GLOBAL ANALYSIS I - WS 2017/2018

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Exercise sheet 5 - due 22-11-2017

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise (Cf. Exercise 16-2 of *Introduction to smooth manifolds*)**.** Let $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ denote the 2-torus,

$$\mathbb{T}^2 := \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 = y^2 + z^2 = 1\}.$$

Let $\omega := xyzdw \wedge dy$. Compute $\int_{\mathbb{T}^2} \omega$.

2. Exercise. Prove *Green's theorem*: Let $D \subset \mathbb{R}^2$ be a compact submanifold with boundary and P and Q smooth functions on D. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} Pdx + Qdy.$$

3. Exercise (Cf. Exercise 16-11 of *Introduction to smooth manifolds*)**.** Let (M, g) be a Riemannian manifold with or without boundary. Show that for any smooth local coordinates (x_i) we have

$$\text{div}\left(X_i \frac{\partial}{\partial x_i} \right) = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x_i} \left(X_i \sqrt{\det(g_{ij})} \right),$$

where $g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$.

4. Exercise. Let (M, g) be an oriented Riemannian manifold and $S \subset M$ a closed embedded submanifold of codimension 1. Show that S is orientable if and only if there exists $N \in \mathscr{X}(M)$ such that for all $p \in S$ and for $X \in T_pS$

- $\langle N, N \rangle_g(p) = 1$
- $\langle X, N \rangle_g(p) = 0$.