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IMPERFECT BIFURCATION WITH A SLOWLY-VARYING CONTROL PARAMETER*

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Abstract. We consider a general class of imperfect bifurcation problems described by the following first order nonlinear differential equation:

$$y_t = ky^p + \lambda(t)y + \delta,$$

where $k = 1$ or -1 and $p = 2$ or 3 are fixed quantities. The solution depends on the values of the "imperfection" parameter $\delta(0<\delta\ll 1)$ and the time-dependent control parameter $\lambda(t)=\lambda_0+\varepsilon t$ ($\lambda_0<0$ and $0<\varepsilon\ll 1$). If $\delta=\varepsilon=0$, this equation admits at $\lambda=0$ a bifurcation from the basic state $y=0$ to nonzero steady states. In the first part of the paper, we analyze the perturbation of the bifurcation solutions produced both by the small imperfection ($\delta\neq 0$) and the slow variation of λ ($\varepsilon\neq 0$). We show that $\lambda=0$ does not correspond to the transition between the two branches of slowly-varying steady states. This transition appears at a larger value of $\lambda=\lambda_1$. Provided that δ is sufficiently small compared to ε, λ_1 is an $0(1)$ quantity which only depends on λ_0, i.e., the initial position of $\lambda(t)$.

Our analysis is motivated by problems appearing in laser physics. In the second part of the paper, we show how the semiclassical equations for the simple laser and the laser with a saturable absorber can be reduced to this simple first-order nonlinear equation. We then discuss the practical interests of our results.

1. Introduction. Bifurcation problems with slowly-varying parameters appear in several areas of practical interest. On one hand, parameters are naturally changing and may lead to an undesirable response of the system. For example, the performance of chemical reactors depends on the efficiency of catalytic agents. As a result of their slow decline, a sudden increase of temperature is possible, causing damage to the plant [1]. On the other hand, the experimeter may decide to investigate the bifurcation diagram by continuously increasing or decreasing a key control parameter. This is the case for most experiments with lasers or other nonlinear optical devices [2]-[6]. Recent numerical studies on the laser with a saturable absorber [7] indicate that the actual (i.e., dynamic) response of the laser may be quite different from the static response. This motivates a systematic study of the effects of slowly-varying bifurcation parameters.

The typical effect of a time-dependent parameter is shown in Fig. 1. If the control parameter λ is constant, the system admits a simple bifurcation from the basic state $y=0$ to a new branch of stable steady states $y_s(\lambda)$. If now $\lambda(t)=\lambda_0+\varepsilon t$, where $\lambda_0<0$ and $0<\varepsilon\ll 1$, the system will follow the basic state until it jumps to the branch $y(t)\simeq y_s(\lambda(t))$. However, the static or steady critical point $(y,\lambda)=(0,0)$ is not the point where the jump occurs. The transition appears after a delay corresponding to $\lambda=\lambda_1$ [1], [8], [13]. If this delay becomes important, the dynamic response diagram will differ considerably from the static bifurcation diagram. The principal purpose of this paper is to investigate the effect of small imperfections on this delay. These imperfections, corresponding to impurities, noise, or other inhomogeneities, are always present in experiments and are known to perturb the bifurcation states when λ is constant [9]. As we shall demonstrate, they also have a considerable effect on the dynamic response diagram when λ is slowly varying in time.

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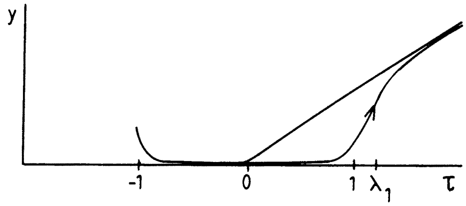

FIG. 1. *The solution* $y(t)$ *of* $y_t=-y^2+\lambda y$, *where* $\lambda\equiv\lambda_0+\varepsilon t$ *is represented as a function of* $\tau=\lambda(t)$. $y(0)=0.5$, $\lambda_0=-1$, *and* $\varepsilon=0.1$. *In the same figure, we represent* $y=\lambda$, *the branch of steady bifurcation states when* $\varepsilon=0$. $\lambda=\lambda_1$ *corresponds to the jump transition between the two branches of slowly-varying steady states*: $y\simeq 0$ *and* $y\simeq\lambda$.

