GLOBAL ANALYSIS I - WS 2017/2018

DR. B. MESLAND, T. KASTENHOLZ (M. SC.) AND S. ROOS (M.SC.)

Exercise sheet 11 - due Wednesday 17-1 -2018

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise (Cf. Exercise 6-4 of *Riemannian manifolds*)**.** Let (M, g) be a Riemannian manifold. A subset $U \subset M$ is *convex* if for every $p, q \in U$ there is a unique minimizing geodesic lying entirely in U. Show that every point has a convex neighborhood as follows:

(1) Let $p \in M$ be fixed and W a uniformly normal neighborhood of p. For $\varepsilon > 0$ small enough so that $B(p, 2\varepsilon) \subset W$. Define $W_\varepsilon \subset TM \times \mathbb{R}$ by

$$W_\varepsilon := \{(q, V, t) \in TM \times \mathbb{R} : q \in B(p, \varepsilon), V \in T_qM, |V| = 1, |t| < 2\varepsilon\}.$$

Define $f : W_\varepsilon \to \mathbb{R}$ by

$$f(q, V, t) := d_g(\exp_q(tV), p)^2,$$

with d_g the Riemannian distance. Show that f is smooth using normal coordinates at p.

(2) Show that for ε small enough

$$\frac{\partial^2 f}{\partial t^2} > 0,$$

on W_ε.

(3) For $q_1, q_2 \in B(p, \varepsilon)$ and γ a minimizing geodesic between them, show that $t \mapsto d_g(\gamma(t), p)$ attains its maximum at one of the endpoints of γ.

(4) Show that $B(p, \varepsilon)$ is convex.

2. Exercise. Let (M, g) be a compact Riemannian manifold. Prove that for all $p \in M$, the exponential map $\exp_p$ is defined on all of T_pM.

3. Exercise (Cf. Exercise 6-5 of *Riemannian manifolds*)**.** Let (M, g) be a complete Riemannian manifold and $N \subset M$ a closed embedded submanifold with the induced Riemannian metric. Show that N is complete. (Note that the restriction of the distance function on M to N need not be equal to the distance function on N.)