Assignment-set 3 Introduction to Perturbation Methods

Deadline to hand in: 28 April 2015, 11.15u

1.) Find a composite expansion (first-term) which is uniformly valid on $x \in (0,1)$ to

$$x^3 \frac{dy}{dx} = \varepsilon[(1+\varepsilon)x + 2\varepsilon^2]y^2$$

with $y(1) = 1 - \varepsilon$.

Make sure to consider all possible layers

2.) In the study of explosions of gaseous mixtures, one finds a model where the (nondimensional) temperature $T(t)$ of the gas satisfies

$$\frac{dT}{dt} = \varepsilon (T_\infty - T)^n \exp\left(\frac{T-1}{\varepsilon T}\right),$$

for $T(0) = 1$. Here $T_\infty > 1$ is a constant known as the adiabatic explosion temperature. Also, n is a positive integer (it is the overall reaction order). If we assume a high activation energy, then the parameter ε is small.

(a) What is the steady-state temperature?

(b) Find the first two terms in a regular expansion of the temperature. This expansion satisfies the initial condition and describes the solution in what is known as the ignition period. Explain why the expansion is not uniform in time. Also, towards the ignition period the solution is known to undergo a rapid transition to the steady state. Use your expansion to estimate when this occurs.

(c) To understand how the solution makes the transition from the rapid rise in the transition region to the steady state, let

$$\tau = \frac{t - t_0}{\mu(\varepsilon)}$$

where t_0 is the time where the transition takes place and $\mu(\varepsilon)$ is determined from balancing in the layer. Assuming that $T = T_\infty - \varepsilon T_1(\tau) + \cdots$, find μ and T_1. Although T_1 is defined implicitly, use its direction field to determine what happens when $\tau \to \infty$ and $\tau \to -\infty$.

It is worth pointing out that there is a second layer in this problem, and it is located between the one from the ignition layer and the layer you found in part (c). The matching of these various layers is fairly involved; the details can be found in Kapila (1983) for the case where $n = 1$. Also, it is actually possible to solve the original problem in closed form, although the solution is not simple (Parang and Jischke, 1975).

3.) In the study of Josephson junctions, the following problem appears (Sanders, 1983):

$$\phi'' + \varepsilon(1 + \gamma \cos\phi)\phi' + \sin\phi = \varepsilon\alpha, \text{ for } t > 0,$$

where $\phi(0) = \phi'(0) = 0$ and γ is a positive constant. Find a first-term approximation of $\phi(t)$ that is valid for large t.

4.) Consider the problem of solving

$$y' = y - y^3$$

for $t > 0$ where $y(0) = \varepsilon$.

(a) Sketch the direction field for this problem and from this determine what happens to the solution as $t \to \infty$.

(b) Suppose one assumes a regular expansion of the form $y = y_0(t) + \varepsilon^\alpha y_1(t) + \cdots$ After finding y_0 and y_1, explain why y_0 is not expected to be an accurate approximation as $t \to \infty$. Also, explain why a multiple-scale expansion should be used but expression (3.9) from the book will not work for this problem.

(c) Suppose $t_2 = \varepsilon^\alpha f(t)$, where $f(t)$ is determined from the secular removing condition and the requirement that $f(0) = 0$. Show that

$$y = \frac{\varepsilon e^t}{\sqrt{1 + \varepsilon^2(e^{2t} - 1)}}.$$

5.) In the study of Raman scattering, one comes across the equation for a forced Morse oscillator with small damping, given as

$$y'' + \varepsilon^2\alpha y' + (1 - e^{-y})e^{-y} = \varepsilon^3 \cos((1 + \varepsilon^2\omega)t),$$

where $y(0) = 0$ and $y'(0) = 0$. Also, $\alpha > 0$ and $\omega \in \mathbf{R}$.

(a) Find a first-term approximation of $y(t)$ that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find the possible steady states (if any) for the amplitude.

(b) Lie and Yuan (1986) used numerical methods to solve this problem. They were interested in how important the value of the damping parameter α is for there to be multiple steady states for the amplitude. They were unable to answer this question because of the excessive computing time it took to solve the problem using the equipment available to them. However, based on their calculations, they hypothesized that multiple steady states for the amplitude are possible even for small values of α. By sketching the graph of A_∞ as a function of ω, for $\alpha > 0$ determine whether or not their hypothesis is correct.