GLOBAL ANALYSIS I - WS 2017/2018

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1. Week 1

1.1. Lecture 1 - Tue 10-10-2017.
Material covered:

[1, Chapter 1 "Smooth manifolds"] and
[1, Chapter 2 "Smooth maps" up to page 40].

Review of defintions of topological manifold, smooth compatibility of charts, smooth manifold, smooth atlas, smooth structure, smooth map, diffeomorphism. Example: the sphere $S^n \subset \mathbb{R}^{n+1}$.

1.2. Lecture 2 - Wed 11-10-2017.
Material covered:

[1, Chapter 2 "Smooth maps" section "Bump functions and partitions of unity", pages 40-47]

Important definitions and results:

1.1. Definition. A collection $\{U_\alpha\}_{\alpha\in A}$ of a topological space M is *locally finite* if every $p \in M$ has a neighborhood V such that $V \cap U_\alpha$ is nonempty for only finitely many α.

1.2. Lemma. *Let $U = \{U_\alpha\}_{\alpha\in A}$ be an open cover for which each U_α is a precompact set. Then U is locally finite if and only if for each α there are at most finitely many β for which $U_\alpha \cap U_\beta$ is nonempty.*

Proof. Is one of this weeks exercises. As noted during the lecture, precompactness is necessary for the equivalence to hold. □

1.3. Lemma. *Every topological manifold admits a locally finite cover by precompact open sets.*

1.4. Definition. Let M a manifold and $W = \{W_i\}_{i\in I}$ an open cover. The cover W is *regular* if

1. the cover W is countable and locally finite;
2. for each i there is a diffeomorphism $\psi_i : W_i \to B(0, 3) \subset \mathbb{R}^n$;
3. the collection $U_i := \psi_i^{-1}(B(0, 1))$ still covers M.

1.5. Proposition. *Let M be a smooth manifold. Then every open cover admits a regular refinement. In particular M is paracompact.*

1.6. Definition. Let $X = \{X_\alpha\}$ be an open cover of the smooth manifold M. A *partition of unity subordinate to* X is a collection of smooth functions $\phi_\alpha : M \to \mathbb{R}, \alpha \in A$ such that

- $0 \leq \phi_\alpha \leq 1$;
- $\text{supp}\phi_\alpha \subset U_\alpha$;
- the set of supports $\{\text{supp}\phi_\alpha\}$ is locally finite;
- for each $x \in M$ we have $\sum_{\alpha\in A} \phi_\alpha(x) = 1$.

Note that the last sum is finite by the condition preceding it.

1.7. Theorem. *Let M be a smooth manifold and $X := \{X_\alpha\}_{\alpha\in A}$ an open cover. Then there exists a partition of unity ϕ_α subordinate to X.*

An important corollary this the above theorem is

1.8. Lemma. *Let M be a smooth manifold, and suppose f is a smooth function defined on a closed subset $A \subset M$. For any open set U containing A, there exists a smooth function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_A = f$ and* $\text{supp} f \subset U$.

2. Week 2

2.1. Lecture 3 - Tue 17-10-2017.

Material covered:

[1, Chapter 3 "The tangent bundle", pages 50-60]

2.1. Definition. A map $v : C^\infty(M) \to \mathbb{R}$ is called a *derivation at* $p \in M$ if it satisfies the Leibniz rule

$$v(fg) = f(p)v(g) + v(f)g(p),$$

for all $f, g \in C^\infty(M)$. The *tangent space at* p is defined to be

$$T_p(M) := \{v : C^\infty(M) \to \mathbb{R} : v \text{ a derivation at } p\}.$$

For $F : M \to N$ a smooth map we define the *differential of* F *at* p to the map

$$(dF)_p : T_p(M) \to T_{F(p)}(N),$$

defined by the rule $(dF)_p(v)(f) := v(f \circ F)$ where $v \in T_p(M)$ and $f \in C^\infty(N)$.

2.2. Proposition (Properties of the differential)**.** *Let M, N, P be smooth manifolds and $F : M \to N$, $G : N \to P$ be smooth maps. For $p \in M$ we have*

(1) $(dF)_p : T_p(M) \to T_p(N)$ *is linear;*

(2) $(dG)_{F(p)} \circ (dF)_p = (d(G \circ F))_p : T_pM \to T_{G\circ F(p)}P$;
(3) $(d\mathrm{Id}_M)_p = \mathrm{Id}_{T_p(M)} : T_pM \to T_p(M)$;
(4) *if F is a diffeomorphism then $(dF)_p : T_p(M) \to T_p(N)$ is an isomorphism with inverse $(dF)_p^{-1} = (dF^{-1})_p$.*

2.3. Proposition (Locality of the differential)**.** *Let M be a smooth manifold with or without boundary, $p \in M$ and $v \in T_p(M)$. Suppose that $f, g \in C^\infty(M)$ are such that there is a neighborhood U of p for which $f|_U = g|_U$. Then $v(f) = v(g)$.*

2.4. Proposition (Open submanifold)**.** *Let M be a smooth manifold with or without boundary, $U \subset M$ an open subset and $i : U \to M$ the inclusion map. For any $p \in U$ the differential $(di)_p : T_p(U) \to T_p(M)$ is an isomorphism.*

2.5. Proposition. *Let M be a smooth n-dimensional manifold with or without boundary. Then for every $p \in M$ the tangent space $T_p(M)$ is an n-dimensional vector space.*

An abstract vector space V carries a canonical topology and smooth structure making in an n-dimensional manifold. Thus the tangent space T_aV is isomorphic to V. The isomorphism is canonical and of the form

$$V \to T_aV, \quad v \mapsto D_v|_a, \quad D_v|_a f = \frac{d}{dt}|_{t=0} f(a+tv),$$

with $f \in C^\infty(V)$. If $L : V \to W$ is a linear map, the above isomorphism satisfies the compatibilty

$$(dL)_a(D_v|_a)f = D_{Lv}|_{La}f,$$

for $f \in C^\infty(V)$.

2.6. Definition. The *tangent bundle* of the manifold M is the set

$$TM := \bigsqcup_{p\in M} T_p(M).$$

The *projection map* $\pi : TM \to M$ is defined by $\pi(p, v) := p$.

2.2. Lecture 4 - Wed 18-10-2017. Material covered:

[1, Chapter 3 "The tangent bundle" pages 60-75]

First we covered a discussion of explicit coordinate expressions for bases of tangent spaces, differentials of smooth maps and change of coordinate maps. This can be found on pages 60-65 of **[1]**.

2.7. Proposition. *For an n-dimensional manifold M, the tangent bundle TM carries a natural topology and smooth structure making it into a $2n$-dimensional manifold and the projection map $\pi : TM \to M$ is smooth.*

2.8. Definition. The *global differential* of a smooth map $F : M \to N$ is the map

$$dF : TM \to TN, \quad dF(p, v) := (F(p), (dF)_p v).$$

2.9. Proposition. *The global differential of a smooth map $F : M \to N$ is a smooth map $dF : TM \to TN$ between the tangent bundles.*

As the pointwise differentials, the global differential satisfies

$$d(F \circ G) = dF \circ dG, d\mathrm{Id}_M = \mathrm{Id}_{TM}$$

and if F is a diffeomorphism then so is dF.

[1, Chapter 10 "Vector bundles" pages 249-252]

2.10. Definition (Vector bundles)**.** Let M be a topological space. A *real vector bundle of rank k over M* is a topological space E together with a continuous map $\pi : E \to M$ satisfying

(1) for each $p \in M$ the fiber $E_p := \pi^{-1}(p)$ is a k dimensional real vector space;
(2) for every $p \in M$ there exists a neighborhood U of p and a homeomorphism
$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$
with the property that $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \to U$ is the coordinate projection, and for every p the restriction
$$\Phi : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^k,$$
is a vector space isomorphism.

In case M, E are manifolds and π, Φ are smooth, then $\pi : E \to M$ is a *smooth vector bundle*.

We often refer to E as the *total space* M as the *base* and π as the *bundle projection*. The maps ϕ are called *local trivializations*. The pertinent example is the tangent bundle $TM \to M$.

2.11. Lemma. *Let $\pi : E \to M$ be a smooth vector bundle and*

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k, \quad \Psi : \pi^{-1}(U) \to U \times \mathbb{R}^k,$$

two local trivializations. There exists a smooth map

$$\tau : U \cap V \to GL(k, \mathbb{R}),$$

such that

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p) \cdot v).$$

Here $\tau(p) \cdot v$ denotes the usual matrix multiplication.

3. Week 3

3.1. Lecture 5, Tue 24-10-2017.
[1, Chapter 10 "Vector bundles" pages 252-255]

3.1. Lemma (Vector bundle chart lemma)**.** *Let M be a smooth manifold (with or without boundary). Suppose that we are given*

(1) *for each $p \in M$ a vector space E_p;*
(2) *an open cover $\{U_\alpha\}_{\alpha \in A}$ of M;*
(3) *a fixed k-dimensional vector space V and for each $\alpha \in A$ a bijection*

$$\Phi_\alpha : \bigsqcup_{p \in U_\alpha} E_p \to U_\alpha \times V,$$

such that the restriction $\Phi_\alpha : E_p \to V$ is a vector space isomorphism;

(4) *for each pair (α, β) with $U_\alpha \cap U_\beta \neq \emptyset$ a smooth map*

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V),$$

such that the map

$$\Phi_\alpha \circ \Phi_\beta^{-1} : U_\alpha \cap U_\beta \times V \to U_\alpha \cap U_\beta \times V,$$

is given by $(u, v) \mapsto (u, \tau_{\alpha\beta}(u) \cdot v)$.

Then $E := \bigsqcup_{p \in M} E_p$ admits a unique topology and smooth structure making it into a manifold with or without boundary and such that

$$\pi : E \to M, \quad (p, v) \mapsto p$$

is a rank k real vector bundle with local trivializations $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A}$.

[1, Pages 276-277]

3.2. Example (The cotangent bundle)**.** Let $E_p := T_p^*(M)(T_p(M))^*$ be the dual of $T_p(M)$ and $\{(U_i, \phi_i)\}_{i \in I}$ a cover of M by coordinate charts. Define

$$\Phi_i : \bigsqcup_{p \in U_i} E_p \to U_i \times \mathbb{R}^n$$

$$\sum_{i=1}^{n} v_i dx_i|_p \mapsto (p, v_1, \cdots v_n),$$

where $dx_i|_p$ is the basis dual to the basis $\frac{\partial}{\partial x_i}|_p$ of T_pM. If (U_j, ψ_j) is another chart with coordinates y_j and $U_i \cap U_j \neq \emptyset$ then

$$\phi_i \circ \phi_j^{-1}(p, v_1, \cdots, v_n) = (p, \sum_j v_j \frac{\partial y_j}{\partial x_1}(p), \cdots, \sum_j v_j \frac{\partial y_j}{\partial x_n}(p)),$$

which is smooth because U_i and U_j are smoothly compatible. The map

$$\tau_{ij} : U_i \cap U_j \to GL(n, \mathbb{R}), \quad p \mapsto (\frac{\partial y_j}{\partial x_i}(p))_{ij}, \tag{1}$$

thus satisfies the axioms of the chart lemma. We so obtain the *cotangent bundle* T^*M of M.

