GLOBAL ANALYSIS I - WS 2017/2018

DR. B. MESLAND AND S. ROOS (M.SC.)

Exercise sheet 3 - due 8-11-2017

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise (Exercise 11.10 in *Introduction to smooth manifolds*)**.** Let $\pi : E \to M$ be a smooth vector bundle with transition functions given by the maps

$$\tau : U \to GL(k, \mathbb{R}).$$

Define the *dual bundle* E^* by setting

$$E^* := \bigsqcup_{p \in M} E_p^*,$$

where E_p^* is the vector space dual of E_p, with the obvious bundle projection. Show that $E^* \to M$ is a smooth vector bundle whose transition functions are given by

$$\tau^*(p) := (\tau(p)^{-1})^T \in GL(k, \mathbb{R}),$$

where A^T denotes the transpose matrix of A.

2. Exercise. For one forms $\omega_1, \cdots, \omega_k$ and vector fields $X_1, \cdots, X_k$ prove that

$$\omega_1 \wedge \cdots \wedge \omega_k(X_1, \cdots, X_k) = \det((\omega_i(X_j))_{ij}).$$

3. Exercise (Cf. Exercise 14-7 in *Introduction to smooth manifolds*)**.** For each of the maps $F : M \to N$ and differential forms $\omega \in \Omega^*(N)$ compute $d\omega$, $F^*\omega$ and verify that $F^*d\omega = dF^*\omega$.

a) $M = N = \mathbb{R}^2$; $\quad F(s,t) = (st, e^t)$; $\quad \omega = xdy$

b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$; $\quad F(\theta, \phi) = ((\cos(\phi)+2)\cos\theta, (\cos(\phi)+2)\sin\theta, \sin\phi)$; $\omega = ydz \wedge dx$.

4. Exercise. Let $F : M \to N$ be a smooth map between smooth manifolds M and N and let $\omega \in \Omega^k(N), \eta \in \Omega^\ell(N)$. Prove the identities

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta, \quad F^*d\omega = dF^*\omega.$$