Delta_{\tilde{h}}(t_0,x) = 0$ the Gronwall inequality gives:

$$\big|\Delta_h(t,x) - \Delta_{\tilde{h}}(t,x)\big| \le \frac{2\epsilon n e^{CT}}{E}(e^{E|t-t_0|} - 1) \le \frac{2\epsilon n e^{CT}}{E}(e^{ET} - 1)$$

Since ϵ was arbitrary this implies that Δ_h is uniformly Cauchy if $h \to 0$ and therefore converges uniformly to a continuous function $\partial\theta^i/\partial y^j$. This finishes the $k = 1$ step.

Now assume that the assumption holds for some $k \ge 1$ and suppose V is in C^{k+1}. By inductive hypothesis θ is of class C^k and therefore $\frac{\mathrm{d}}{\mathrm{dt}}\,\theta^i$ is also in C^k. By differentiating the equation $\theta^i(t,x) = x^i + \int_{t_0}^t V^i(\theta(s,x))ds$ we get

$$\frac{\partial\theta^i}{\partial x^j}(t,x) = \delta^i_j + \int_{t_0}^t \frac{\partial V^i}{\partial y^k}(\theta(s,x))\frac{\partial\theta^k}{\partial x^j}(s,x)ds$$

Therefore the fundamental theorem of calculus implies that $\partial\theta^i/\partial x^j$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\theta^i}{\partial x^j}(t,x) = \frac{\partial V^i}{\partial y^k}(\theta(t,x))\frac{\partial\theta^k}{\partial x^j}(t,x)$$

Consider the following initial value problem for the $n + n^2$ unknown functions (α^i, β^i_j):

$$\begin{aligned}(\alpha^i)'(t) &= V^i(\alpha(t))\\ (\beta^i_j)'(t) &= \frac{\partial V^i}{\partial y^k}(\alpha(t))\beta^k_j(t)\\ \alpha^i(t_0) &= a^i\\ \beta^i_j(t_0) &= b^i_j\end{aligned}$$

One easily checks that the functions on the right-hand side are in C^k. Furthermore it is easy to see that $\alpha^i(t) = \theta^i(t,x)$ and $\beta^i_j(t) = \partial\theta^i/\partial x^j(t,x)$ solve this system with initial conditions $a^i = x^i$, $b^i_j = \delta^i_j$. This implies, that the derivatives of θ are in C^k, which completes the proof. □