3.3. Example (Alternating tensors)**.** The *bundle of alternating tensors of degree* k is defined to be

$$\bigwedge^k T^*M := \bigsqcup_{p\in M} \bigwedge^k T_p^*M,$$

where $\bigwedge^k T_p^*M$ is the k-th exterior power of $T_p^*(M)$. To a cover of coordinate charts $\{(U_i, \phi_i)\}_{i\in I}$ of M we associate the maps

$$\Phi_i : \bigsqcup_{p\in U_i} \bigwedge^k T_p^*M \to U_i \times \bigwedge^k \mathbb{R}^n$$

$$\sum_J \omega_I dx_{j_1} \wedge \cdots \wedge dx_{j_k}|_p \mapsto (p, \sum_J \omega_J(p) e_{j_1} \wedge \cdots e_{j_k}),$$

where e_j is the standard basis of $\mathbb{R}^n$. The transition maps for this bundle are given by the functions

$$\tau_{ij}^k : U_i \cap U_j \to GL\left(\bigwedge^k \mathbb{R}^n\right),$$

defined through $\tau_{ij}^k(p)(v_1 \wedge \cdots \wedge v_k) := \tau_{ij}(p)v_1 \wedge \cdots \wedge \tau_{ij}(p)v_k$, where τ_{ij} is as in (1).

[1, Chapter 10, pages 255-261]

3.4. Definition. Let $\pi : E \to M$ be a vector bundle. A *global section* of E is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. In case $E \to M$ is a topological vector bundle, we denote by $\Gamma(M, E)$ the space of continuous sections of E.
In case $E \to M$ is a smooth vector bundle we denote by $\Gamma^\infty(M, E)$ the space of smooth sections of E.
A *local section* over an open set $U \subset M$ is a map $s : U \to E$ such that

$\pi \circ s = \mathrm{id}_U$ and we adopt the same notational conventions for continuous and smooth local sections.

3.5. Example. $\Gamma(M, TM)$ is the space of continuous vector fields on M;

$\mathscr{X}(M) := \Gamma^\infty(M, TM)$ is the space of smooth vector fields on M.

For a trivial bundle $E := M \times \mathbb{R}^k$ we have $\Gamma(M, E) \simeq C(M, \mathbb{R}^k)$ and $\Gamma^\infty(M, E) \simeq C^\infty(M, \mathbb{R}^k)$.

3.6. Definition. A *smooth covector field* or *differential 1-form* is a smooth section of the contangent bundle T^*M.

3.7. Definition. Let $E \to M$ be a vector bundle. A k-tuple of local sections $(\sigma_i)_{i=1}^k$ over an open U is a *local frame* over U if for all $p \in U$ the vectors $(\sigma_i(p))_{i=1}^k$ form a basis for E_p.

3.8. Example (Frames and trivializations)**.** Given a trivialization

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k,$$

over U and e_i the standard basis of $\mathbb{R}^k$ the maps $\sigma_i(u) := \Phi^{-1}(u, e_i)$ define a local frame over U.

3.9. Proposition. *Any smooth local frame over U is associated with a local trivialization as in the previous example.*

The trivialization associated with the local frame (σ_i) is defined by

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k, \quad v_p \mapsto (p, v_1(p), \cdots, v_k(p)),$$

where the functions v_i are defined by $v_p = \sum v_i(p)\sigma_i(p)$.

3.10. Corollary. *If the bundle $E \to M$ admits a frame defined on all of M then $E \simeq M \times \mathbb{R}^k$ and this is identification is continuous or smooth whenever the σ_i are continuous or smooth.*

3.11. Corollary. *Let (V, ϕ) be a smooth chart for M and (σ_i) a smooth local frame over V. Then*

$$\tilde{\phi} : \pi^{-1}(V) \to \phi(V) \times \mathbb{R}^k$$
$$\sum v_i \sigma_i(p) \mapsto (x_1(p), \cdots, x_n(p), v_1, \cdots, v_k),$$

is a smooth chart for $\pi^{-1}(V) \subset E$.

3.12. Proposition. *Let $\pi : E \to M$ be a smooth vector bundle, (σ_i) a smooth local frame and $\tau : M \to E$ a section. Then τ is smooth if and only if the coordinate functions $\tau_i : M \to \mathbb{R}$ defined by $\tau(p) = \sum_i \tau_i(p)\sigma_i(p)$ are smooth.*

3.2. Lecture 6, Wed 25-10-2017. A vector field $X \in \mathscr{X}(M)$ associates to a smooth function on M a new function Xf on M via $(Xf)(p) := X_p(f)$, since $X_p \in T_p(M)$ is a derivation at p.

[1, Chapter 8, pages 180-181 and 185-186]

3.13. Proposition (Smoothness criterion for vector fields)**.** *Let M be a smooth manifold and $X : M \to TM$ a vector field. The following are equivalent:*

(1) *X is smooth;*
(2) *for every $f \in C^\infty(M)$ the function Xf is smooth;*
(3) *for every open set $U \subset M$ and $f \in C^\infty(U)$ the function Xf is smooth on U.*

We thus have that a smooth vector field X induces a map

$$\begin{aligned} X : C^\infty(M) &\to C^\infty(M) \\ (Xf)(p) &:= X_p f, \end{aligned}$$

and this map is a *derivation*, that is, it satisfies the Leibniz rule $X(fg) = (Xf)g + f(Xg)$. The converse is true as well.

3.14. Proposition. *Let $D : C^\infty(M) \to C^\infty(M)$ be a derivation. Then there is a vector field $X : M \to TM$ such that $Xf = Df$.*

The *Lie bracket* of vector fields is the map

$$\mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M), (X, Y) \mapsto [X, Y],$$

where $[X, Y]$ is defined by its action on functions

$$[X, Y]f = X(Yf) - Y(Xf).$$

It is straightforward to check that $[X, Y]$ is a derivation and thus defines a vector field.

[1, Chapter 11, pages 278-282]

3.15. Proposition (Smoothness of covector fields)**.** *Let M a smooth manifold with or without boundary and $\omega : M \to T^*M$ a 1-form. The following are equivalent:*

(1) *ω is smooth;*
(2) *in every chart the component functions with respect to the local frame dx_i are smooth;*
(3) *every point of M is contained in some chart for which the component functions with respect to the local frame dx_i are smooth;*
(4) *for every vector field $X : M \to TM$ the function $\omega(X)$ is smooth;*
(5) *for every open set $U \subset M$ and vector field $X : U \to TM$, the function $\omega(X)$ is smooth in U.*

3.16. Definition (The differential of a function)**.** Let $f \in C^\infty(M)$ and $v_p \in T_p(M)$. We define the *differential of f at p* to be the covector

$$(df)_p(v_p) := v_p(f).$$

3.17. Proposition. *The differential of a smooth function is a smooth covector field.*

[1, Chapter 14, pages 259-372]

Pages 249-259 contain a review multilinear algebra on vector spaces. We did not review this material in the lecture but it is recommended reading.

3.18. Definition. A *differential k-form* is a section of the bundle $\bigwedge^k T^*M$. We introduce the notation

$$\Omega^k(M) := \Gamma^\infty(M, \bigwedge^k T^*M), \quad \Omega^*(M) := \bigoplus_{k=0}^{\dim M} \Omega^k(M).$$

The *wedge product* of differential forms is defined as follows:

For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ and vector fields $X_1, \cdots, X_{n+k}$ we define

$$\omega \wedge \eta(X_1, \cdots X_{\ell+k}) := \sum_{\sigma \in S_{\ell+k}} \operatorname{sgn}(\sigma)\omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)})\eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(\ell+k)}).$$

The summation runs over all permutations σ in the symmetric group $S_{\ell+k}$ on $\ell + k$ elements. The wedge product satisfies

$$(\lambda\omega_1 + \mu\omega_2) \wedge \eta = \lambda\omega_1 \wedge \eta + \mu\omega_2 \wedge \eta,$$

$$\omega \wedge \eta = (-1)^{k\ell}\eta \wedge \omega.$$

Given a smooth map $F : M \to N$, the *pull back* of a form $\omega \in \Omega^k(N)$ is the k-form

$$(F^*\omega)(X_1, \cdots, X_k) := \omega(dF(X_1), \cdots, (dF)(X_k)),$$

or more comapctly $F^*\omega := \omega \circ dF$. Also recall that for a k-form

$$\omega = \sum \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

defined on an open set of $\mathbb{R}^n$ is *exterior derivative* is defined by

$$d\omega = \sum d\omega_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

3.19. Lemma. *Suppose $F : M \to N$ is a smooth map. Then*

(1) $F^* : \Omega^k(N) \to \Omega^k(M)$ *is linear;*
(2) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$*;*

(3) *in any chart on N with coordinates y_i*

$$F^*(\sum_I \omega_I dy_{i_1} \wedge \cdots \wedge dy_{i_k}) = \sum_I (\omega_I \circ F) d(y_{i_1} \circ F) \wedge \cdots \wedge d(y_{i_k} \circ F)$$

(4) *if $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are open sets then $F^*(d\omega) = d(F^*\omega)$;*
(5) *if $G : P \to M$ is another smooth map then $(F \circ G)^* = G^* \circ F^*$.*

3.20. Theorem. *Let M be a smooth manifold with or without boundary. There are operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$, uniquely determined by*

(1) *d is $\mathbb{R}$-linear*
(2) *for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ we have*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta;$$

(3) *$d^2 = 0$;*
(4) *for $f \in C^\infty(M)$ and $X \in \mathscr{X}(M)$ it holds that $df(X) = Xf$.*

In any chart we have

$$d(\sum_J \omega_J dx_{j_1} \wedge \cdots \wedge dx_{j_k}) = \sum_J d\omega_J \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

3.21. Theorem. *For a k-form ω and vector fields $X_1, \cdots, X_{k+1}$ it holds that*

$$\begin{aligned} d\omega(X_1, \cdots x_{k+1}) = &\sum_{1 \leq i \leq k+1} (-1)^{i-1} X_i \omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}). \end{aligned}$$

In particular, for a 1-form ω we have

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

4. Week 4

No lectures in week 4.

