GLOBAL ANALYSIS I - WS 2017/2018

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Exercise sheet 4 - due 15-11-2017

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise (Exercise 15.13 in *Introduction to smooth manifolds*)**.** Let M and N be oriented smooth manifolds of positive dimension (with or without boundary), and $F : M \to N$ a local diffeomorphism. Show that the following are equivalent:

(1) F is orientation preserving
(2) with respect to any two oriented charts for M and N, the Jacobian matrix of F has positive determinant
(3) for any positively oriented orientation form ω on N, the form $F^*\omega$ is a positively oriented orientation form on M.

2. Exercise. A manifold is *parallelizable* if it admits a smooth global frame or equivalently if TM is isomorphic to a trivial bundle.

(1) Prove that every parallelizable manfiold is orientable
(2) More generally, prove that an n-dimensional manifold M is orientable if and only if $\bigwedge^n TM$ and $\bigwedge^n T^*M$ are isomorphic to a trivial bundle.

3. Exercise (Cf. Exercise 15-3 of *Introduction to smooth manifolds*)**.** Let $n \geq 1$ and let $\mathbb{S}^n$ be the n-sphere. Then $\mathbb{S}^n = \partial\mathbb{B}^{n+1}$ is the boundary of the ball

$$\mathbb{B}^{n+1} := \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}.$$

(1) Show that $N = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$ is an outward pointing vector field and thus induces an orientation on $\mathbb{S}^n = \partial\mathbb{B}^{n+1}$.
(2) Show that the *antipodal map*

$$\alpha : \mathbb{S}^n \to \mathbb{S}^n, \quad \alpha(x) := -x,$$

is orientation preserving if and only if n is odd.

4. Exercise (Cf. Exercise 16-1 of *Introduction to smooth manifolds*)**.** Let $v_1, \cdots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. The n-dimensional paralellepiped P spanned by the v_i is

$$P := \{\sum_{i=1}^{n} t_i v_i : 0 \leq t_i \leq 1\}.$$

Show that the volume $\mathrm{Vol}(P)$ of P satisfies $\mathrm{Vol}(P) = |\det V|$, where $V := (v_1 \cdots v_n)$ is the $n \times n$ matrix whose columns are the vectors v_i.