To analyze the perturbation of the slowly-varying steady branches produced by the imperfections, we shall consider the following nonlinear problem:

$$y_t=ky^p+\lambda(t)y+\delta,\qquad y(0)=y_i \tag{1.1}$$

where λ is the control parameter given by

$$\lambda(t)=\lambda_0+\varepsilon t\qquad(\lambda_0<0, 0<\varepsilon\ll 1) \tag{1.2}$$

and $k=\pm 1$, $p=2$ or 3. The additional parameter $\delta>0$ characterizes the magnitude of the imperfections. Similar equations with a different constant coefficient for y^p or $-\delta$ instead of $+\delta$ can be reduced to (1.1) by redefining t and y. In most experimental situations, the system is initially at a stable steady state. Thus we assume that y_i corresponds to a stable equilibrium solution of (1.1) with $\varepsilon=0$. The possible bifurcation diagrams of the steady states of (1.1) with $\varepsilon=0$ are presented in Fig. 2. In §§ 2 and 3, we then consider $\varepsilon\neq 0$ and analyze the delayed bifurcation or jump transition. Since this delay critically depends on the ratio δ/ε, we study this question in detail. Other initial conditions may, however, lead to different long-time behaviors. In § 5, we assume that the stable steady state is initially perturbed and analyze the response of the system in the phase plane (y, t).

Equations (1.1) and (1.2) could be obtained by investigating the solutions of the more complicated nonlinear problem

$$F(u,\lambda',\delta')=0 \tag{1.3}$$

in the vicinity of a steady bifurcation point $\lambda'=\lambda'_c$. In (1.3), u represents a vector of the dependent variables, $\lambda'=\lambda'(\varepsilon' t)$ $(0<\varepsilon'\ll 1)$ is the control parameter, and δ' measures the size of the imperfections. Then, the quantities λ_0, ε and δ in (1.1) and (1.2) are determined by inner products of the derivatives of F with the critical mode, and the initial value y_i is obtained from a projection of the initial data for (1.3) onto the critical mode. Section 4 provides two examples in laser physics leading to an amplitude equation of the form (1.1). The nonlinear problem (1.3) could also be studied by the method of matched asymptotic expansions as $\delta'\to 0$ [9]. We then expect that the solution of the perturbed problem approaches the slowly-varying bifurcation branches except in the vicinity of a singular point. The direct application of this method presents two difficulties. First, as noticed by Haberman [8], the singular point of the slowly-varying solution does not always correspond to the bifurcation point of the steady states. Second, numerical studies of a laser model [7] suggest that different behaviors for the slowly-varying solution are possible if the ratio δ'/ε' is small or large. As we shall demonstrate in §§ 2 and 3, these two difficulties already appear for the simple equation (1.1).

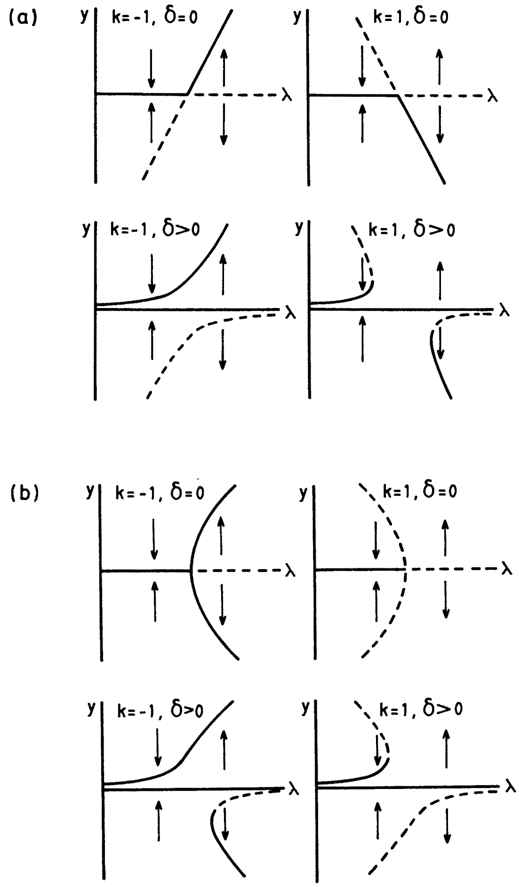

FIG. 2. *Bifurcation diagrams of the steady state solutions of* (1.1). *Figs.* 2(a) *and* 2(b) *correspond to the cases* $p=2$ *and* 3, *respectively. The full and broken lines represent the branches of stable and unstable steady states, respectively.*