5. Week 5

5.1. Lecture 7, Tue 07-11-2017. This lecture coves the material in **[1, Chapter 15, pages 377-384]** concerning orientations of manifolds.

5.1. Definition. Let $e := (e_1, \cdots, e_n)$ and $E := (E_1, \cdots, E_n)$ be two ordered bases for the vector space V. We say that e and E are *consistently oriented* if the transition matrix (B_i^j) defined by $e_i = \sum_j B_i^j E_j$ has positive determinant.

The above definition gives an equivalence relation on the set of ordered basis with exactly two equivalence classes. The *orientation* determined by the basis e is denoted $[e]$. If $[E] = [e]$ we say that E is *positively oriented with respect to* e. Otherwise it is *negatively oriented.*

5.2. Proposition. *Let V be a vector space of dimension $n \geq 1$. Then any $\omega \in \bigwedge^n V^*$ with $\omega \neq 0$ determines a unique orientation $\mathscr{O}_\omega$ on V. An ordered basis $(e_1, \cdots, e_n)$ is* positively oriented with respect to ω *if*

$$\omega(e_1, \cdots e_n) > 0,$$

and negatively oriented with respect ω *if*

$$\omega(e_1, \cdots, e_n) < 0.$$

Two elements $\omega, \eta \in \bigwedge^n V^*$ define the same orientation on V if and only if $\omega = \lambda\eta$ for some $\lambda > 0$.

5.3. Example. Let $(e_1, \cdots, e_n)$ be an ordered basis for V with dual basis $(\varepsilon_1, \cdots, \varepsilon_n)$. Then $(e_1, \cdots, e_n)$ and $\omega := \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ define the same orientation.

For manifolds the situation is more complicated. A *pointwise orientation* of a manifold M is a choice of orientation on each tangent space T_pM, $p \in M$. A local frame $(X_1, \cdots, X_n)$ is *psotively oriented* if $(X_{1,p}, \cdots, X_{n,p}) \in T_p(M)$ is positively oriented.

5.4. Definition. A pointwise orientation for M is *continuous* if every $p \in M$ has a neighborhood U such that there exists a positively oriented local frame over U. A manifold M is *oriented* if it is equipped with a continuous pointwise orientation.

5.5. Definition. Let M be an n-dimensional manifold. An n-form $\omega \in \Omega^n(M)$ is *non-vanishing* if for for every $p \in M$ there exists a local frame $(X_1, \cdots, X_n)$ with dual coframe $(dx_1, \cdots, dx_n)$ such that $\omega = f dx_1 \wedge \cdots dx_n$ and $f(p) \neq 0$.

5.6. Proposition. *Let M be a smooth manifold of dimension n (with or without boundary). Any non-vanishing n-form $\omega \in \Omega^n(M)$ determines a unique orientation on M. Conversely if M is oriented there exists a non-vanishing n-form defining the orientation.*

5.7. Definition. Let M be an oriented manifold (with or without boundary). A chart (U, φ) is *positively oriented* if the frame $\frac{\partial}{\partial x_i}$ is positively oriented. An atlas $\{(U_i, \varphi_i)\}$ is positively oriented if the transition maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j),$$

have positive Jacobian determinant at each point of $\varphi_j(U_i \cap U_j)$.

5.8. Proposition. *Let M be a smooth manifold of dimension $n \geq 1$. Suppose M admits a consistently oriented smooth atlas $\{(U_i, \varphi_i)\}$. Then M is orientable and there is a unique orientation for which each (U_i, φ_i) is positively oriented. Conversely if M is oriented and $n > 1$ or $\partial M = \emptyset$ then the collection of all positively oriented smooth charts is a consistently oriented atlas.*

Now suppose $F : M \to N$ is a local diffeomorphism between smooth manifolds M and N. If M and N are oriented we say that F is *orientation preserving* if $(dF)_p : T_pM \to T_pN$ maps positively oriented bases to positively oriented bases. F is *orientation reversing* if $(dF)_p$ maps positively oriented bases to negatively oriented bases.

5.9. Proposition. *Let $F : M \to N$ be a local diffeomorphism and suppose that N is oriented. Then M is orientable and there exists a unique orientation on M for which F is orientation preserving.*

In this case, if ω is an orientation form for N then $F^*\omega$ is an orientation form for M.

5.2. Lecture 8. We return to some structural results about manifold with boundary, see **[1, Chapter 1, pages 27-29]**.

5.10. Theorem. *Let M be a smooth manifold with boundary and $p \in M$. Suppose that there is a boundary chart (U, φ) with $p \in U$ such that $\varphi(U) \subset \overline{\mathbb{H}}^n$ and $\varphi(p) \in \partial\overline{\mathbb{H}}^n$. Then for any other chart (V, ψ) with $p \in V$ it holds that $\varphi(V) \subset \overline{\mathbb{H}}^n$ and $\varphi(p) \in \partial\overline{\mathbb{H}}^n$.*

5.11. Corollary. *A manifold with boundary decomposes as a disjoint union $M = \mathrm{Int}M \sqcup \partial M$.*

To equip ∂M with a smooth structure we use the results from **[1, Chapter 5, pages 101-104]**.

5.12. Definition. Let M be a manifold and $S \subset M$ a subset. Then S is an *embedded submanifold* if, equipped with the subspace topology, it has a smooth structure such that the inclusion map $i : S \to M$ is a *smooth embedding*. That is, i is a homeomorphism onto its image and $(di)_p : T_pS \to T_pM$ is injective for all $p \in M$.

5.13. Theorem. *Let M be a manifold of dimension n with boundary ∂M. Then ∂M is a manifold of dimension $n - 1$ with charts (V, ψ) given by*

$$V := U \cap \partial M, \quad \psi := \pi_{n-1} \circ \varphi,$$

where (U, φ) is a chart for M and $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is given by

$$\pi_{n-1}(x_1, \cdots, x_n) := (x_1, \cdots, x_{n-1}).$$

With these charts the inclusion $i : \partial M \to M$ becomes a smooth embedding.

To equip ∂M with an orientation, we return to **[1, Chapter 15, pages 384-387]**.

For an oriented manfiold with boundary the tangent space at a boundary point $p \in \partial M$ decomposes as a disjoint union

$$T_pM = T_p^{\text{in}}M \sqcup T_p^{\text{out}}M \sqcup T_p\partial M,$$

where

$$T_p^{\text{in}}M := \{\sum_{k=1}^{n} v_k \frac{\partial}{\partial x_k} : v_n > 0\}, \quad T_p^{\text{out}}M := \{\sum_{k=1}^{n} v_k \frac{\partial}{\partial x_k} : v_n < 0\}.$$

We call $T_p^{\text{in}}M$ the *inward pointing* tangent vectors and $T_p^{\text{out}}M$ the *outward pointing* tangent vectors.

5.14. Lemma. *Let M be an oriented manifold with boundary. There exists a vector field $X \in \mathscr{X}(M)$ such that for every $p \in \partial M$, X_p is an outward pointing vector. Similarly there exists a vector field $Y \in \mathscr{X}(M)$ such that for every $p \in \partial M$, X_p is an inward pointing vector.*

5.15. Proposition. *Let $n \geq 1$ and M an oriented smooth n-dimensional manifold. Then ∂M is oriented and all outward pointing vector fields define the same orientation on ∂M.*

[1, Chapter 16, pages 400-404] Recall that a *domain of integration* in $\mathbb{R}^n$ is a subset $D \subset \mathbb{R}^n$ whose topological boundary $\overline{D} \setminus D^\circ$ has Lebesgue measure zero. A continuous n-form ω on $\overline{D}$ can be written

$$\omega = f dx_1 \wedge \cdots \wedge dx_n.$$

We define the integral of ω over D by

$$\int_D \omega := \int_D f dx_1 \cdots dx_n.$$

5.16. Lemma. *Let $U \subset \mathbb{R}^n$ be an open set and $K \subset U$ a compact set. Then there exists a domain of integration D such that $K \subset D \subset \overline{D} \subset U$.*

If ω is an n-form with compact support contained in an open set U we define

$$\int_U \omega := \int_D \omega,$$

where D is any domain with $\operatorname{supp} \omega \subset D \subset \overline{D} \subset U$.

5.17. Proposition. *Let D and E be domains of integration in $\mathbb{R}^n$ or $\overline{\mathbb{H}}^n$ and $G : \overline{D} \to \overline{E}$ a smooth map which restricts to an orientation preserving diffeomorphism $G : D \to E$. Then for an n-form ω on $\overline{E}$*

$$\int_D G^*\omega = \int_E \omega.$$

In case $G : D \to E$ is orientation reversing, we have

$$\int_D G^*\omega = -\int_E \omega.$$

5.18. Proposition. *Let U, V be open subsets of $\mathbb{R}^n$ or $\overline{\mathbb{H}}^n$ and $G : U \to V$ an orientation preserving diffeomorphism. If ω is a compactly supported n-form on V then*

$$\int_V \omega = \int_U G^*\omega,$$

and if G is orentation reversing then

$$\int_V \omega = -\int_U G^*\omega.$$

6. Week 6

6.1. Lecture 9, Tue 14-11-2017. [1, Chapter 16, pages 404-408 and 411-415]

We are now ready to define integration of n-froms on an oriented manifold M. First suppose ω is an n-form whose support is contained in a single positively oriented chart (U, φ). For such ω we set

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

If the chart (U, φ) is negatively oriented we set

$$\int_M \omega := -\int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

6.1. Proposition. *Suppose ω is a compactly support n-form on an oriented manifold M, and (U, φ), (V, ψ) are charts such that* $\operatorname{supp}\omega \subset U \cap V$. *Then*

$$\int_{\varphi(U)} (\varphi^{-1})^*\omega = \int_{\psi(V)} (\psi^{-1})^*\omega,$$

and in particular $\int_M \omega$ is independent of the choice of chart.

6.2. Definition. Let M be an oriented smooth manifold and ω a compactly supported n-form. Let $\{(U_i, \varphi)\}$ be an atlas of oriented charts and χ_i a partition of unity subordinate to U_i. The *integral of ω over M* is defined to

$$\int_M \omega := \sum_i \int_M \chi_i\omega.$$

6.3. Proposition. *The definition of $\int_M \omega$ is independent of the choice of cover and the choice of partition of unity.*

The integral so defined has the following properties:

(1) for $a, b \in \mathbb{R}$ and ω, η compactly supported n-forms,
$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$$
(2) if $-M$ denotes M with the opposite orientation then
$$\int_{-M} \omega = -\int_M \omega$$
(3) if ω is a positively oriented orientation form then $\int_M \omega > 0$
(4) if $F : M \to N$ is an orientation preserving diffeomorphism between oriented manifolds M and N then $\int_M \omega = \int_N F^*\omega$.

