GLOBAL ANALYSIS I - WS 2017/2018

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Exercise sheet 1 - due 18-10-2018

1. Exercise. Let $N = (0, ..., 0, 1)$ be the "north pole" and $S = -N$ the "south pole" of the n-sphere

$$S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Define stereographic projection $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_1, ..., x_{n+1}) = \frac{(x_1, ..., x_n)}{1 - x_{n+1}}.$$

Let $\tau(x) := -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(a) Show that σ is bijective, and $\sigma^{-1}(u_1, ..., u_n) = \frac{(2u_1, ..., 2u_n, |u|^2 - 1)}{|u|^2 + 1}$

(b) Compute the transition map $\tau \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus N, \sigma)$ and $(S^n \setminus S, \tau)$ defines a smooth structure on S^n. (The coordinates defined by σ and τ are called stereographic coordinates.)

(c) Show that this smooth structure is the same as the one defined in the lecture, using the charts

$$U_i^{\pm} : \{(x_1, \cdots x_{n+1}) : \pm x_i > 0\}, \quad \phi_i^{\pm}(x) := (x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}).$$

2. Exercise. Show that an open cover $\{U_\alpha\}_{\alpha \in A}$ of precompact sets is locally finite if and only if every U_α has nonempty intersection with only finitely many U_β. Give a counterexample in case the cover is not consisting of open sets and in case the cover is not consisting of precompact sets.

3. Exercise. Write down a paritition of unity subordinate to the cover $(S^n \setminus \setminus N, \sigma), (S^n \setminus S, \tau)$ of S^n.

4. Exercise. Recall the extension Lemma:

Let M be a smooth manifold, and suppose f is a smooth function defined on a closed subset $A \subset M$. For any open set U containing A, there exists a *smooth function* $\tilde{f} \in C^\infty(M)$ *such that* $\tilde{f}|_A = f$ *and* $\text{supp} f \subset U$.

Show that the condition that A be closed is necessary by giving an example of a smooth function on a nonclosed subset of a manifold that admits no smooth extension to the whole manifold.