Assignment-set 5 Introduction to Perturbation Methods

Deadline to hand in: 29 June 2017, 11.15u

1.) For the following equations, describe the steady states, classify the bifurcation points, and determine the stability of each branch. Also, sketch the bifurcation diagram.

(a)

$$y'' + y' + y[2y^2 + \lambda(\lambda - 2)] = 0.$$

(b)

$$y'' + y' + e^{\lambda y} - 1 - (y-1)^2 = 0.$$

2.) In the theory for the buckling of an initially straight rod subjected to an axial load λ, the following problem arises (Euler, 1744):

$$\theta'' + \lambda \sin\theta = 0, \text{ for } 0 < x < 1,$$

where $\theta'(0) = \theta'(1) = 0$. The variable $\theta(x)$ is the angle the tangent to the rod makes with the horizontal at the spatial position x. This variable is related to the vertical displacement $w(x)$ of the rod through the relation $w' = \sin(\theta)$, where $w(0) = w(1) = 0$.

(a) Find the solutions that bifurcate from the equilibrium solution $\theta_s = 0$. The values of λ where bifurcation occurs are called buckling loads.

(b) In mechanics, the principle of minimum potential energy states that a system will move into a state with the minimum potential energy. For this problem, the potential energy is proportional to

$$V = \int_0^1 [\theta_x^2 + 2\lambda(\cos\theta - 1)]dx.$$

Use this principle to determine the preferred configuration of the rod near the first bifurcation point. c) Setting $\lambda_2 = (n\pi)^2$, find a first-term approximation of V for the solutions that bifurcate from (λ_n, θ_s). Comparing these with the value of V when $\theta_s = 0$, comment on what configuration the rod might take when $\lambda_n < \lambda < \lambda_{n+1}$. Also, explain why this may not actually be the configuration.

3.) The van der Pol equation is

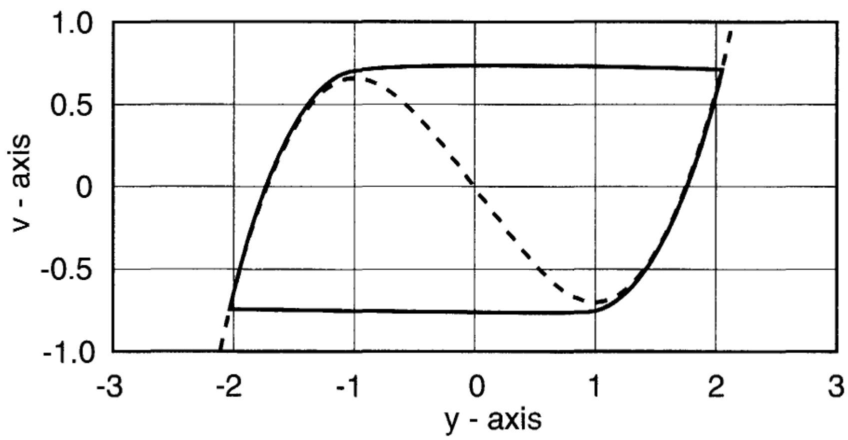
$$\varepsilon y'' - (1 - y^2)y' + y = 0, \text{ for } t > 0.$$

Assume that $y(0) = \sqrt{3}$ and $y'(0) = 0$.

(a) Letting $y = -v'$, show that the equation can be written as the first-order system

$$\begin{aligned} v' &= -y \\ \varepsilon y' &= v + y - \frac{1}{3}y^3. \end{aligned}$$

What are the initial conditions? The numerical solution of this system is shown in the figure below for $\varepsilon = 10^{-3}$.



The solution is periodic, and the solid curve in the graph is the trajectory through one cycle. Note that this path consists of two fast components, which are the horizontal segments, and slow components, which follow the dashed (cubic) curve. Together they form what is known as a relaxation oscillation.

(b) Use the first term in the outer expansion of y and v to determine the equation of the cubic curve shown in the figure. Also, use it to determine where the first corner layer is located (give the value of t, y, and v).

(c) Find the first term in the corner-layer expansion of y and v. With this, determine the location of the first transition layer. The analysis of the transition layers is rather involved and can be found in MacGillivray (1990).

(d) In each transition layer, the solution very quickly switches from one branch of a cubic to another. Use this observation to find a first- term approximation of the period T of the oscillation. For comparison, the period found numerically is $T = 1.91$ when $\varepsilon = 10^{-2}$ and $T = 1.68$ when $\varepsilon = 10^{-3}$.

(e) Use the idea from part (d) to sketch the first-term approximation of $y(t)$ for $0 < t < 3T$.

4.) Consider the modified Rayleigh equation

$$y'' - \lambda(1 - \alpha(y')^{2n})y' + y = 0, \text{ for } t > 0.$$

where α is a positive constant and n is a positive integer.

(a) Find the steady state and determine for what values of λ it is stable.

(b) Find a first-term approximation of the limit cycle that appears at the Hopf bifurcation point.