For an oriented manifold M with boundary ∂M we always equip ∂M with the induced (or Stokes) orientation. Given an $n-1$-form ω we set
$$\int_{\partial M} \omega := \int_{\partial M} i^*\omega,$$
with $i : \partial M \to M$ the embedding. Note that $d\omega$ is a n-form on M

6.4. Theorem (Stokes' theorem)**.** *Let M be an oriented manifold with boundary ∂M ad ω a compactly supported $n-1$-form on M. Then*
$$\int_M d\omega = \int_{\partial M} \omega,$$
where ∂M carries the Stokes orientation.

We now turn to the discussion of Riemannian metrics **[1, Chapter 13, pages 327-337]**.

6.5. Definition. Let M be a smooth manifold with or without boundary. A *Riemannian metric* on M is a paring
$$\begin{aligned} g : \mathscr{X}(M) \times \mathscr{X}(M) &\to C^\infty(M) \\ (X, Y) &\mapsto g(X, Y), \end{aligned}$$
with the following properties.

- (symmetry) for all $X, Y \in \mathscr{X}(M)$ we have $g(X, Y) = g(Y, X)$
- (bilinearity) for all $f_1, f_2 \in C^\infty(M)$ and $X, Y_!, Y_2$ we have
$$g(X, f_1Y_1 + f_2Y_2) = f_1 g(X, Y_1) + f_2 g(X, Y_2),$$
- (nondegeneracy) for $p \in M$ and all $X \in \mathscr{X}(M)$ it holds that $g(X, X)(p) > 0$.

The pair (M, g) is called a *Riemannian manifold*.

For each $p \in M$ the metric g defines an inner product on the tangent space $T_p(M)$ denoted $\langle \cdot, \cdot \rangle_g$. It is defined by

$$\langle X_p, Y_p \rangle_g := g(X, Y)(p).$$

6.6. Lemma. *For any manifold there exists a Riemannian metric.*

For 1-forms $\omega, \eta \in \Omega^1(M)$ we define their *symmetric product* to be the two form $\omega \cdot \eta$ given on vector fields X, Y by

$$\omega \cdot \eta(X, Y) := \frac{1}{2}(\omega(X)\eta(Y) + \omega(Y)\eta(X)).$$

6.7. Example (Euclidean metric on $\mathbb{R}^n$)**.** The expression

$$g = \sum_{i=1}^{n} dx_i \cdot dx_i = \sum_{i=1}^{n} (dx_i)^2,$$

defines a Riemannian metric on $\mathbb{R}^n$ called the *Euclidean metric*.

6.8. Example (Round metric on $\mathbb{S}^n$)**.** The restriction of the Euclidean metric on $\mathbb{R}^{n+1}$ to $\mathscr{X}(M)$ gives the *round metric* on $\mathbb{S}^n$.

6.9. Example (Hyperbolic metric on $\mathbb{H}^n$)**.** Recall that $\mathbb{H}^n$ is the upper half space

$$\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

The *hyperbolic metric* on $\mathbb{H}^n$ is given by

$$g = \frac{\sum_{i=1}^{n} (dx_i)^2}{x_n^2}.$$

6.2. Lecture 10. A Riemannian metric defines an inner product on each tangent space. This allows us to talk about the length of tangent vectors and angles between them:

Two vector fields X, Y are *orthogonal* over a set U if $g(X, Y)(p) = 0$ for all $p \in U$. For a vector field X we denote by $|X|$ the function $p \mapsto \sqrt{g(X, X)(p)}$ on M.

6.10. Definition. A smooth local frame $(X_1, \cdots, X_n)$ over U is *orthonormal* if

$$g(X_i, X_j)(p) = \delta_{ij}, \quad \text{for all } p \in U.$$

In particular $X_i(p)$ is an orthonormal basis for T_pM for all $p \in U$. It is in general not true that the coordinate frame $\frac{\partial}{\partial x_i}$ associated to a chart (U, φ) is orthonormal.

6.11. Proposition. *For every $p \in M$ there is a neighborhood U of p and a smooth orthonormal frame over U.*

The following discussion of the Riemannian volume form can found in **[1, Chapter 15, pages 388-390]**

6.12. Proposition. *On an oriented Riemannian manifold* (M, g) *there is a unique positive orientation form* ω_g *such that*

$$\omega_g(E_1, \cdots, E_n) = 1,$$

for every orthonormal frame E_i.

6.13. Proposition (Volume form in a coordinate frame)**.** *Let* (M, g) *be an oriented Riemannian manifold of dimension* $n \geq 1$ *and* (U, φ) *a positively oriented chart with coordinates* x_i. *The volume form* ω_g *in these cooridnates is given by*

$$\omega_g = \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n,$$

with $g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$.

The normal bundle and its orthonormal frames are introduced in **[1, Chapter 13, page 337]**.

For an embedded submanifold $S \subset M$ of a Riemannian manifold (M, g). For $p \in S$ the tangent space T_pS is a subspace T_pM. We define the *normal space* to be

$$N_pS := \{v \in T_pM : \forall w \in T_pS \langle v, w \rangle g = 0\}.$$

The *normal bundle* is the collection

$$NS := \bigsqcup_{p \in S} N_pS \subset TM,$$

and the bundle projection $\pi : TM \to M$ restricts to a bundle projection $NS \to S$. The normal bundle is a vector bundle over S of rank dim $M-$dim S. For every $p \in S$ we have $T_pM = T_pS \oplus N_pS$.

6.14. Proposition. *Let* (M, g) *be a Riemannian manifold of dimension* n *and* $S \subset M$ *an embedded submanifold of dimension* k. *For each* $p \in S$ *there exists a neighborhood* U *of* p *a smooth local orthonormal frame* $(E_1, \cdots E_n)$ *over* U *such that* $(E_1, \cdots, E_k)$ *is a local orthonormal frame for* TS *over* $S \cap U$ *and* $(E_{k+1}, \cdots, E_n)$ *is a local orthonromal frame for* NS *over* $S \cap U$.

The integration of functions and the divergence theorem are discussed in **[1, Chapter 16, pages 421-424]**.

The *volume integral* of a compactly supported continuous function $f \in C(M)$ on a Riemannian manifold (M, g) is defined to be

$$\int_M f dV_g := \int_M f \omega_g.$$

The volume integral has the property that if $f \geq 0$ then $\int_M f dV_g \geq 0$. For a codimension 1 submanifold $S \subset M$, we define a *normal vector field* to be a vector field $N \in \mathscr{X}(M)$ such that for all $p \in S$ we have $N(p) \in N_pS$ and $g(N,N)(p) = 1$. If N is outward pointing at each point of S, then it defines an orientation on S. In fact

$$\omega_{\tilde{g}}^S(X_1, \cdots, X_{n-1}) := \omega_g^M(N, X_1, \cdots, X_{n-1}),$$

defines the volume form on S with the induced metric $\tilde{g}$ for the orientation determined by N.

Consider the map

$$\alpha : C^\infty(M) \to \Omega^n(M), \quad f \mapsto f\omega_g,$$

as well as the map

$$\beta : \mathscr{X}(M) \to \Omega^{n-1}(M),$$

defined by $\beta(X)(X_1, \cdots, X_{n-1}) = \omega_g(X, X_1, \cdots, X_{n-1})$.

6.15. Lemma. *Let (M, g) be a Riemannian manifold and $S \subset M$ an embedded submanifold of codimension 1 with $i : S \to M$ the inclusion and normal vector field N. Then for all $X \in \mathscr{X}(S)$ it holds that*

$$i_S^*\beta(X) = \langle X, N\rangle_g \omega_{\tilde{g}}^S,$$

where $\omega_{\tilde{g}}^S = \beta(N)$ as above is the volume form on S determined by N.

We define the *divergence* of a vector field to be $\text{div}(X) := \alpha^{-1}d\beta(X)$. Equivalently $d\beta(X) = \text{div}(X)\omega_g$.

6.16. Theorem (Divergence theorem)**.** *Let (M, g) be an oriented Riemannian manifold with boundary ∂M and outward pointing normal vector field N. For any compactly supported smooth vector field $X \in \mathscr{X}(M)$ it holds that*

$$\int_M \text{div}(X) dV_g = \int_{\partial M} \langle X, N\rangle_g dV_{\tilde{g}},$$

where $\tilde{g}$ denotes the induced metric on S.

It should be noted here we equip ∂M with the Stokes orientation, which creates the need to work with an outward pointing normal. However, the divergence theorem holds in this form whenever S is equipped with the orientation inherited from N.

7. Week 7

7.1. Lecture 11, Tue 21-11-2017. The tangent cotangent isomorphism **[1, Pages 340-343]**.

Given a Riemannian manifold (M, g) we can define an isomorphism

$$\hat{g} : TM \to T^*M$$

defined on vector fields X via the formula

$$\hat{g}(X)(Y) := g(X, Y),$$

so indeed $\hat{g}(X) \in \Omega^1(M)$. The map $\hat{g}$ is injective by nondegeneracy of g and because the fibers of TM and T^*M are finite dimensional, $\hat{g}$ is fibrewise surjective. In coordinates $\hat{g}$ has the expression

$$\hat{g}(X) = \sum_{i,j} g_{ij} X_i dx_j,$$

where X_i are the component functions of X and $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ in the coordinates x_i. Because the matrix g_{ij} is invertible, the inverse

$$\hat{g}^{-1} : \Omega^1(M) \to \mathscr{X}(M),$$

takes the coordinate form

$$\hat{g}^{-1}(\omega) = \sum_{i,j} (g^{-1})_{ij} \omega_j \frac{\partial}{\partial x_i},$$

with $(g^{-1})_{ij}$ the components of the inverse matrix of (g_{ij}). The existence of the inverse proves that $\hat{g}$ is an isomorphism.

7.1. Definition. Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. The *gradient* of f is the vector field $\text{grad} f := \hat{g}^{-1}(df)$. Equivalently $\text{grad} f$ is determined by the equality

$$\langle \text{grad} f, X \rangle_g = Xf,$$

for all smooth vector fields $X \in \mathscr{X}(M)$.

The coordinate form of the gradient is

$$\text{grad} f = \sum_{i,j} (g^{-1})_{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$$

Let (M, g) be an oriented manifold with boundary. We wish to show there always exists an outward pointing normal vector field along ∂M. See **[1, Pages 118-119]**.

