Assignment-set 5 Introduction to Dynamical Systems 2013

Deadline to hand in: 22 January 2014, 17.00u, in mailbox Corine

1.) Consider the 2-dimensional system

$$\begin{cases} \dot{x} &= \mu + x^2 - xy, \\ \dot{y} &= y^2 - x^2 - 1. \end{cases} \tag{1}$$

(a) Take $\mu = 0$. Show that (1) has two saddle points $P^+(0)$ and $P^-(0)$ and that these points are connected by a heteroclinic orbit. Give a sketch of the phase portrait.

(b) Now consider $\mu \neq 0$ and small. Determine a Taylor/perturbation expansion of the saddles $P^+(\mu)$ and $P^-(\mu)$ up to quadratic terms in μ. What happens to the heteroclinic connection? Give a sketch of the phase portraits, both for $\mu > 0$ as well as for $\mu < 0$.

2.) Consider for $\alpha \in \mathbf{R}$ the 2-dimensional system

$$\begin{cases} \dot{x} &= \alpha xy - x^3 + y^2, \\ \dot{y} &= -y + x^2 + xy. \end{cases}$$

Determine the center manifold $W^c((0,0))$ up to and including terms of order three. Determine the (approximate) flow on $W^c((0,0))$ near $(0,0)$. Determine the stability of $(0,0)$ for all $\alpha \in \mathbf{R}$.

3.) Consider for $\beta \in \mathbf{R}$ the 2-dimensional system

$$\begin{cases} \dot{x} &= -x^3, \\ \dot{y} &= -y + x^2 + \beta x^4. \end{cases}$$

(a) Take $\beta = -2$. Determine the stable manifold $W^s((0,0))$ and the center manifold(s) $W^c((0,0))$ *explicitly* by solving the appropriate equations. Is $W^c((0,0))$ uniquely determined? Is it analytic? If so, give an expression of $W^c((0,0))$ in terms of a power series.

(b) Sketch, for β still equal to -2, the phase portrait, including the manifolds $W^s((0,0))$ and $W^c((0,0))$.

(c) Consider the general case $\beta \in \mathbf{R}$. What can you say about $W^c((0,0))$? Is it unique? Is it analytic? Can you give an explicit expression, or a power series expansion?

4.) Consider the 3-dimensional system

$$\begin{cases} \dot{x} &= -y + xz - x^4, \\ \dot{y} &= x + yz + xyz, \\ \dot{z} &= -z - x^2 - y^2 + z^2 + \sin x^3. \end{cases}$$

Determine the stability of the critical point $(0,0,0)$.
Hint: Determine the flow on the center manifold and use polar coordinates.