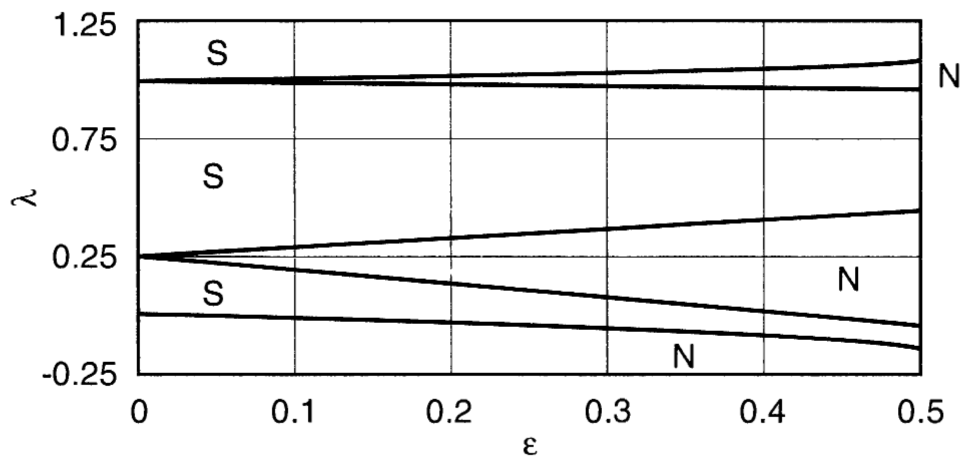
Assignment-set 4 Introduction to Perturbation Methods

Deadline to hand in: 8 June 2017, 11.15u

1.) Mathieu's equation is given by

$$y'' + [\lambda + \varepsilon \cos(t)]\, y = 0, \text{ for } t > 0,$$

where $y(0) = a$ and $y'(0) = b$. Also, λ is a positive constant. This equation describes the small amplitude oscillations of a pendulum whose length varies periodically with time. If the pendulum's natural frequency is a particular multiple of the frequency of the length variation, then instability can occur. This is indicated in the figure below, which shows the regions in the (ε, λ)-plane where the motion is stable (S) and unstable (N). Equations for the boundaries of these regions are derived in this exercise.



(a) Assuming λ is independent of ε, use a regular expansion to show that secular terms appear in the second term of the expansion if $\lambda = \frac{1}{4}$ and in the third term no matter what the value of λ.

(b) In the case where $\lambda = \frac{1}{4}$, use multiple scales to remove the secular term in the second term of the expansion. Use this to explain why the solution can grow exponentially in time and is therefore unstable. By generalising this analysis, it is possible to show that the solutions may be unbounded, and hence unstable, if $\lambda = \frac{n^2}{4}$, where $n = 0, 1, 2, 3, ...$ (you do not need to show this).

(c) Assuming $\lambda \neq \frac{n^2}{4}$, where $n = 0, 1, 2, 3, ...$, use multiple scales to show that

$$y = a_0 \cos\left(\sqrt{\lambda} + \frac{\varepsilon^2 t}{4\sqrt{\lambda}(1 - 4\lambda)} + \theta_0\right),$$

to leading order, where a_0 and θ_0 are constants. This expression indicates what the solution looks like in the stable regions in the figure.

(d) To investigate what happens for λ values near $\frac{1}{4}$, suppose suppose $\lambda = \frac{1}{4}(1 + 2\varepsilon\lambda_1) + \cdots$. Find a first-term approximation of the solution that is valid for large t. From this, show that the solution may be unbounded, depending on the initial conditions, if $|\lambda_1| < 1$. Moreover, irrespective of the initial conditions, the solution is bounded if $|\lambda_1| > 1$ Because of this, the curves $\lambda = \frac{1}{4}(1 \pm 2\varepsilon) + \cdots$ form the stability boundaries in this region.

(e) To investigate what happens near $\lambda = 1$, suppose $\lambda = 1 + \varepsilon^2\lambda_1 + \cdots$ Find a first-term approximation of the solution that is valid for large t. From this, show that the solution may be unbounded, depending on the initial conditions, if $-\frac{1}{12} < \lambda_1 < \frac{5}{12}$.

2.) Use the WKBJ method to find an approximate solution of the following problem

$$\varepsilon^2 y'' + \varepsilon x y' - y = -1,$$

for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 3$. Compare your answer with the composite expansion obtained using matched asymptotic expansions.

3.) Consider the eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - r(x)y = -\lambda^2 q(x)y, \text{ for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. Here $p(x), q(x)$, and $r(x)$ are given, smooth, positive functions and $\lambda \geq 0$ is the eigenvalue.

(a) Show that if one lets $y(x) = h(x)w(x)$, where $h(x) = \frac{1}{\sqrt{p(x)}}$, then

$$p(x)w'' + \left[\lambda^2 q(x) - f(x)\right] w = 0, \text{ for } 0 < x < 1,$$

where $f(x) = r - \frac{(ph')'}{h}$.

(b) For the problem in part (a), use a first-term WKBJ approximation to show that for the large eigenvalues, $\lambda \sim \lambda_n$, where $\lambda_n = \frac{n\pi}{\kappa}$ and

$$\kappa = \int_0^1 \sqrt{\frac{q(x)}{p(x)}}dx.$$

What is the corresponding WKBJ approximation of the eigenfunctions?

(c) To find the second term in the expansion of the eigenvalue, one starts with the expansion $\lambda = \lambda_n + \frac{1}{n}\lambda_c + \cdots$ Extending your argument in part (b), show that

$$\lambda_c = \frac{1}{2\pi}\int_0^1 \left[-g(x)g''(x) + (pq)^{-\frac{1}{2}}f(x)\right] dx,$$

where $g(x) = (\frac{p}{q})^{\frac{1}{4}}$.

(d) When $p = q = 1$, explain why $\lambda \sim \lambda_n$ gives an accurate approximation to all the eigenvalues as long as r_M is small compared to $2\pi^2$, where $r_M = \max_{0 \leq x \leq 1} r(x)$.

(e) To investigate the accuracy of the WKBJ result, consider the eigenvalue problem

$$y'' + \lambda^2 q(x) y = 0, \text{ for } 0 < x < 1,$$

where $y(0) = y(1) = 0$ and $q(x) = (x+1)^4$. Calculate the two-term expansion for λ derived in parts (b) and (c), and then compare it with the numerical values given below.

n	1	2	3	4	5	10	20
λ^2	0.924915	3.89727	8.88444	15.8774	24.8732	99.8654	399.863

4.) The motion of planetary rings is described using the theory of self- gravitating annuli orbiting a central mass. For circular motion in the plane, with the planet at the origin, one ends up having to find the circumferential velocity $V(r, \theta, t) = v(r)e^{i(\omega t + m\theta)}$. The function $v(r)$ satisfies

$$\frac{d}{dr}\left(r\frac{d}{dr}(rv)\right) = m^2(1 - \kappa^2 r^2)v \text{ for } 0 < r < \infty,$$

where $\kappa = \frac{\alpha + \beta m}{m}$. Here r is the radial coordinate and α and β are positive constants. The parameter m is positive and is a mode number. Find a first-term approximation of the solution for large m.