7.2. Definition. Let M be a smooth manifold with boundar. A *boundary defining function* is a smooth function $f : M \to \mathbb{R}$ with properties

- $f^{-1}(0) = \partial M$;

- for all $p \in \partial M$ the differential $df_p \neq 0$

7.3. Proposition. *Every manifold with boundary admits a boundary defining function.*

The following result is found on **[1, Page 391]**:

7.4. Corollary. *Every manifold with boundary admits an outward pointing unit normal vector field.*

Given a boundary defining function f one sets $N := -\text{grad}f/|\text{grad}f|_g$. This is well defined in a neighborhood

$$\partial M \subset \{p \in M : |df_p|_g > \varepsilon\},$$

and can thus be extended to all of M.

Line integrals **[1, Pages 287-292]**

7.5. Definition. By a *piecewise smooth curve* in a manifold M we mean a smooth map $\gamma : [a, b] \to M$ such that there exists a partition

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

such that the restrictions $\gamma|_{[a_i,a_{i+1}]} : [a_i, a_{i+1}] \to M$ are smooth.

7.2. Lecture 12, Wed 22-11-2017. For a one form ω on M we define the *integral of ω over γ* as

$$\int_\gamma \omega := \sum_i \int_{[a_i,a_{i+1}]} \gamma^*\omega.$$

By a *reparametrization* of the curve γ we mean a curve of the form

$$\tilde{\gamma} := \gamma \circ \phi : [c, d] \to M,$$

with $\phi : [c, d] \to [a, b]$ a diffeomorphism. The integral is invariant for reparametrizations in the following sense:

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega,$$

when ϕ is increasing. When ϕ is decreasing the integrals differ by a minus sign. The line integral has the usual linearity properties and if $F : M \to N$ is a smooth map and $\omega \in \Omega^1(N)$ then

$$\int_\gamma F^*\omega = \int_{F\circ\gamma} \omega.$$

The *tangent vector field* to γ is defined to be the map

$$\gamma' : [a, b] \to TM, \quad t \mapsto d\gamma(\frac{d}{dx}|_t),$$

with x the coordinate on $[a,b]$. The line integral admits the epxression

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t)).$$

The Riemannian distance function **[1, Pages 337-341]**.

7.6. Proposition. *If M is a connected manifold then for any two points p,q there exists a piecewise smooth curve $\gamma : [a,b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$.*

On a Riemannian manifold (M,g) we define the *length* of a piecewise smooth curve γ as

$$L_g(\gamma) := \int_a^b |\gamma'(t)|_g dt = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_g} dt$$

7.7. Proposition. *Let (M,g) be a Riemannian manifold and $\gamma : [a,b] \to M$ a piecewise smooth curve in M. If $\tilde{\gamma} : [c,d] \to M$ is a reparametrization of γ then $L_g(\gamma) = L_g(\tilde{\gamma})$.*

The Riemannian distance function of (M,g) is defined for points $p,q \in M$ as

$$d_g(p,q) := \inf\{L_g(\gamma) : \gamma : [a,b] \to M, \gamma(a) = p, \gamma(b) = q.\},$$

the infimum of lengths of piecewise smooth curves joinging p and q. To prove that the distance function is a metric we use the following local result.

7.8. Lemma. *Let g be a Riemannian metric on an open subset $U \subset \mathbb{R}^n$ and let $\overline{g}$ denote the Euclidean metric. Then for any compact subset $K \subset U$ there exist $c, C \in \mathbb{R}_{>0}$ such that for all $x \in K$ with $v \in T_x\mathbb{R}^n$ it holds that*

$$c|v|_{\overline{g}} \leq |v|_g \leq C|v|_{\overline{g}}.$$

7.9. Theorem. *The Riemannian distance function defines a metric on M whose metric topology coincides with the manifold topology.*

8. Week 8

8.1. Lecture 13, Tue 28-11-2017.

Review of tensor bundles.

8.1. Definition. Let V be a vector space. A *covariant k-tensor* on V is an element of $(V^*)^{\otimes k} := V^* \otimes \cdots \otimes V^*$ (k-fold tensor product). A *contravariant k-tensor* is an element of $V^{\otimes k} := V \otimes \cdots \otimes V$ (k-fold tensor product).

A covariant tensor ξ can be viewed as a multilinear functional $V^k \to \mathbb{R}$ via

$$(\xi_1 \otimes \cdots \otimes \xi_k)(v_1, \cdots, v_k) := \prod_{i=1}^{k} \xi_i(v_i).$$

Similarly a contravariant k-tensor gives a multilinear functional $(V^*)^k \to \mathbb{R}$, by essentially the same formula.

A k-tensor α is *symmetric* if for any permutation $\sigma \in S_k$ we have

$$\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \alpha(v_1, \cdots, v_k).$$

It is alternating if

$$\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\alpha(v_1, \cdots, v_k).$$

The *symmetrization* of a k-tensor α is the k-tensor

$$\mathrm{Sym}(\alpha)(v_1, \cdots, v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}).$$

The *anti-symmetrization* of α is the k-tensor

$$\mathrm{A}(\alpha)(v_1, \cdots, v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}).$$

Clearly $\mathrm{A}(\alpha)$ is *alternating*, that is

$$\mathrm{A}(\alpha)(v_{\tau(1)}, \cdots, v_{\tau(k)}) = \mathrm{sgn}(\tau)\mathrm{A}(V_1, \cdots, v_k),$$

for any $\tau \in S_k$. In general, if α, β are (anti)-symmetric tensors, then $\alpha \otimes \beta$ is in general neither symmetric nor anti-symmetric. We have seen that the wedge product of alternating tensors is again alternating. Similarly the *symmetric product* of a symmetric k-tensor α and a symmetric ℓ-tensor β, defined by

$$\alpha \cdot \beta(v_1, \cdots, v_{k+\ell}) := \frac{1}{(k+\ell)!}\sum_{\sigma \in S_{k+\ell}} \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+\ell)}),$$

is a symmetric $k + \ell$-tensor. The symmetric product is commutative,

$$\alpha \cdot \beta = \beta \cdot \alpha,$$

and satisfies the distributive law

$$(a\alpha + b\beta) \cdot \gamma = a\alpha \cdot \gamma + b\beta \cdot \gamma, \quad a, b \in \mathbb{R}.$$

8.2. Definition. Let M be a manifold. The *bundle of covariant k-tensors* on M is

$$T^k M := (T^* M)^{\otimes k} = \bigsqcup_{p \in M} (T_p^* M)^{\otimes k},$$

and the *bundle of contravariant k-tensors* is

$$T_k M := (TM)^{\otimes k} = \bigsqcup_{p \in M} (T_p^* M)^{\otimes k}.$$

The bundle of *mixed tensors of type* (k, ℓ) is

$$T_\ell^k M := (T^* M)^{\otimes k} \otimes (TM)^{\otimes \ell}.$$

Using the vector bundle chart lemma, we define maps

$$\tau_{ij} : U_i \cap U_j \to GL(\mathbb{R}^{nk} \otimes \mathbb{R}^{*\otimes n\ell}),$$

by

$$\tau_{ij}(p)(v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_\ell) := \\ \tau_{ij}^{TM}(p) v_1 \otimes \cdots \otimes \tau_{ij}^{TM}(p) \otimes \tau_{ij}^{T^*M}(p) \omega_1 \otimes \cdots \otimes \tau_{ij}^{T^*M}(p) \omega_\ell.$$

In this way $T_\ell^k M$ becomes a vector bundle over M. A *tensor field* of type (k, ℓ) is a section of $T_\ell^k M$.

By applying he duality map $\hat{g} : TM \to T^*M$ to any index we get maps $T_\ell^k M \to T_{\ell-1}^{k+1} M$ and by applying $\hat{g}^{-1}$ we obtain maps $T_\ell^k M \to T_{\ell+1}^{k-1} M$. Lastly,for a contravariant 2-tensor on a Riemannian manifold we define its *trace* to be the map

$$T_2 M \to M \times \mathbb{R},$$

determined on vector fields X, Y by

$$X \otimes Y \mapsto g(X, Y).$$

Connections. To address the problem of differentiating vector fields we introduce the notion of connection.

8.3. Definition. Let $\pi : E \to M$ be a smooth vector bundle over a manifold M. A *connection* is a linear map $\nabla : \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(M)} \Omega^1(M)$ satisfying the *Leibniz rule*:

$$\nabla(Y \cdot f) = \nabla(Y) f + Y \otimes df,$$

for all sections $Y \in \Gamma^\infty(E)$ and functions $f \in C^\infty(M)$.

Using the pairing

$$\mathscr{X}(M) \times \Omega^1(M), \quad (X, \omega) \mapsto \omega(X),$$

we obtain a pairing

$$\mathscr{X}(M) \times \Gamma^\infty(E) \otimes \Omega^1(M), \quad (X, Y \otimes \omega) \mapsto Y \cdot \omega(X).$$

Writing this pairing as $(Y \otimes \omega)(X)$ we can view a connection as a map

$$\mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M), \quad (X, Y) \mapsto \nabla(Y)(X).$$

The common notation for $\nabla(Y)(X)$ is $\nabla_X(Y)$. Connections are local in the following sense.

8.4. Lemma. *The value of vector field $\nabla_X Y$ at $p \in M$ depends only on the value of X at p and the values of Y in a neighborhood of p.*

Due to this lemma we write $\nabla_{X_p} Y$ for $\nabla_X(Y)(p)$ and think of it as the directional derivative of Y in the direction X_p.

8.5. Definition. An *affine* or *linear* connection is a connection in the vector bundle TM.

If E_i is a local frame for TM in a neighborhood U we can write any section $Y \in \mathscr{X}(TM)$ as $Y = \sum_i Y_i E_i$, with $Y_i \in C^\infty(M)$. In particular for $X \in \mathscr{X}(M)$ the section $\nabla_{E_i} E_j \in \Gamma^\infty(E)$ can be written

$$\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k,$$

for certain functions $\Gamma_{ij}^k : U \to \mathbb{R}$. These functions are referred to as the *Christoffel symbols* of the connection ∇ relative to the frame E_j. The Christoffel symbols determine the linear connection ∇ locally:

8.6. Lemma. *Let ∇ be a linear connection on a manifold M and E_i a local frame over the open set U. For vector fields $X, Y \in \mathscr{X}(M)$ we have*

$$\nabla_X Y = \sum_k \left(X(Y_k) + \sum_{i,j} X_i Y_j \Gamma_{ij}^k \right) E_k,$$

over U.

