GLOBAL ANALYSIS I - WS 2017/2018

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Exercise sheet 10 - due Wednesday 10-1 -2018

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise (Cf. Exercise 5.3 of *Riemannian manifolds*)**.** Let $M \subset \mathbb{R}^n$ be an embedded submanifold with Riemannian metric g induced from the Euclidean metric on $\mathbb{R}^n$. Denote by π^t the orthogonal projection $T_p\mathbb{R}^n \to T_pM$ and by $\overline{\nabla}$ the Euclidean connection on $\mathbb{R}^n$. Show that the operator

$$\nabla^t : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$$

given by $\nabla^t_X Y := \pi^t \overline{\nabla}_X(Y)$ is a connection that is compatible with g.

Hint: use that $[X, Y]$ is tangent to M whenever X and Y are tangent to M.

2. Exercise. Let $(U, (x_i))$ be a normal coordinate chart centered at p.

(1) Prove that for $V = \sum_i V_i \frac{\partial}{\partial x_i} \in T_pM$ the geodesic γ_V through p with initial velocity V is given, in normal coordinates, by

$$\gamma_V(t) = (tV_1, \cdots, tV_n),$$

as long as γ_V stays within U.

(2) Prove that the first partial derivatives of g_{ij} and the Christoffel symbols Γ^k_{ij} vanish at p (in these coordinates).

3. Exercise (Cf. Exercise 5-5 of *Riemannian manifolds*)**.** Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Show that if $\|\mathrm{grad} f\|_g = 1$ then the inegral curves of $\mathrm{grad} f$ are geodesics.

4. Exercise (Cf. Exercise 5.7 of *Riemannian manifolds*)**.** Let S^2_R be the sphere of radius R in $\mathbb{R}^3$. Define spherical coordinates (θ, φ) on the subset

$$S^2_R \setminus \{(x, y, z) : x \leq 0, y = 0\}$$

of the sphere by

$$(x, y, z) = (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi), \quad \pi < \theta < \pi, \quad 0 < \varphi < \pi.$$

(1) Show that the round metric of radius R is

$$\mathring{g}_R = R^2 d\varphi^2 + R^2 \sin^2 \varphi d\theta^2$$

in spherical coordinates.

(2) Compute the Christoffel symbols of $\mathring{g}_R$.

(3) Using the geodesic equation in spherical coordinates, verify that each meridian $(\theta(t), \varphi(t)) = (\theta_0, t)$ is a geodesic.