The imperfect bifurcation problem described by (1.1) with $p=2$ is also associated with the following bifurcation problem

$$\frac{dx}{dt}=-(x-\lambda'(t))(x-a\lambda'(t)), \tag{1.4}$$

where $\lambda'(t)=\lambda_0'+\varepsilon' t$ and $a<1$. This equation describes the exchange of stabilities between $x=a\lambda'$ and $x=\lambda'$, i.e., two states which both depend on the control parameter λ'. By defining $y\equiv x-a\lambda'$, (1.4) reduces to

$$\frac{dy}{dt}=-y^2+y(1-a)\lambda'-a\varepsilon', \tag{1.5}$$

which corresponds to (1.1) with $k=-1$, $\lambda\equiv(1-a)\lambda'$ and $\delta=-a\varepsilon'$.

In § 2 we analyze the behavior of the solution $y(t)$ of (1.1) when $p=2$, i.e., when the nonlinearity is quadratic. In particular, we shall investigate the behavior of the solution when ε is fixed and δ approaches zero. Section 3 is devoted to the analysis of equation (1.1) with $p=3$ (cubic nonlinearity). The analysis of § 3 is motivated by transition problems in laser physics which are described in § 4. Section 5 considers the role of the initial condition and § 6 discusses practical interests of slowly-varying control parameters.

2. Quadratic nonlinearities. We first analyze equation (1.1) with $p=2$. Introducing the slow time variable $\tau \equiv \lambda_0 + \varepsilon t$, this equation may be rewritten as

$$\varepsilon y_\tau = ky^2 + \tau y + \delta, \qquad y(\lambda_0) = y_i > 0 \tag{2.1}$$

where $\varepsilon > 0$, $\delta > 0$, $\lambda_0 < 0$, $k = \pm 1$, and $\tau \geqq \lambda_0$. We intend to solve this equation by seeking asymptotic expansions as $\varepsilon \to 0$ of the solution $y(\tau, \varepsilon, \delta)$. Since the additional limit $\delta \to 0$ leads to interesting conclusions, we shall benefit from the fact that (2.1) is an exactly solvable Riccati equation. Its general solution is given by

$$y(\tau) = -\frac{\tau}{2k} - \frac{\varepsilon}{k}\left[\frac{\alpha D'_\nu(\varepsilon^{-1/2}\tau) + D'_\nu(-\varepsilon^{-1/2}\tau)}{\alpha D_\nu(\varepsilon^{-1/2}\tau) + D_\nu(-\varepsilon^{-1/2}\tau)}\right] \tag{2.2}$$

where $D_\nu(\xi)$ is the parabolic cylinder function with $\nu \equiv k\delta/\varepsilon$ [10] and $'$ denotes differentiation with respect to τ. The coefficient α is determined by the initial conditions. Both for $k=1$ and $k=-1$, our study will consist of three parts: first we obtain an expansion of $y(\tau, \varepsilon, \delta)$ that is valid as $\varepsilon \to 0$ and $\delta = O(1)$. Then, we investigate the singular behavior of this expansion as $\delta \to 0$. Finally, we analyze the transition from $\delta = O(1)$ to $\delta = 0$.

2.1. $k = -1$. When $\varepsilon = 0$, there exists a unique branch of stable and positive steady states $y = y_s(\lambda_0, \delta)$:

$$y_s(\lambda_0, \delta) = \tfrac{1}{2}[\lambda_0 + (\lambda_0^2 + 4\delta)^{1/2}]. \tag{2.3}$$

When $\varepsilon \ll 1$, and since $y_i > 0$, we expect that $y(\tau, \varepsilon, \delta)$ rapidly changes ($\tau = \lambda_0 + O(\varepsilon)$) until the branch of slowly-varying steady states $y \simeq y_s(\tau, \delta)$ is reached. Therefore, we seek an asymptotic expansion as $\varepsilon \to 0$ of this solution in the form

$$y(\tau, \varepsilon, \delta) = y_0(\tau, \delta) + \varepsilon y_1(\tau, \delta) + \cdots. \tag{2.4}$$