8.2. Lecture 14, Wed 29-11-2017.

8.7. Lemma (Existence of connections on charts)**.** *Let $U \subset \mathbb{R}^n$ be an open set. There is a bijective correpondence between connections on TU and the choice of n^3 functions Γ_{ij}^k via*

$$\nabla_X Y = \sum_k \left(X(Y_k) + \sum_{i,j} X_i Y_j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k},$$

for vector fields $\sum_i X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_i Y_i \frac{\partial}{\partial x_i}$.

8.8. Proposition. *Every manifold admits a linear connection.*

A connection is constructed using the connections ∇_i on charts U_i and gluing through a partition of unity χ_i to set $\nabla := \sum_i \chi_i \nabla_i$. Here it is important to note that the space of connections is not a vector space: a linear combination $\lambda_1 \nabla_1 + \lambda_2 \nabla_2$ of connections ∇_i is not a connection in general. It satisfies the Leibniz rule only if $\lambda_1 + \lambda_2 = 1$.

8.9. Lemma. *Let ∇ be a linear connection on M. There is a unique connection ∇ in each tensor bundle $T^k_\ell M$ with the properties*

(1) ∇ *agrees with the given connection on* TM
(2) *on* $T^0M = M \times \mathbb{R}$ ∇ *is given by* $\nabla(f) = df$, $\nabla_X f = X(f)$
(3) ∇ *obeys the follwoing Leibniz rule for tensor products:*

$$\nabla_X(F \otimes G) = \nabla_X(F) \otimes G + F \otimes \nabla_X(G)$$

(4) *if* (M, g) *is Riemannian,* ∇ *commutes with all contractions: if* Tr_g *denotes the trace on any pair of indices then*

$$\nabla_X(\mathrm{Tr}Y) = \mathrm{Tr}\nabla_X(Y).$$

The connection ∇ *satisifes the following additional properties:*

- *for all* $\omega \in \Omega^1(M)$ *and* $X, Y \in \mathscr{X}(M)$

$$\nabla_X(\omega(Y)) = \nabla_X(\omega)(Y) + \omega(\nabla_X(Y))$$

- *for any* $F \in T^k_\ell M$*, vector fields* Y_i *and 1-forms* ω_j *we have*

$$\begin{aligned}\nabla_X(F)(\omega_1, \cdots, \omega_\ell, Y_1, \cdots, Y_k) &= X(F(\omega_1, \cdots, \omega_\ell, Y_1, \cdots, Y_k) \\ &- \sum_j F(\omega_1, \cdots, \nabla_X \omega_j, \cdots, \omega_\ell, Y_1, \cdots, Y_k) \\ &- \sum_{i=1}^k F(\omega_1, \cdots, \omega_\ell, Y_1, \cdots, \nabla_X Y_i, \cdots, Y_k)\end{aligned}$$

We now construct the *total derivative* of a (k, ℓ) tensor field.

8.10. Lemma. *Let* ∇ *be a linear connection on a manifold* M *and* $F \in T^k_\ell(M)$*. The map*

$$\nabla F : \Omega^1(M)^\ell \times \mathscr{X}(M)^{k+1} \to C^\infty(M)$$

$$\nabla F(\omega_1, \cdots, \omega_\ell, X_1, \cdots, X_{k+1}) := \nabla_{X_{k+1}} F(\omega_1, \cdots, \omega_\ell, X_1, \cdots, X_k),$$

defines a $(k + 1, \ell)$ *tensor field.*

For $f \in C^\infty(M)$, $\nabla f = df$ and the 2-tensor field $\nabla(\nabla(f))$ is called the *covariant Hessian* of the function f.

Tangent vector fields along curves.

8.11. Definition. Let $\gamma : [a, b] \to M$ be a smooth curve. A *vector field along* γ is a map $V : [a, b] \to TM$ such that $V(t) \in T_{\gamma(t)}M$. We write $T(\gamma)$ for the space of all vector fields along γ.

The tangent vector field $\gamma'(t)$ is the most important example of a vector field along a curve.

8.12. Example. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth curve and let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation over $\frac{\pi}{2}$. Set $N(t) := J\gamma'(t)$. Then $N(t)$ is normal to $\gamma'(t)$. In coordinates $N(t) = (-\gamma_2'(t), \gamma_1'(t))$.

8.13. Example. Let $\tilde{X} \in \mathscr{X}(M)$ and define $X(t) := \tilde{X}_{\gamma(t)}$.

A vector field X along γ is *extendible* if there exists $\tilde{X} \in \mathscr{X}(M)$ such that $X = \tilde{X}|_\gamma$. Not all vector fields are extendible, e.g. if $\gamma(t_0) = \gamma(t_1)$ and $\gamma'(t_0) \neq \gamma'(t_1)$, then $\gamma'(t)$ is not extendible.

8.14. Lemma. *Let ∇ be a linear connection on a manifold M. For each smooth curve $\gamma : [a, b] \to M$, ∇ determines a unique operator $D_t : T(\gamma) \to T(\gamma)$ satisfying*

(1) $D_t(aV + bW) = aD_tV + bD_tW$
(2) *for all $f \in C^\infty([a, b])$ $D_t(fV) = f'V + fD_tV$*
(3) *if V is extendible then for any extension $\tilde{V}$ we have $D_tV = \nabla_{\gamma'(t)}\tilde{V}$.*

The operator D_t is called the *covariant derivative* along γ. The *acceleration* of a smooth curve $\gamma : [a, b] \to M$ is the vector field $D_t\gamma'$ along γ.

8.15. Definition. A smooth curve γ is a *geodesic* with respect to ∇ if $D_t\gamma' = 0$.

8.16. Theorem (Existence and uniqueness of geodesics)**.** *Let M be a manifold with a linear connection ∇. For any $p \in M$, $V \in T_p(M)$ and $t_0 \in \mathbb{R}$ there exists an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = V$. Any two such geodesics agree on their common domain.*

8.17. Corollary. *For any $p \in M$ and $V \in T_pM$ there exists a unique maximal geodesic $\gamma : I \to M$, that is, a geodesic that cannot be extended to any larger interval, such that $\gamma'(0) = p$ and $\gamma'(0) = V$. This geodesic is denoted γ_V.*

9. Week 9

9.1. Lecture 15, Tue 5-12-2017. A vector field V along γ is said to be *parallel* if $D_tV = 0$. A vector field $X \in \mathscr{X}(M)$ is parallel if it is parallel along every curve. It is easy to check that X is parallel if and only if $\nabla(X) = 0$.

9.1. Theorem (Parallel translation)**.** *Given $\gamma : [a, b] \to M$, $t_0 \in [a, b]$ and $V_0 \in T_{\gamma(t_0)}M$ there exists a unique parallel vector field V along γ such that $V(t_0) = V_0$.*

This theorem relies on the following existence and uniqueness result of linear ODE's.

9.2. Theorem. *Let $I \subset \mathbb{R}$ be an interval and $A^k_j : I \to \mathbb{R}$ be smooth functions, $1 \leq j, k \leq n$. The linear initial value problem*

$$V'_k(t) = \sum_j A^k_j V_j(t), \quad V_k(t_0) = B_k,$$

has a unique solution on all of I for any $t_0 \in I$ and any $B = (B_1, \cdots, B_n) \in \mathbb{R}^n$.

The Riemannian connection **[2, Chapter 5, pages 65-76]**.

Let $M \subset \mathbb{R}^n$ be an embedded submanifold. Denote by π^t the orthogonal projection $T_p\mathbb{R}^n \to T_pM$ and $\overline{\nabla}$ the Euclidean connection on $\mathbb{R}^n$.

9.3. Lemma. *Let $M \subset \mathbb{R}^n$ be an embedded submanifold. The operator $\nabla^t : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ given by $\nabla^t_X Y := \pi^t \nabla_X(Y)$ is a connection on M. This connection is called the* tangential connection *and satisfies*

$$\langle \nabla^t_X Y, Z\rangle + \langle Y, \nabla^t_X Z\rangle = \nabla^t_X \langle Y, Z\rangle,$$

with respect to the induced Riemannian metric.

Using the deep *Nash embedding theorem*, which states that any Riemannian manifold can be relaized as an embedded submanifold of some $\mathbb{R}^n$ with the induced metric, one could study any manifold as an embedded submanifold. This sheds no light on *intrinsic* properties. It turns out that the above connection can be characterized by two properties that relate it to the Riemannian metric.

9.4. Definition. Let (M, g) be a Riemannian manifold and ∇ a linear connection on M. The connection ∇ is *compatible* with the Riemannian metric if we have

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \nabla_X \langle Y, Z\rangle,$$

for all $X, Y, Z \in \mathscr{M}$.

9.5. Proposition. *For a linear connection on (M, g) the following are equivalent:*

- *∇ is compatible with g;*
- *$\nabla g = 0$;*
- *for any curve γ and vector fields V, W along γ we have*

$$\frac{d}{dt}\langle V, W\rangle = \langle V, D_t W\rangle + \langle D_t V, W\rangle$$

- *if V, W are parallel along γ then $D_t\langle V, W\rangle$ is constant*
- *parallel translation $P_{t_0 t_1} : T_{\gamma(t_0)} \to T_{\gamma(t_1)}$ is an isometry.*

The second intrinsic property of connections involves the *torsion tensor*

$$\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

We say that ∇ is *torsion free* if $\tau(X,Y)=0$ for all $X,Y\in\mathscr{X}(M)$.

