GLOBAL ANALYSIS I - WS 2017/2018

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Exercise sheet 2 - due 25-10-2018

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr.

1. Exercise. Let $F : M \to N$ and $G : N \to P$ be smooth maps between smooth manifolds M and N, and $p \in M$. Prove that:

a.) $(dF)_p : T_pM \to T_pN$ is linear;
b.) $(dG)_{F(p)} \circ (dF)_p = (d(G \circ F))_p : T_pM \to T_{G\circ F(p)}P$;
c.) $(d\mathrm{Id}_M)_p = \mathrm{Id}_{T_p(M)} : T_pM \to T_p(M)$.

2. Exercise (Exercise 3.17 in *Introduction to smooth manifolds*)**.** Let (x, y) be the standard coordinates on $\mathbb{R}^2$. Verify that

$$\tilde{x} = x, \quad \tilde{y} = y + x^3,$$

are global smooth coordinates on $\mathbb{R}^2$. Let $p = (1, 0)$ in standard coordinates and show that

$$\frac{\partial}{\partial x}|_p \neq \frac{\partial}{\partial \tilde{x}}|_p,$$

even though x and $\tilde{x}$ are identically equal.

3. Exercise (Exercise 3-4. in *Introduction to smooth manifolds*)**.** Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Show that the tangent bundle TS^1 is diffeomorphic to the trivial bundle $S^1 \times \mathbb{R}$.

Hint: use that the tangent vectors to $(x, y) \in S^1$ are perpendicular to (x, y) and consider the map $S^1 \times \mathbb{R} \to TS^1$ given by

$$((x, y), t) \mapsto ((x, y), -ty\frac{\partial}{\partial x} + tx\frac{\partial}{\partial y}),$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the standard derivations on $\mathbb{R}^2$.

4. Exercise (See Example 10.3 in *Introduction to smooth manifolds*)**.** Define an equivalence relation on $\mathbb{R}^2$ by

$$(x, y) \sim (x', y') \Leftrightarrow \exists n \in \mathbb{Z} \quad (x', y') = (x + n, (-1)^n y).$$

Let $E := \mathbb{R}^2/\sim$ be the quotient space and let

$$S^1 = \{e^{it} \in \mathbb{C} : t \in \mathbb{R}\} \subset \mathbb{C} \simeq \mathbb{R}^2$$

be the unit circle.

a.) Show that the map

$$\tilde{\pi} : \mathbb{R}^2 \to S^1, \quad (x, y) \mapsto e^{2\pi i x},$$

descends to a continuous map $\pi : E \to S^1$;
b.) Show that $\pi : E \to S^1$ is a topological vector bundle of rank 1;
c.) Show that E is not homeomorphic to the trivial bundle $S^1 \times \mathbb{R}$.