Assignment-set 5 Introduction to Perturbation Methods

Deadline to hand in: 16 June 2015, 11.15u

1.) For the following equations, describe the steady states, classify the bifurcation points, and determine the stability of each branch. Also, sketch the bifurcation diagram.

(a)

$$y'' + y' + y[\lambda - 3y + 2y^2] = 0.$$

(b)

$$y'' + y' + e^{\lambda y} - 1 - (y-1)^2 = 0.$$

2.) The WazewskaCzyzewska and Lasota model for the survivability of red blood cells is

$$y'(t) = -\mu y(t) + e^{-y(t-T)} \text{ for} t > 0.$$

Here $y(t)$ is the number of cells at time t and T is the time required to produce a red blood cell. The purpose of this exercise is to see if the delay in cell production can produce an instability in the system, and for this reason T is the bifurcation parameter. Assume that μ is a positive constant and $y(t) \geq 0$.

(a) In the case where there is no delay, so that $T = 0$, show that there is a nonzero steady state y_s and show that it is asymptotically stable. Also, explain what happens to the value of y_s as μ varies from 0 to ∞.

(b) The steady state in part (a) is also a steady state when $T > 0$. Show that it is asymptotically stable if $\mu > \frac{1}{e}$.

(c) Assume that $0 < \mu < \frac{1}{e}$. Show that y_s is asymptotically stable if $0 < T < T_c$, where

$$T_c = \frac{\pi - \arctan(\sqrt{y_s^2 - 1})}{\mu\sqrt{y_s^2 - 1}},$$

and it is unstable if $T > T_c$.

3.) Rayleigh (1883) undertook a study of what he called 'maintained vibrations' that occur in organ pipes, singing flames, finger glasses, and other such systems. The equation he used to model such vibrations was

$$\varepsilon y'' - (1 - \frac{1}{3}(y')^2)y' + y = 0 \text{ for} t > 0.$$

Assume that $y(0) = 0$ and $y'(0) = -\sqrt{3}$.

(a) Letting $v = y'$, write the problem as a first-order system. Make sure to give the initial conditions.

(b) Find the first term in the outer expansion of y and v. From this determine where the first corner layer is located (give the value of t, y, and v).

(c) Find the first term in the corner-layer expansion of y and v. With this, determine the location of the first transition layer.

(d) In the transition layers, the solution very quickly switches from one branch of a cubic to another. Use this observation to find a first-term approximation of the period of the oscillation. Use this same idea to sketch the first-term approximation of $y(t)$ for $0 \leq t \leq 3T$.

4.) Consider the modified Rayleigh equation

$$y'' - \lambda(1 - \alpha(y')^{2n})y' + y = 0, \text{ for } t > 0.$$

where α is a positive constant and n is a positive integer.

(a) Find the steady state and determine for what values of λ it is stable.

(b) Find a first-term approximation of the limit cycle that appears at the Hopf bifurcation point.