9.6. Theorem. *Let (M,g) be a Riemannian manifold. There exists a unique linear connection ∇ on M that is compatible with g and torsion free.*

The above connection is called the *Riemannian connection*. Its Christoffel symbols are given by the explicit formula

$$\Gamma^k_{ij} = \sum_{\ell} \frac{1}{2}(g^{-1})_{k\ell}\left(\frac{\partial}{\partial x_i} g_{j\ell} + \frac{\partial}{\partial x_j} g_{i\ell} - \frac{\partial}{\partial x_\ell} g_{ij}\right).$$

9.7. Lemma. *Any Riemannian geodesic is a constant speed curve.*

9.8. Proposition. *Suppose that $\varphi : (M,g)\to(\tilde{M},\tilde{g})$ is an isometry and $\nabla,\tilde{\nabla}$ the respective Riemannian connections. Then*

- *φ intertwines the Riemannian connections:*
$$\varphi_*(\nabla_X Y)) = \tilde{\nabla}_{\varphi_* X}\varphi_* Y$$
- *If V is a vector field along a curve γ in M then*
$$\varphi_* D_t V = \tilde{D}_t \varphi_* V$$
- *φ takes geodescis to geodesics, that is, if γ_V is the geodesic through p with initial velocity V then $\varphi\circ\gamma_V$ is the geodesic through $\varphi(p)$ with initial velocity $\varphi_* V$.*

10. Week 10, lectures 16 and 17, see the notes by Kastenholz

11. Week 11

11.1. Lectures 18-19, Tue 19-12-2017, Wed 20-12-2017. The exponential map, **[2, Chapter 5, pages 72-76]**.

The exponential map is a map defined on an open subset $\mathscr{E}$ of the tangent bundle into M. Its restriction to to specific tangent spaces gives a diffeomorphism exp : $\mathscr{E}_p \to M$ onto its image. To be precise, set

$$\mathscr{E} := \{V\in TM : \gamma_V \text{ is defined on an interval containing } [0,1]\},$$

and define exp : $\mathscr{E}\to M$ by $V\mapsto \gamma_V(1)$. Furthermore, for $p\in M$ define $\mathscr{E}_p := T_pM\cap\mathscr{E}$ and $\exp_p : \mathscr{E}_p \to M$ the restriction of exp to $\mathscr{E}_p$. Recall that a subset X of a vector space is *star-shaped with respect to* $x\in X$ if for all $y\in X$ the line segment connecting x to y lies entirely within X.

11.1. Proposition (Properties of the exponential map)**.** *For a Riemannian manifold (M,g) we have that*

- *$\mathscr{E} \subset TM$ is open, contains the zero section, and each $\mathscr{E}_p$ is star-shaped with respect to 0;*
- *for each $V \in TM$ the geodesic γ_V is given by $\gamma_V(t) = \exp(tV)$ whenever either side is defined;*
- *the exponential map is smooth.*

The proof of the above statement relies on

11.2. Lemma (Rescaling lemma)**.** *For any $V \in TM$ and $c, t \in \mathbb{R}$ it holds that*

$$\gamma_{cV}(t) = \gamma_V(ct),$$

whenever either side is defined.

The exponential map is natural with respect to Riemannian isometries.

Normal neighborhood and normal coordinates **[2, Section 5, pages 76-81]**.

11.3. Lemma. *For any $p \in M$ there is a neighborhood V of $0 \in T_pM$ and a neighborhood U of p such that* exp : $U \to V$ *is a diffeomorphism.*

11.4. Definition. A neighborhood U of $p \in M$ is called a *normal neighborhood* if U is the image of a star-shaped (with respect to 0) open set $V \subset T_pM$ under $\exp_p$. If $\exp_p$ is a diffeomorphism on the ball $B_g(0,\varepsilon)$, then $\exp_p(B_g(0,\varepsilon))$ is a *geodesic ball* in M. If the closed ball $\overline{B_g(0,\varepsilon)}$ is contained in an open set V on which $\exp_p$ is a diffeomorphism, then $\exp_p(\overline{B_g(0,\varepsilon)})$ is called a *closed geodesic ball* and $\exp_p(\partial\overline{B_g(0,\varepsilon)})$ is a *geodesic sphere*.

Any orthonormal basis E_i of T_pM gives a diffeomorphism $E : \mathbb{R}^n \to T_pM$ by $(x_i) \mapsto \sum_i x_i E_i$ and so gives rise to a coordinate chart by considering $E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$. Such charts are called *normal coordinates at* p and they are in 1-1 correspondence with with orthonormal bases of T_pM.

In a normal coordinate chart at p we define the *radial distance function* by

$$r(x) := \left(\sum x_i^2\right)^{\frac{1}{2}},$$

and the *unit radial vector field* by

$$\frac{\partial}{\partial r} := \sum_i \frac{x_i}{r(x)} \frac{\partial}{\partial x_i}.$$

We emphasize that these objects depend on the normal coordinate chart at hand.

11.5. Proposition. *Let $(U, (x_i)$ be a normal coordinate chart at p.*

- *for any* $V = \sum_i V_i \frac{\partial}{\partial x_i}$ *the geodesic* γ_V *starting at* p *is given in coordinates by*

$$\gamma_V(t) = (tV_1, \cdots, tV_n),$$

as long as γ_V *stays within* U.
- *the coordinates of* p *are* $(0, \cdots, 0)$*;*
- *the components of the metric at* p *are* $g_{ij}(p) = \delta_{ij}$*;*
- *any Euclidean ball* $\{x : r(x) < \varepsilon\}$ *contained in* U *is a geodesic ball;*
- *for any* $q \in U \setminus p$ *the radial vector field* $\frac{\partial}{\partial r}$ *gives the velocity vector of the unit speed geodesic from* p *to* q *and thus has unit length with respect to* g*;*
- *the first partial derivatives of* g_{ij} *and the Christoffel symbols vanish at* p.

An open set $W \subset M$ is called a *uniformly normal neighborhood* of $p \in W$ if there exists $\delta > 0$ such that for every $q \in W$ the geodesic ball of radius δ around q contains W.

11.6. Lemma. *For any* $p \in M$ *and any open neighborhood* U *of* p *there exists a uniformly normal neighborhood* W *of* p *contained in* U.

12. Week 12

12.1. Lecture 20, Tue 9-1-2018. Material discussed can be found in **[2, Chapter 6, pages 96-98 and 102-106]**.

12.1. Definition. A piecwise smooth curve $\gamma : [a, b] \to M$ is *minimizing* if for any curve $\tilde{\gamma}$ between $p = \gamma(a)$ and $q = \gamma(b)$ we have $L(\gamma) \leq L(\tilde{\gamma})$.

If γ is minimizing it must hold that $L(\gamma) = d_g(p, q)$.

12.2. Definition. An *admissible family* of curves is a map $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ for which there is a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that $\Gamma : (-\varepsilon, \varepsilon) \times [a_{i-1}, a_i] \to M$ is smooth and for all $s \in (-\varepsilon, \varepsilon)$ $\Gamma_s(t) := \Gamma(s, t)$ is an admissible curve.

The curves Γ_s are called the *main curves*. The *transverse curves* are $\Gamma^t(s) := \Gamma(s, t)$ for t fixed and are smooth.

A *vector field along an admissible curve* Γ is a map $V : (-\varepsilon, \varepsilon) \times [a, b] \to TM$ such that $V_{(s,t)} \in T_{\Gamma(s,t)}M$. Moreover there should a (possibly finer) subdivision $a = b_0 < b_1 < \cdots < b_\ell = b$ for which $V_{(-\varepsilon,\varepsilon)\times[b_{i-1},b_i]}$ is smooth.

The most important examples of such vector fields are

$$\partial_t \Gamma(s, t) := \frac{d}{dt}\Gamma_s(t), \quad \partial_s \Gamma(s, t) := \frac{d}{ds}\Gamma^t(s).$$

The vector field $\partial_s\Gamma$ is continuous, but $\partial_t\Gamma$ is in general not continuous at the points a_i. For a vector field V along Γ we denote by D_tV the derivative of V along Γ_s and by D_sV the derivative of V along Γ^t.

12.3. Lemma (Symmetry Lemma)**.** *Let $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be an admissible family of curves. On each rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ where Γ is smooth it holds that $D_s\partial_t\Gamma = D_t\partial_s\Gamma$.*

12.4. Theorem (Gauss Lemma)**.** *Let (M, g) be a Riemannian manifold and U be a geodesic ball centered at $p \in M$. The unit radial vector field $\frac{\partial}{\partial r}$ is g -orthogonal to the geodesic spheres in U.*

12.5. Corollary. *Let (x_i) be normal coordinates on a geodesic ball centered at p and $r(x)$ the radial distance function. Then* $\operatorname{grad} r = \frac{\partial}{\partial r}$ *on $U \setminus p$.*

12.6. Proposition. *Suppose that q is contained in a geodesic ball around p. Then (up to reparametrization) the radial geodesic from p to q is the unique minimizing curve from p to q.*

12.7. Corollary. *Within a geodesic ball around p we have $r(x) = d_g(p, x)$.*

12.2. Lecture 21, Wed 10-1-2018. Material discussed can be found in **[2, Chapter 6, pages 107-111]**.

12.8. Definition. A piecewise smooth curve $\gamma : I \to M$ is *locally minimizing* if every $t_0 \in I$ has an open neighborhood U such that $\gamma_{|U}$ is minimizing between each pair of points in $\gamma(U)$.

12.9. Theorem. *Every Riemannian geodesic is locally mimimizing.*

12.10. Theorem. *Every minimizing curve is a geodesic.*

12.11. Definition. A Riemannian manifold (M, g) is *geodesically complete* if every maximal geodesic is defined for all $t \in \mathbb{R}$.

12.12. Example. An open ball in $\mathbb{R}^n$ is not geodesically complete.

Note that geodesic completeness implies that the exponential map is defined on all of T_pM for all $p \in M$.

12.13. Theorem (Hopf-Rinow)**.** *A connected Riemannian manifold without boundary is geodesically complete if and only if it is complete as a metric space.*

In fact our proof showed that if $\exp_p$ is defined on all of T_pM for some $p \in M$, then M is complete.

13. Week 13

13.1. Lecture 22, Tue 16-1-2018. [2, Chapter 7].

13.1. Definition. The *curvature endomorphism* of the Riemannian manifold (M, g) is the map

$$R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M),$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

R is a $(3, 1)$ tensor field and admits the local expression

$$R = \sum_{i,j,k,\ell} R^{\ell}_{ijk} dx_i \otimes dx_j \otimes dx_k \otimes \partial_\ell.$$

13.2. Definition. The *Riemann curvature tensor* Rm is the covariant 4-tensor field

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle_g.$$

Locally this is written as

$$Rm = \sum_{i,j,k,\ell} R_{ijk\ell} dx_i \otimes dx_j \otimes dx_k \otimes dx_\ell,$$

with

$$R_{ijk\ell} = \sum g_{\ell m} R^m_{ijk}.$$

13.3. Lemma. *The curvature endomorphism and Riemann tensor are local isometry invariants of* (M, g)*. That is if* $\phi : M \to \tilde{M})$ *is a local isometry then*

$$\phi^*(\widetilde{Rm}) = Rm, \quad \tilde{R}(\phi_* X, \phi_* Y)\phi_* Z = \phi_*(R(X, Y)Z).$$

13.4. Definition. A Riemannian manifold is *flat* if it is locally isometric to $\mathbb{R}^n$ with its Euclidean metric.

It is clear that for flat manifolds, $Rm = 0$. The converse is true as well. In order to prove this we need some facts about vector fields. A point $p \in M$ is a *regular point* of the vector field V if $V_p \neq 0$. The following canonical form result is **[1, Theorem 9.22]**.