The coefficients $y_0, y_1, \cdots$, are determined by inserting (2.4) into (2.1) and equating to zero the resulting coefficients of each power of ε. This leads to the following result:

$$y(\tau, \varepsilon, \delta) = y_s(\tau, \delta)\left[1 - \frac{\varepsilon}{\tau^2 + 4\delta} + O(\varepsilon^2)\right]. \tag{2.5}$$

Thus, $y(\tau, \varepsilon, \delta)$ corresponds in first approximation to the branch of slowly-varying steady states $y_s(\tau, \delta)$. From (2.5), we observe that this expansion becomes singular if

$$\delta = O(\varepsilon) \tag{2.6}$$

and

$$-\tau = O(\varepsilon^{1/2}) \qquad (\tau < 0). \tag{2.7}$$

We also note from (2.5) that $y = O(\varepsilon^{1/2})$ in the critical regime defined by (2.6)-(2.7). Assuming an expansion of $\delta(\varepsilon)$ of the form

$$\delta(\varepsilon) = \varepsilon(\delta_0 + \varepsilon\delta_1 \cdots), \tag{2.8}$$

the method of matched asymptotic expansion then implies that the inner solution is given by

$$y = \varepsilon^{1/2}(Y_0(s) + \varepsilon Y_1(s) \cdots) \tag{2.9}$$

where s, defined by

$$s \equiv \varepsilon^{-1/2}\tau, \tag{2.10}$$

is still a slow time variable on the time scale t, since $s = \varepsilon^{-1/2}\lambda_0 + \varepsilon^{1/2}t$. To leading order, we find the following equation for Y_0:

$$Y_{0s} = -Y_0^2 + sY_0 + \delta_0 \tag{2.11}$$

which corresponds to the original equation (2.1) with $\tau = s$ and $\delta = \delta_0$. As for the solution of (2.1), the general solution of (2.11) can be expressed in terms of the parabolic cylinder functions:

$$Y_0(s) = \frac{s}{2} + \frac{\alpha D'_\nu(s) + \beta D'_\nu(-s)}{\alpha D_\nu(s) + \beta D_\nu(-s)}, \tag{2.12}$$

where $\nu \equiv -\delta_0$ and $'$ denotes differentiation with respect to s. The coefficients α and β must be determined from the matching condition. From (2.5), we find that

$$Y_0 \to -\frac{\delta_0}{s}, \quad \text{as } s \to -\infty. \tag{2.13}$$

Using the asymptotics of the parabolic cylinder functions, we find that (2.13) can be satisfied only if $\alpha = 0$. Thus, Y_0 is given by

$$Y_0(s) = \frac{s}{2} + \frac{D'_\nu(-s)}{D_\nu(-s)}. \tag{2.14}$$

From (2.14), we observe that

$$Y_0(s) \to s, \quad \text{as } s \to \infty \tag{2.15}$$

and therefore, $Y_0(s)$ reconnects the outer solution (2.5), since $y(\tau, \varepsilon, \varepsilon\delta_0) \simeq \tau$ as $\tau\varepsilon^{-1/2} \to 1$.

In summary, if $\delta = 0(1)$, $y(\tau, \varepsilon, \delta)$ follows the branch of slowly-varying steady states, i.e., $y \simeq y_s(\tau, \delta)$. However, if $\delta = O(\varepsilon)$, then $y \simeq y_s$ except near $\tau = 0$. Indeed, as $|\tau| = O(\varepsilon^{1/2})$, y slightly deviates from y_s (see Fig. 3b). This deviation can be analyzed by comparing the expressions (2.3) and (2.9) for y_s and y, respectively. For example, at $\tau = 0$ and $\delta_0 = 1$, we find that

$$y(0, \varepsilon, \varepsilon) \simeq \varepsilon^{1/2}\sqrt{\frac{2}{\pi}}, \qquad y_s(0, \varepsilon) \simeq \varepsilon^{1/2}. \tag{2.16}$$

