# GLOBAL ANALYSIS I - WS 2017/2018

DR. B. MESLAND, T. KASTENHOLZ (M. SC.) AND S. ROOS (M.SC.)

## Exercise sheet 12 - due Wednesday 24-1 -2018

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

**1. Exercise** (Cf. Exercise 7-1 of *Riemannian manifolds*)**.** Let $(M, g)$ be a Riemannian manifold.

(1) Compute the components of the Riemann curvature tensor in terms of the Christoffel symbols in coordinates.

(2) Suppose that $(x_i)$ are normal coordinates centered at $p$ and

$$Rm = \sum_{i,j,k,\ell} R_{ijk\ell} dx_i \otimes dx_j \otimes dx_k \otimes dx_\ell,$$

the local expression of the Riemann tensor. Show that the following holds at $p$:

$$2R_{ijk\ell} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell} g_{ik} + \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} g_{j\ell} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_\ell} g_{jk} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} g_{i\ell}$$

**2. Exercise.** Let $\nabla$ be the Riemannian connection and $\omega_{ij}$ the connection 1-forms with respect to a local frame $E_i$. Let $\phi_i$ denote the dual coframe and define $\Omega_{ij}$ the *curvature 2-forms* by

$$\Omega_{ij} = \frac{1}{2} R^j_{k\ell i} \phi_k \wedge \phi_\ell,$$

with $R^j_{k\ell i}$ the components of the curvature endomorphism. Show that they satisfy *Cartan's second structure equation*

$$\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}.$$

**3. Exercise** (Cf. Exercise 9-16 of *Introduction to smooth manifolds*)**.** Give an example of smooth vector fields $V$, $\tilde{V}$, and $W$ on $\mathbb{R}^2$ such that $V = \tilde{V} = \frac{\partial}{\partial x}$ along the $x$-axis but $\mathscr{L}_V W \neq \mathscr{L}_{\tilde{V}} W$ at the origin. (Remark: this shows that it is really necessary to know the vector field $V$ to compute $(\mathscr{L}_V W)_p$ ; it is not sufficient just to know the vector $V_p$ , or even to know the values of $V$ along an integral curve of $V$ .)