13.5. Theorem. *Let* V *be a smooth vector field on* M *and* p *a regular point of* V*. There there exists a neighborhood of* p *and coordinates* (x_i) *such that* $V = \frac{\partial}{\partial x_1}$.

13.6. Definition. Let $D \subset M \times \mathbb{R}$ and $\theta : D \to M$ be a smooth flow. We say that the vector field W is *invariant under* θ if

$$(d\theta_t)_p(W_p) = W_{\theta_t(p)},$$

for all $(t, p) \in D$.

We define the *Lie derivative of* W *with respect to* V as

$$\begin{aligned}(\mathscr{L}_V W)_p :&= \frac{d}{dt}|_{t=0} d(\theta^V_{-t})_{\theta^V_t(p)}(W_{\theta^V_t(p)}) \\ &= \lim_{t\to 0} \frac{d(\theta^V_{-t})_{\theta^V_t(p)}(W_{\theta^V_t(p)}) - W_p}{t}.\end{aligned}$$

Here θ^V denotes the flow of V.

13.7. Lemma. *$\mathscr{L}_V(W)_p$ exists for all $p \in M$ and defines a smooth vector field.*

13.8. Theorem. $\mathscr{L}_V(W) = [V, W]$.

13.2. Lecture 23, Wed 17-1-2018.

13.9. Theorem. *For vector fields $V, W \in \mathscr{X}(M)$ the following are equivalent:*

(1) $[V, W] = 0$;
(2) *V is invariant under the flow of W;*
(3) *W is invariant under the flow of V.*

Two flows θ and ψ are said to *commute* if whenever one of the expressions

$$\theta_t \circ \psi_s(p), \quad \psi_s \circ \theta_t(p),$$

is defined then both are defined and they are equal.

13.10. Theorem. *Two vector fields V, W commute if and only if their flows commute.*

We now provide a criterion for when a given frame can be regarded as a coordinate frame.

13.11. Theorem. *Let M be an n-dimensional manifold and $(E_1, \cdots, E_n)$ a local frame over an open set W such that $[E_i, E_j] = 0$ on W. Then for each $p \in W$ there exists a smooth chart $(U, (x_i))$ around p such that $E_i = \frac{\partial}{\partial x_i}$.*

The above results are needed to proof the following characterization of flat manifolds.

13.12. Theorem. *A Riemannian manifold is flat if and only if $Rm = 0$.*

14. Week 14

14.1. Lecture 24, Tue 23-1-2018. We collect some symmetries of the Riemann tensor Rm which can be found **[2, Chapter 7]**.

14.1. Proposition. *The identities*

(1) $Rm(W,X,Y,Z) = -Rm(X,W,Y,Z)$
(2) $Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)$
(3) $Rm(W,X,Y,Z) = Rm(Y,Z,W,X)$
(4) $Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0$

The last identity is known as the first Bianchi identity.

14.2. Proposition (Second Bianchi identity)**.**

$$\nabla_W Rm(X,Y,Z,V) + \nabla_Z Rm(X,Y,V,W) + \nabla_V Rm(X,Y,W,Z) = 0.$$

We now consider some simpler tensors derived from the Riemann tensor.

14.3. Definition. The *Ricci tensor* is the covariant 2-tensor field

$$Rc : (X,Y) \mapsto \mathrm{Tr}_g(Z \mapsto R(Z,Y)X).$$

In coordinates

$$Rc = \sum_{i,j} R_{ij} dx_i \otimes dx_j = \sum_{i,j,k,\ell,m} g^{km} R_{kijm} dx_i \otimes dx_j.$$

The *scalar curvature* is the function $S := \mathrm{Tr}_g Rc = \sum g^{ij} R_{ij}$, where the last expression is a local one. The following result is **[2, Lemma 8.7]**

14.4. Proposition. *Let (M,g) be a 2-dimensional manifold. Then*

$$Rm(X,Y,Z,W) = \frac{1}{2} S(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$$
$$Rc(X,Y) = \frac{1}{2} S\langle X,Y\rangle$$
$$S = 2\frac{Rm(E_1,E_2,E_2,E_1)}{|E_1|^2|E_2|^2 - \langle E_1,E_2\rangle^2},$$

where in the last expression, E_1, E_2 is any basis of T_pM.

14.2. Lecture 25, Wed 24-1-2018. The discussion of the Gauss-Bonnet theorem is to be found in **[2, Chapter 9]**.

Suppose that $\gamma : [a,b] \to \mathbb{R}^2$ is a smooth unit speed closed curve. The *tangent angle function* is the map $\theta : [a,b] \to \mathbb{R}$ satisfying $\theta(a) \in (-\pi,\pi]$ and $\gamma'(t) = (\cos\theta(t), \sin\theta(t))$. This map is smooth as it is the lift of γ to the universal cover $\mathbb{R}$ of the unit circle.

14.5. Definition. If $\gamma : [a,b] \to \mathbb{R}^2$ is a unit speed smooth closed curve satisfying $\gamma'(a) = \gamma'(b)$ we define its *rotation angle* to be $\mathrm{Rot}(\gamma) := \theta(b) - \theta(a)$.

It is clear that $\mathrm{Rot}(\gamma) = 2k\pi$ for some integer k. We now extend the definition of rotation angle to piecewise smooth closed curves. Let

$$a = a_0 < a_1 < \cdots < a_k = b,$$

be the subdivision for which $\gamma|_{[a_{i-1},a_i]}$ is smooth. We call the points $\gamma(a_i)$ *vertices* and the segments $\gamma|_{[a_{i-1},a_i]}$ *edges*. Note that the limits

$$\gamma'(a_i^+) := \lim_{t\downarrow a_i} \gamma'(t), \quad \gamma(a_i^-) = \lim_{t\uparrow a_i} \gamma'(t),$$

both exist. We define the *exterior angle* ε_i between $\gamma'(a_i^+)$ and $\gamma'(a_i^-)$ to be chosen in $[-\pi, \pi]$ with a positive sign if $(\gamma'(a_i^-), \gamma'(a_i^+))$ is an oriented basis of $\mathbb{R}^2$ and a negative sign otherwise. If $\gamma(a_i^+) = -\gamma(a_i^-)$ there is no way to choose between π and $-\pi$ and we leave this case undefined.

14.6. Definition. A *curved polygon* in $\mathbb{R}^2$ is a simple closed piecewise smooth unit speed curve $\gamma : [a, b] \to \mathbb{R}^2$ such that

- None of the exterior angles equals $\pm\pi$;
- γ is the boundary of a bounded open set $\Omega \subset \mathbb{R}^2$.

A curved polygon γ is *positively oriented* if γ' is compatible with the Stokes orientation of $\partial\Omega$.

The tangent angle function can now be defined as follows: choose $\theta(a) \in (-\pi, \pi]$ and $\theta(t)$ as before for $t \in (a, a_1)$. Then set

$$\theta(a_1) := \lim_{t\uparrow a_1} \theta(t) + \varepsilon_1,$$

and proceed as before for $t \in (a_1, a_2)$. Inductively we then set

$$\theta(a_i) := \lim_{t\uparrow a_i} \theta(t) + \varepsilon_i.$$

We so obtain the tangent angle function $\theta : [a, b] \to \mathbb{R}^2$ and define the *rotation angle* of the curved polygon γ as $\mathrm{Rot}(\gamma) = \theta(b) - \theta(a)$.

14.7. Theorem (Hopf)**.** *If γ is a positively oriented curved polygon in $\mathbb{R}^2$ then* $\mathrm{Rot}(\gamma) = 2\pi$.

14.8. Definition. Let (M, g) be a Riemannian 2-manifold. A *curved polygon* in M is a piecewise smooth unit speed curve $\gamma : [a, b] \to M$ that is the boundary of an open set Ω with compact closure. Moreover we require that γ is contained in a single chart (U, φ) such that $\varphi \circ \gamma$ is a curved polygon in $\mathbb{R}^2$.

Becuase of the above definition, to define the tangent and exterior angles of a curved polygon in a 2-manifold, it is enough to do so for curved polygons contained in an open set of $\mathbb{R}^2$ with an arbitrary metric g. Using the Stokes orientation we define the exterior angle $\varepsilon_i \in [-\pi, \pi]$ at a_i by

$$\cos \varepsilon_i := \langle \gamma(a_i^+), \gamma(a_i^-) \rangle_g.$$

The tangent angle θ at smooth smooth points can be defined relative to $\frac{\partial}{\partial x_1}$, so this definition may depend on the chart chosen. As before we obtain $\theta : [a, b] \to \mathbb{R}$ and set $\mathrm{Rot}_g(\gamma) := \theta(b) - \theta(a)$.

14.9. Lemma. *If γ is a positively oriented polygon in M then* $\mathrm{Rot}_g(\gamma) = 2\pi$.

We denote by $N(t)$ the normal vector field to γ at smooth points that makes $(\gamma'(t), N(t))$ into an oriented basis. The *signed curvature* at smooth points is defined as

$$\kappa_N(t) := \langle D_t\gamma'(t), N(t)\rangle_g.$$

Since $D_t\gamma'(t)$ is orthogonal to $\gamma'(t)$ we obtain that $D_t\gamma'(t) = \kappa_N(t)N(t)$.

14.10. Theorem. *Let (M, g) be an oriented Riemannian 2-manifold and γ a positively oriented curved polygon in M. Then*

$$\frac{1}{2}\int_M S dV_g + \int_\gamma \kappa_N ds + \sum_{i=1}^{k} \varepsilon_i = 2\pi.$$

14.11. Definition. Let (M, g) be a Riemannian 2-manifold. A *triangulation* of M is a finite collection $\mathscr{T} = \{T_i\}$ of curved triangles T_i such that

- $T_i = \partial\Omega_i$ for precompact open sets Ω_i;
- $\bigcup_i \overline{\Omega_i} = M$;
- the intersections $T_i \cap T_j$ consist of at most a single vertex or a single edge.

Every smooth compact surface admits a triangulation and if N_v is the number of vertices, N_e the number of edges and N_f the number of faces (all counted once, that is without multiplicities) in the triangulation the the *Euler characteristic*

$$\chi(M, \mathscr{T}) = N_v - N_e + N_f$$

is independent of the triangulation and is in fact a topological invariant of M.

14.12. Theorem (Gauss-Bonnet)**.** *Let (M, g) be a compact oriented Riemannian 2-manifold. Then*

$$\int_M S dV_g = 4\pi\chi(M).$$

References

[1] John. M. Lee, *Introduction to smooth manifolds*, 2nd edition, Springer 2013.
[2] John. M. Lee, *Riemannian manifolds - An introduction to curvature*, Springer.