We now analyze the behavior of $y(\tau, \varepsilon, \delta)$ as $\delta \to 0$. As for $\delta = O(\varepsilon)$, we observe from (2.5) that the outer expansion of $y(\tau, \varepsilon, \delta)$ becomes singular if (2.7) is satisfied. Thus, as $\delta \to 0$,

$$y(\tau, \varepsilon, \delta) = \left[-\frac{\delta}{\tau} + O(\delta^2)\right](1 + O(\varepsilon)) \tag{2.17}$$

provided that $|O(-\tau)| > O(\varepsilon^{1/2})$ and $\tau < 0$. On the other hand, the deviation $|y(\tau, \varepsilon, \delta) - y_s(\tau, \delta)|$ may become much more important than in the case $\delta = O(\varepsilon)$ as τ approaches zero and becomes positive. This can be seen from a study of the exact solution of (2.1) given by (2.2) with $k = -1$. For example, at $\tau = 0$, we find that for ε fixed, as $\delta \to 0$:

$$y(0, \varepsilon, \delta) \simeq \sqrt{\frac{\pi}{2}}\frac{\delta}{\varepsilon^{1/2}} \ll y_s(0, \delta) \simeq \delta^{1/2}. \tag{2.18}$$

Moreover, when $\delta=0$, problem (2.1) reduces to a Bernoulli equation and its solution is given by

$$y(\tau,\varepsilon,0)=\left[\beta\, e^{-\tau^2/2\varepsilon}+\frac{1}{\varepsilon}\, e^{-\tau^2/2\varepsilon}\int_0^\tau e^{s^2/2\varepsilon}\, ds\right]^{-1} \tag{2.19}$$

where the coefficient β is determined by the initial condition at $\tau=\lambda_0<0$. Evaluating Dawson's integral for small ε, we clearly note four different stages for $y(\tau,\varepsilon,0)$ (see Fig. 1):

1. the initial layer at $\tau=\lambda_0+O(\varepsilon)$ where y changes from y_i to almost zero values;
2. the slowly-varying solution $y\sim\exp(\tau^2-\lambda_0^2)/2\varepsilon\ll 1$ during the interval $\lambda_0<\tau<-\lambda_0$;
3. the transition layer located at $\tau=-\lambda_0+O(\varepsilon)$ where y jumps from zero to $y\sim-\lambda_0$;
4. the slowly-varying solution $y\sim\tau$.

Thus, we observe a considerable delay for the transition to the new branch of slowly-varying solutions. Furthermore, we note that the position of the jump only depends on the initial value of $\lambda(\varepsilon t)$ (i.e., λ_0), but not on its rate of change (i.e., $\varepsilon\ll 1$). We conclude that the primary result of decreasing δ/ε is to progressively increase the deviation $|y(\tau,\varepsilon,\delta)-y_s(\tau,\delta)|$ leading to a delay for the transition to the new branch of solutions. In Figs. 3a to 3c, $y(\tau,\varepsilon,\delta)$ and $y_s(\tau,\delta)$ are represented for different values of δ.

This large delayed transition is typical when the system is initially near the basic (zero) solution and τ progressively increases. By contrast, if the system is initially near the bifurcated state (i.e., $\lambda_0>0$, $y(0)=y_i>0$) and τ slowly decreases (i.e., $\tau=\lambda_0-\varepsilon t$), we only observe a small deviation near $\tau=0$. Indeed, under these conditions, the exact

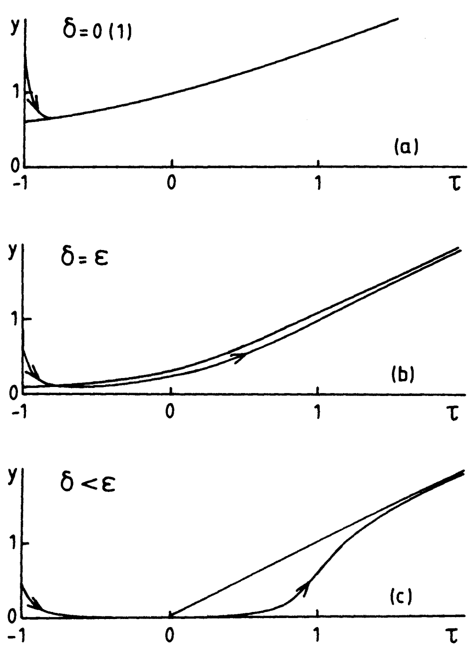

FIG. 3. *The solution $y(t)$ of $y_t=-y^2+\tau y+\delta$, where $\tau\equiv\lambda_0+\varepsilon t$ is represented as a function of $\tau(t)$ for three different values of δ. We also represent the steady state solutions when $\varepsilon=0$.* (a) $\delta=1$, $\varepsilon=10^{-1}$, $y(0)=1.5$, $\lambda_0=-1$; (b) $\delta=\varepsilon=10^{-1}$, $y(0)=0.5$, $\lambda_0=-1$; (c) $\delta=10^{-3}$, $\varepsilon=10^{-1}$, $y(0)=0.5$, $\lambda_0=-1$.

