Assignment-set 5 Bifurcations and Chaos

Deadline to hand in: 20 June 2016, 17.00u

1.) Exercise 6.1.1 p.290 Guckenheimer & Holmes.

2.) Consider the system

$$\begin{aligned} \frac{dx}{dt} &= 2y \\ \frac{dy}{dt} &= 2x - 3x^2 + \lambda y(x^3 - x^2 + y^2 - c) \end{aligned}$$

where $a, \mu \in \mathbf{R}$.

(a) First, choose $c = 0$. Note that for $\lambda = 0$, the system has a Hamiltonian H and that there exists a homoclinic orbit. Now, show that this homoclinic orbit persists for all λ.
Hint: How does H vary along the orbits?

(b) Now, take $\lambda = -1$ and $c \neq 0$. Determine all fixed points, their character and stability.

(c) Take c close to $c = 0$ and determine for which values of c a periodic solution exists. Is this periodic solution stable or unstable? What happens to the periodic orbit as $c \to 0$?

3.) Consider the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \mu_1 + \mu_2 y + x^2 - xy \end{aligned}$$

where $\mu_1, \mu_2 \in \mathbf{R}$.

(a) Determine the fixed points, their character and stability.

(b) Determine the curves on which saddle-node and Hopf bifurcations take place. In the case of a Hopf bifurcation, determine whether it is sub- or supercritical.

Now, we want to analyse possible global behaviour that might occur. For this assume that

$$\mu_1 = -\varepsilon^4 \text{ and } \mu_2 = \varepsilon^2 \nu_2$$

where $0 < \varepsilon \ll 1$.

(c) Rescale x, y and t to u, v and τ, resp, such that the resulting system is suitable to apply the Melnikov method to.

(d) Set $\varepsilon = 0$ in the (u, v)-system, determine the Hamiltonian and sketch the phase-plane.

(e) Use the Melnikov method to show that for $\varepsilon \neq 0$ the (u, v)-system has a heteroclinic connection for

$$\mu_1 = a\mu_2^2 + \mathcal{O}(\mu_2^\alpha),$$

and determine a and α. This higher order term is important for determining the behaviour of the bifurcation curves in the (μ_1, μ_2)-plane close to the origin.

(f) Sketch in the (μ_1, μ_2)-plane all bifurcation curves. Sketch in each of the regions (seperated by these curves) and on each of the curves the corresponding phase plane.