GLOBAL ANALYSIS I - WS 2017/2018

DR. B. MESLAND, T. KASTENHOLZ (M. SC.) AND S. ROOS (M.SC.)

Exercise sheet 7 - due TUESDAY 5-12 -2017
NO LECTURE ON WEDNESDAY DECEMBER 6th (Dies Academicus)

Please hand in the exercises stapled or paperclipped together, with name AND Matrikelnr. If you work in groups, one sheet per group suffices (at most 3 people per group).

1. Exercise (Cf. Exercise 4.2 of *Riemannian manifolds*)**.** Let ∇ be a linear connection on M and consider the map

$$\tau : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M),$$

defined by

$$\tau(X,Y) := \nabla_X(Y) - \nabla_Y(X) - [X,Y].$$

(1) Show that τ is a $(2,1)$ tensor field. τ is called the *torsion tensor* of ∇.
(2) We say that ∇ is *symmetric* if $\tau \equiv 0$. Show that ∇ is symmetric if and only if its Christoffel symbols Γ_{ij}^k with respect to any coordinate frame satisfy $\Gamma_{ij}^k = \Gamma_{ji}^k$.

2. Exercise. Let $f \in C^\infty(M)$ and ∇ a linear connection on M.

(1) Show that the covariant Hessian $\nabla^2(f) = \nabla(\nabla(f))$ is given by
$$\nabla^2(f)(X,Y) = Y(Xf) - (\nabla_Y X)f, \quad X, Y \in \mathscr{X}(M).$$
(2) Show that ∇ is symmetric if and only if for any $f \in C^\infty(M)$ the covariant Hessian $\nabla(\nabla(f))$ is a symmetric 2-tensor field.

3. Exercise (Cf. Exercise 4.4 of *Riemannian manifolds*)**.** Let ∇^0 and ∇^1 be linear connections on M.

(1) Show that the *difference tensor*, defined on vector fields X and Y by
$$A(X,Y) := \nabla^0_X Y - \nabla^1_X Y,$$
is a $(2,1)$ tensor field (that is, show that A is $C^\infty(M)$-linear in Y). Conclude that the set of all linear connections on M is equal to
$$\{\nabla^0 + A : A \in \Gamma^\infty(T^2_1 M)\}.$$
(2) Show that ∇^0 and ∇^1 determine the same geodesics if and only if their difference tensor is antisymmetric, ie. $A(X,Y) = -A(Y,X)$.
(3) Show that ∇^0, ∇^1 have the same torsion tensor if and only if their difference tensor is symmetric, that is $A(X,Y) = A(Y,X)$.