solution of (2.1) with $\delta=0$ is given by:

$$y(\tau, \varepsilon, 0)=\left[\beta\, e^{\tau^2/2\varepsilon}-\frac{1}{\varepsilon}\, e^{\tau^2/2\varepsilon}\int_0^{\tau} e^{-s^2/2\varepsilon}\, ds\right]^{-1} \tag{2.20}$$

where β is now related to the initial conditions at $\tau=\lambda_0>0$. The integral in (2.20) corresponds to the error function and a study of (2.20) when $\varepsilon\to 0$ leads to the following conclusions. We again observe four distinct stages for $y(\tau, \varepsilon, 0)$ as τ decreases:

1. the initial layer at $\tau=\lambda_0+O(\varepsilon)$, where y changes from y_i to $y\sim\lambda_0$;
2. the slowly-varying solution $y\sim\tau+O(\varepsilon)$;
3. the transition near $\tau=0$, where $y=O(\varepsilon^{1/2})$;
4. the slowly-varying solution $y\sim\varepsilon^{1/2}\, e^{\tau^2/2\varepsilon}\ll 1$ as $\tau\to-\infty$.

We conclude that the maximum deviation $|y(\tau, \varepsilon, 0)-y_s(\tau, 0)|$ occurs near $\tau=0$ and in an $O(\varepsilon^{1/2})$ quantity.

2.2. $k=1$. When $\varepsilon=0$, we find two distinct branches of positive steady states given by

$$y_{\pm}(\lambda_0, \delta)=\tfrac{1}{2}[-\lambda_0\pm(\lambda_0^2-4\delta)^{1/2}], \qquad \lambda_0<0. \tag{2.21}$$

y_- and y_+ correspond to stable and unstable solutions, respectively. Moreover, they exist only if

$$\lambda_0\leqq\lambda_c\equiv-2\delta^{1/2} \tag{2.22}$$

and $(y, \lambda_0)=(\delta^{1/2}, \lambda_c)$ correspond to the coordinates of a limit point. When $0<\varepsilon\ll 1$ but $\delta=O(1)$, we again seek an expansion of a slowly-varying solution in the form (2.4). Then, similarly to (2.5), we find that

$$y(\tau, \varepsilon, \delta)=y_-(\tau, \delta)\left[1-\frac{\varepsilon}{\tau^2-4\delta}+O(\varepsilon^2)\right]. \tag{2.23}$$

As $\tau\to\tau_c\equiv-2\delta^{1/2}$, (2.23) becomes singular if $(\tau_c-\tau)=O(\varepsilon^{2/3})$. Moreover, in this critical regime (2.23) indicates that $y-\delta^{1/2}=O(\varepsilon^{1/3})$. Then, a new expansion of the slowly-varying solutions near the limit point reveals that the transition to a rapid jump solution occurs at (see [1], [8] for a detailed study):

$$\tau-\tau_c\simeq\varepsilon^{2/3}\delta^{-1/6}(-s_0), \tag{2.24}$$

where $s_0\simeq-2.3381$ is the first zero of the Airy function. Thus, in the vicinity of the limit point, the deviation $|y(\tau, \varepsilon, \delta)-y_s(\tau, \delta)|$ increases from ε to $\varepsilon^{1/3}$ and the jump appears at an $O(\varepsilon^{2/3})$ distance from the limit point.

