GLOBAL ANALYSIS I - WS 2017/2018

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Exercise sheet 6 - due 29-11-2017

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise. Let (M,g) be a Riemannian manifold and $\gamma : [a,b] \to M$ be a *smooth* curve such that $\gamma'(t) \neq 0$ for all $t \in [a,b]$. Prove that γ admits a reparametrization $\tilde{\gamma} : [c,d] \to M$ with $|\tilde{\gamma}'(s)|_g = 1$ for all $s \in [c,d]$. What can we say about the length of of γ?

2. Exercise. Let $F : M \to N$ be a smooth map between manifolds and g a Riemannian metric on N. Define the *pull back* of the metric g to be the pairing

$$F^*g : \mathscr{X}(M) \times \mathscr{X}(M) \to C^\infty(M), \quad F^*g(X,Y)(p) := g(dF_p(X_p), dF_p(Y_p)).$$

Show that F^*g is a Riemannian metric if and only if F is a smooth immersion, that is, $dF_p : T_p(M) \to T_{F(p)}(N)$ is injective for all $p \in M$.

3. Exercise (Cf. Exercise 13.27 of *Introduction to smooth manifolds*)**.** Let (M,g) and $(\tilde{M},\tilde{g})$ be Riemannian manifolds with Riemannian distance function d_g and $d_{\tilde{g}}$ respectively. Let $F : M \to \tilde{M}$ be a *Riemannian isometry*, that is, F is a diffeomorphism and $F^*\tilde{g} = g$. Show that $d_{\tilde{g}}(F(p),F(q)) = d_g(p,q)$.

4. Exercise (Upper half-space, Poincare ball and upper hyperboloid sheet)**.** Consider the Riemannian manifolds:

- $\mathbb{H}^n := \{(x_1,\cdots,x_n) \in \mathbb{R}^n : x_n > 0\}, \quad g = \sum_{i=1}^n \frac{(dx_i)^2}{x_n^2}$,
- $\mathbb{B}^n := \{(x_1,\cdots,x_n) \in \mathbb{R}^{n+1} : \|x\|^2 := \sum_{i=1}^n x_i^2 < 1\}$,

  $h = 4\sum_{i=1}^n \frac{(dx_i)^2}{(1-\|x\|^2)^2}$,
- $\mathbb{U}^n := \{(x_1,\cdots,x_n,\tau) \in \mathbb{R}^{n+!} : \tau > 0, \tau^2 - \|x\|^2 = 1\}$,

  $k = \sum_{i=1}^n (dx_i)^2 - (d\tau)^2$.

(1) The *hyperbolic stereographic projection*

$$\pi : \mathbb{U}^n \to \mathbb{B}^n, \quad (x,\tau) \mapsto \frac{x}{1+\tau}$$

is a diffeomorphism with inverse

$$\pi^{-1} : \mathbb{B}^n \to \mathbb{U}^n, \quad x \mapsto \left(\frac{2x}{1 - \|x\|^2}, \frac{1 + \|x\|^2}{1 - \|x\|^2} \right),$$

Prove that $(\pi^{-1})^* k = h$ so that π and π^{-1} are isometries.

(2) Write $(x_1, \cdots, x_n) = (y, x_n)$. The *generalized Cayley transform* $\sigma : \mathbb{B}^n \to \mathbb{H}^n$

$$\sigma : (y, x_n) \mapsto \left(\frac{2y}{\|y\|^2 + (x_n - 1)^2}, \frac{1 - \|y\|^2 - x_n^2}{\|y\|^2 + (x_n - 1)^2} \right)$$

is a diffeomorphism with inverse

$$\sigma : (x, \tau) \mapsto \left(\frac{2x}{\|x\|^2 + (\tau + 1)^2}, \frac{\|x\|^2 + \tau^2 - 1}{\|x\|^2 + (\tau + 1)^2} \right).$$

Show that $\sigma^* g = h$ and proving that σ and σ^{-1} are Riemannian isometries.

Hint: for both problems, consider the case $n = 2$ first. In complex coordinates $z = y + ix_2$, the map σ in (2) takes the form $z \mapsto -i\frac{z+i}{z-i}$.