Assignment-set 4 Introduction to Dynamical Systems 2013

Deadline to hand in: 19 December 2013, 11.15u, in mailbox Corine

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1.) Consider the integrable problem

$$\ddot{x} + x - x^3 + C = 0, \quad \text{with } \; C \in \mathbf{R}. \tag{1}$$

(a) Determine the set $\mathcal{I}_{\text{hom}}$ such that system (1) has a homoclinic orbit if $C \in \mathcal{I}_{\text{hom}}$. For which $C = C_{\text{het}}$ does (1) have heteroclinic orbits? Give sketches of the phase portraits of (1) for C such that $C_{\text{het}} > C \in \mathcal{I}_{\text{hom}}$, $C = C_{\text{het}}$, $C_{\text{het}} < C \in \mathcal{I}_{\text{hom}}$, and $C_{\text{het}} \neq C \notin \mathcal{I}_{\text{hom}}$.

(b) Now consider a more general version of (1),

$$\ddot{x} + x + Ax^2 + Bx^3 + C = 0, \quad \text{with } \; A, B, C \in \mathbf{R}. \tag{2}$$

For which A, B, C does (2) have heteroclinic orbits?

2.) Let $V : \mathbf{R}^2 \to \mathbf{R}$ be given as a smooth (at least C^2) map; consider the associated *gradient flow*

$$\dot{x} = -\nabla V(x), \quad \text{or } \; \dot{x}_i = -\frac{\partial V}{\partial x_i}(x), \; i = 1, 2. \tag{3}$$

Prove that system (3) cannot have a homoclinic solution.
Hint: Consider (and use) $\dot{V}$.

3.) Consider the 2-dimensional system,

$$\begin{cases} \dot{x} &= 1 + y - x^2 - y^2, \\ \dot{y} &= 1 - x - x^2 - y^2. \end{cases} \tag{4}$$

(a) Determine the critical points of (4) and their local character; show that the flow generated by (4) is symmetric with respect to the line $\{x + y = 0\}$.

(b) Show that system (4) is integrable by constructing an integral $K(x, y)$.
Hint: Introduce new variables $u = x - y$ and $v = x + y$ that exploit the symmetry found in (a), write (4) as a system in u and v, and determine an integral $\tilde{K}(u, v)$ for this system by introducing $w = v^2$ and solving the equation for $\frac{dw}{du}$.

(c) Sketch the phase portrait associated to (4) and conclude that system (4) has a homoclinic solution.

Now consider a more general version of (4),

$$\begin{cases} \dot{x} &= 1 + y - x^2 - y^2 + h(x, y), \\ \dot{y} &= 1 - x - x^2 - y^2 + h(x, y), \end{cases} \quad \text{with } \; h : \mathbf{R}^2 \to \mathbf{R}, h(0, 0) = 0, \; \text{smooth enough.} \tag{5}$$

(d) Take $h(x, y) = \varepsilon(x + y)$ with $0 < \varepsilon \ll 1$: show that the homoclinic orbit of system (4) does not survive the perturbation of (5).

Hint: Determine $\dot{K}$ or $\dot{\tilde{K}}$.

(e) Take $h(x, y) = \alpha(x - y)^3$, $\alpha \in \mathbf{R}$: show that system (5) is integrable by deriving an integral $K_\alpha(x, y)$ (or $\tilde{K}_\alpha(u, v)$) such that $K_0(x, y) = K(x, y)$, with $K(x, y)$ as in (b).

(f) Take $h(x, y)$ as in (e) with $\alpha = \varepsilon$ and $0 < \varepsilon \ll 1$: show that system (5) has a homoclinic orbit and give a sketch of the phase portrait.

(g) Take $h(x, y)$ as in (e) with $\alpha = A \gg 1$: show that system (5) does not have a homoclinic orbit and give a sketch of the phase portrait.