We now investigate the behavior of $y(\tau, \varepsilon, \delta)$ as $\varepsilon\to 0$ and $\delta\to 0$. From (2.24), we note that the distance between the jump solution and the static limit point increases as $\delta\to 0$. This motivates a systematic study of this limit. We first examine the singular behavior of (2.23) as $\delta\to 0$. As in the previous case, the nonuniformity appears when (2.6) and (2.7) are satisfied. A similar analysis leads to the following results for the inner solution. First, we assume the expansion (2.8) for δ and seek a solution of the form (2.9). Then, we find that the inner equation for Y_0 satisfies

$$Y_{0s}=Y_0^2+sY_0+\delta_0 \tag{2.25}$$

where s is defined by (2.10). Using the matching condition (2.13), the solution of (2.25) is given by

$$Y_0(s)=-\frac{s}{2}-\frac{D_\nu'(-s)}{D_\nu(-s)} \tag{2.26}$$

where $\nu \equiv \delta_0$ and $'$ denotes differentiation with respect to s. Since the parabolic cylinder function $D_\nu(-s)$ is an oscillatory function of s if $\nu > 0$, $|y_0(s)| \to \infty$ at a finite value of $s = s_0$. From the fact that $D_\nu(-s) = (-1)^n He_n(s)$ when $\nu = n = 0, 1, 2, \cdots$, where $He_n(s)$ is a Hermite polynomial of degree n, it can be shown that $s_0 \geqq 0$ if $0 < \nu \leqq 1$; $-1 \leqq s_0 < 0$ if $1 < \nu \leqq 2$; $-3 \leqq s_0 < -1$ if $2 < \nu \leqq 3$; etc. In summary, the analysis of the case $\delta = O(\varepsilon)$ reveals that the transition from the slowly-varying solution to the rapid jump solution appears at a finite value of $\tau = \tau_0 \equiv \varepsilon^{1/2} s_0$. τ_0 is a negative (positive) quantity if $\nu \equiv \delta_0 > 1$ ($\delta_0 < 1$). Therefore, we expect that the distance between the position of the jump and the limit point will further increase as $\delta \simeq \delta_0$ decreases.

We now examine the case $\delta = 0$. Equation (2.1) reduces to a Bernoulli equation and its solution is similar to (2.19). From an asymptotic study of this solution as $\varepsilon \to 0$, we observe three distinct stages for $y(\tau, \varepsilon, 0)$:

1. the initial layer located at $\tau = \lambda_0 + O(\varepsilon)$ where y changes from $y_i < -\lambda_0$ to $y \sim 0$;
2. the slowly-varying solution $y \sim 0$;
3. the transition to the jump solution ($y \to \infty$) located at $\tau = -\lambda_0 + O(\varepsilon)$.

Thus, the distance between the jump and the bifurcation point is an $O(1)$ quantity equal to $|\lambda_0|$ where λ_0 is the initial position of τ.

In Fig. 4, we represent $y(\tau, \varepsilon, \delta)$ for different values of δ. As predicted by our analysis, we observe that the deviation between the jump and the limit point starts increasing when $\delta = O(\varepsilon)$ and is maximum when $\delta = 0$.

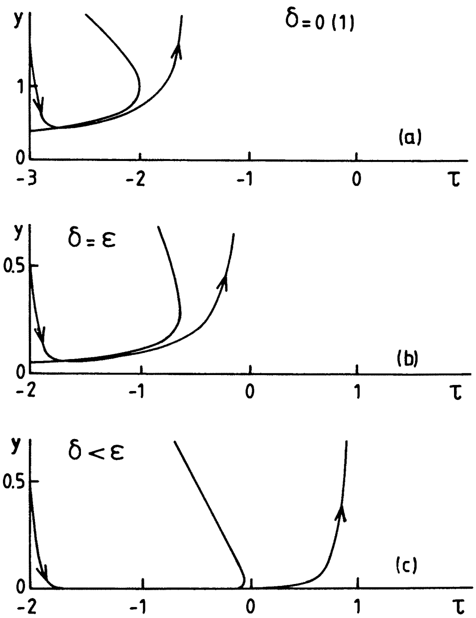

FIG. 4. *The solution $y(t)$ of $y_t = y^2 + \tau y + \delta$, where $\tau \equiv \lambda_0 + \varepsilon t$ is represented as a function of $\tau(t)$ for three different values of δ. We also represent the steady state solutions when $\varepsilon = 0$.* (a) $\delta = 1$, $\varepsilon = 10^{-1}$, $y(0) = 1.5$, $\lambda_0 = -3$; (b) $\delta = \varepsilon = 10^{-1}$, $y(0) = 0.5$, $\lambda_0 = -2$; (c) $\delta = 10^{-3}$, $\varepsilon = 10^{-1}$, $y(0) = 0.5$, $\lambda_0 = -2$.

