Assignment-set 2 Introduction to Dynamical Systems 2013

Deadline to hand in: 31 October 2013, 11.15u

1.) (a) Show that $\bar{x} = 0$ is an asymptotically stable solution of $\dot{x} = -x^3$ ($\in \mathbf{R}$). Use the definition! (see page 118 of the book).

(b) Is this also true for the *complex* equation $\dot{z} = -z^3$ (with $z \in \mathbf{C}$)? Use again the definition.
Hint: Write this complex equation as a 2-dimensional real system by introducing the real and imaginary parts of z, i.e. by setting $z = x + iy$ and deriving equations for $\dot{x}$ and $\dot{y}$. Make a sketch of the phase portrait. Note that *in*stability is defined as *not* being stable (page 116 book).

2.) Consider the linear system,

$$\begin{cases} \dot{x} &= Ax + B(t)x, \quad x \in \mathbf{R}^n, n \geq 1, \\ x(0) &= x_0, \end{cases} \tag{1}$$

with A a $n \times n$ matrix (with constant coefficients) and $B(t)$ a $n \times n$ matrix with coefficients that depend continuously on time t; define $\phi(t; x_0)$ as the solution of (1). Moreover, it is given that there is a (positive) continuous function $C : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\|B(t)x\| < C(t)\|x\|$ for all $x \in \mathbf{R}^n$ and $t \geq 0$.
(*Note*: For given $B(t)$, functions $C(t)$ like defined here always exist; the infimum over all possible $C(t)$'s defines the norm $\|B(t)\|$ of the matrix $B(t)$.)

(a) Explain why $\tilde{\phi}(t; x_0) = e^{At + \int_0^t B(s)ds} x_0$ is, in general, not a solution of (1).

Assume first that,

(Ai) all eigenvalues λ_j of A satisfy $\mathrm{Re}\lambda_j < 0$;
(Bi) $\lim_{t\to\infty} C(t) = 0$.

(b) Show that it follows from assumptions *(Ai)* and *(Bi)* that $\lim_{t\to\infty} \phi(t; x_0) \to 0$ for all $x_0 \in \mathbf{R}^n$ and that the critical point $x^* = 0$ of (1) is asymptotically stable (use the definition!).
Hint. Apply the arguments of the proof of Theorem 4.6 (in the book) with $g(y)$ replaced by $B(t)x$.

Now assume that,

(Aii) all eigenvalues λ_j of A satisfy $\mathrm{Re}\lambda_j \leq 0$;
(Aiii) the eigenvalues λ_j of A with $\mathrm{Re}(\lambda_j) = 0$ do not coincide
(i.e. these eigenvalues have – algebraic and geometric – multiplicity 1);
(Bii) $C(t)$ is integrable: $\int_0^\infty C(t)\, dt = D < \infty$.

In the subsequent exercises it may be used that for matrices A that satisfy assumptions *(Aii)* and *(Aiii)*, there is a $K > 0$ such that $|e^{At}x_0| < K|x_0|$ for all $x_0 \in \mathbf{R}^n$ and $t \geq 0$.

(c) Show that it follows from assumptions *(Aii)*, *(Aiii)*, and *(Bii)* that the critical point $x^* = 0$ of (1) is stable. Use the definition.

(d) Assumption *(Bii)* is (only slightly) stronger than *(Bi)*. Does the statement of (c) also hold if *(Bii)* replaced by *(Bi)*? If yes: give a proof; if no: give a counterexample.

(e) Is additional assumption *(Aiii)* really necessary for the proof of (c)? Explain!

3.) Consider the n-dimensional system

$$\dot{X} = AX + F(X), \tag{2}$$

with A an $n \times n$ constant coefficients matrix and $F(X)$ a nonlinear expression that satisfies $\|F(X)\| \leq C\|X\|^2$ as $\|X\| \to 0$, for some $C > 0$. Recall that we cannot draw any conclusions on the stability of the critical point $\bar{X} = 0$ of (2) if A has eigenvalues λ with $\mathrm{Re}(\lambda) = 0$. Take $n = 2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and write X as (x, y). Show that the critical point $\bar{X} = (\bar{x}, \bar{y}) = (0, 0)$ of (2) is

(a) unstable for $F(X) \equiv 0$.

(b) stable for $F(X) = F(x, y) = \begin{pmatrix} 0 \\ -x^3 \end{pmatrix}$.

Hint: Write (2) as equation in $\ddot{x}$ and determine a *first integral* $V(x, y)$ of this equation, i.e. a function $V(x, y)$ such that $\frac{d}{dt}V(x, y) = 0$ for solutions of the equation. Use this $V(x, y)$ as Lyapunov function.

(c) asymptotically stable for $F(X) = F(x, y) = \begin{pmatrix} -x^3 \\ -x^3 - y^3 \end{pmatrix}$.

Hint: Use the Lyapunov function constructed in (b).

(d) Keep n and A as above and consider $F(x, y) = \begin{pmatrix} 0 \\ \alpha x^\ell \end{pmatrix}$ for $\alpha \in \mathbf{R}$ and $\ell = 2, 3, 4, ...$ What can you say about the stability of $(0, 0)$ as function of α and ℓ?

(e) Now consider $F(x, y) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$ with nonlinearities $f, g : \mathbf{R} \to \mathbf{R}$ smooth analytic functions of x such that the limits $\lim_{x\to 0} \frac{f(x)}{x^k}$ and $\lim_{x\to 0} \frac{g(x)}{x^\ell}$ both exist and are $\neq 0$ for some $k, \ell \geq 2$. Keep n and A still as above. Determine necessary and sufficient conditions on f and g so that $(0, 0)$ is stable/asymptotically stable.