Assignment-set 1 Introduction to Dynamical Systems 2013

Deadline to hand in: 10 October 2013, 11.15u

1.) Exercise 7, page 26 book.

Note: Essential background information on the Volterra-Lotka model (and its analysis) can be found on pages 14-17 of the book.

2.) (a) Consider $g : [0, \infty) \to \mathbf{R}$ given by

$$g(t) = \frac{\sin t^2}{t+1}.$$

Show that $\lim_{t\to\infty} g(t)$ exists, while $\lim_{t\to\infty} \dot{g}(t) (= \frac{dg}{dt}(t))$ does not.

(b) Consider the autonomous ODE $\dot{x} = f(x)$, $x \in \mathbf{R}^n$, with initial condition $x(0) = x_0$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ (at least) continuously differentiable. Let $\phi(t; x_0)$ be a solution such that

$$\lim_{t\to\infty} \phi(t; x_0) = a$$

for a certain $a \in \mathbf{R}^n$. Prove that a must be a critical point of the system.
Warning: Be aware of functions that behave like $g(t)$ in (a).

(c) Explain why the function $g(t)$ that is given in (a) cannot be a solution of a system as described in (b) (with $n = 1$).

3.) Consider the two-dimensional system,

$$\left\{ \begin{array}{rcl} \frac{du}{d\xi} & = & v \\ \frac{dv}{d\xi} & = & Av - u(1-u), \end{array} \right. \tag{1}$$

with parameter $A > 0$.

(a) Determine the critical points E_1 and E_2 of (1) and their character (as function of $A > 0$). Sketch the local linearized phase portraits near the critical points E_1 and E_2, depending on A.

The aim of this exercise is to establish the existence of a *positive heteroclinic orbit* $(u_h(\xi), v_h(\xi))$ that connects the critical point E_1 to E_2, i.e. a solution $(u_h(\xi), v_h(\xi))$ of (1) that satisfies $\lim_{\xi\to-\infty}(u_h(\xi), v_h(\xi)) = E_1$ and $\lim_{\xi\to+\infty}(u_h(\xi), v_h(\xi)) = E_2$, while $u_h(\xi), v_h(\xi) > 0$ for all $\xi \in \mathbf{R}$.

(b) Explain that $A \geq 2$ is a necessary condition for the existence of such an orbit $(u_h(\xi), v_h(\xi))$. Is $(u_h(\xi), v_h(\xi))$ uniquely determined (if it exists)?

To construct $(u_h(\xi), v_h(\xi))$, we consider the ODE (1) in 'backwards time' $\tilde{\xi} = -\xi$ and consider the well-defined orbit $(u_s(\tilde{\xi}), v_s(\tilde{\xi}))$ (by the nature of E_2) that satisfies $\lim_{\tilde{\xi}\to-\infty}(u_s(\tilde{\xi}), v_s(\tilde{\xi})) = E_2$. Within this framework, proving the existence of the positive heteroclinic orbit $(u_h(\xi), v_h(\xi))$ is equivalent to establishing that $\lim_{\tilde{\xi}\to\infty}(u_s(\tilde{\xi}), v_s(\tilde{\xi})) = E_1$ (while $u_s(\tilde{\xi}), v_s(\tilde{\xi}) > 0$ for all $\tilde{\xi} \in \mathbf{R}$).

(c) Formulate the equivalent of (1) in terms of $\tilde{\xi} = -\xi$ and show that for $\alpha > \frac{1}{4A}$, $(u_s(\tilde{\xi}), v_s(\tilde{\xi}))$ can only leave the rectangular region with vertices $(0, 0)$, $(1, 0)$, $(1, \alpha)$ and $(0, \alpha)$ through the edge between $(0, 0)$ and $(0, \alpha)$.

(d) Prove the existence of a positive heteroclinic orbit $(u_h(\xi), v_h(\xi))$ for every $A \geq 2$. *Hint:* Show that there exists a $k > 0$ such that $(u_s(\tilde{\xi}), v_s(\tilde{\xi}))$ cannot cross through the (half)line $\{v = ku, u > 0\}$ and apply exercise 2.

(e) A (positive) traveling wave solution to the PDE

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + U(1 - U)$$

with $U(x, t) : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$, is a positive bounded solution of the PDE that is stationary in a co-moving frame that travels with speed $c \in \mathbf{R}$ – the latter implies that $U(x, t)$ can be written as $u(x - ct)$ for a certain $c \in \mathbf{R}$. The function $U(x, t) = u_h(\xi)$ defines such a traveling wave. Explain! What is the relation between ξ and (x, t), and between A and c? Sketch the traveling wave $U(x, t) = u_h(\xi)$ for several values of t and A or c.