3. Cubic nonlinearities. We now consider (1.1) with $p = 3$. Introducing the new time variable $\tau \equiv \lambda_0 + \varepsilon t$, this equation is given by

$$\varepsilon y_\tau = ky^3 + \tau y + \delta, \qquad y(\lambda_0) = y_i > 0 \tag{3.1}$$

where $\varepsilon > 0$, $\delta > 0$, $\lambda_0 < 0$, $k = \pm 1$, and $\tau \geqq \lambda_0$. In contrast to problem (2.1), the Abel equation (3.1) cannot be solved exactly. However, our previous analysis of equation

(2.1) suggests the following strategy to find the approximate behavior of $y(\tau, \varepsilon, \delta)$. First, we seek an expansion as $\varepsilon \to 0$ and $\delta = O(1)$ of the slowly-varying solution. Then, we analyze the validity of this expansion as $\delta \to 0$. Finally, we explore numerically and analytically (for $\delta = 0$) the behavior of $y(\tau, \varepsilon, \delta)$ as $\delta \to 0$ and when ε is fixed. As for § 2, we investigate separately the cases $k = -1$ and $k = 1$.

3.1. $k = -1$. When $\varepsilon = 0$, (3.1) admits a unique positive and stable branch of steady state solutions $y_s(\lambda_0, \delta)$ given by

$$\lambda_0 = -\frac{\delta}{y_s} + y_s^2, \qquad y_s > 0. \tag{3.2}$$

If $0 < \varepsilon \ll 1$ and when $\tau \gg \lambda_0$, we expect that $y(\tau, \varepsilon, \delta)$ approaches a slowly-varying regime of the form $y \simeq y_s(\tau, \delta)$. In order to find this slowly-varying solution, we seek as $\varepsilon \to 0$ a solution of the form (2.4). Introducing (2.4) into (3.1), the perturbation analysis then leads to the following results:

$$y(\tau, \varepsilon, \delta) = y_s(\tau, \delta)\left[1 - \frac{\varepsilon y_s^2(\tau, \delta)}{[\delta + 2y_s^3(\tau, \delta)]^2} + O(\varepsilon^2)\right]. \tag{3.3}$$

Thus, provided that $\delta = O(1)$, we find that $y_s(\tau, \delta) = O(1)$ and therefore $y(\tau, \varepsilon, \delta)$ corresponds to the slowly-varying steady-state solution. On the other hand, if $\delta \to 0$, the expansion (3.3) may become nonuniform. We find that this singularity appears if

$$\delta = O(\varepsilon^{3/4}) \tag{3.4}$$

and

$$\tau = O(\varepsilon^{1/2}), \qquad \tau < 0, \tag{3.5}$$

or equivalently if $y_s = O(\varepsilon^{1/4})$. Assuming the expansion of δ of the form

$$\delta(\varepsilon) = \varepsilon^{3/4}(\delta_0 + \varepsilon\delta_1 + \cdots), \tag{3.6}$$

we investigate the critical regime defined by (3.4)–(3.5) by seeking an inner expansion of the solution $y(\tau, \varepsilon, \delta)$ of the form:

$$y = \varepsilon^{1/4}(Y_0(s) + Y_1(s) + \cdots) \tag{3.7}$$

where s is now defined by

$$s \equiv \varepsilon^{-1/2}\tau. \tag{3.8}$$

To leading order, we then find the following inner equation for Y_0

$$Y_{0s} = -Y_0^3 + sY_0 + \delta_0 \tag{3.9}$$

which corresponds to the original equation (3.1) with $\tau = s$ and $\delta = \delta_0$. This equation must be solved with the matching condition obtained from (3.3):

$$Y_0 \to -\frac{\delta_0}{s} \quad \text{as } s \to -\infty. \tag{3.10}$$

The analytical solution of (3.9)–(3.10) is still unknown and numerical investigations of this equation are required in order to analyze the behavior of Y_0. Figure 5 presents the numerical solution of (3.1) when $\delta = O(1)$, $\delta = O(\varepsilon^{3/4})$, and $\delta \ll \varepsilon^{3/4}$.

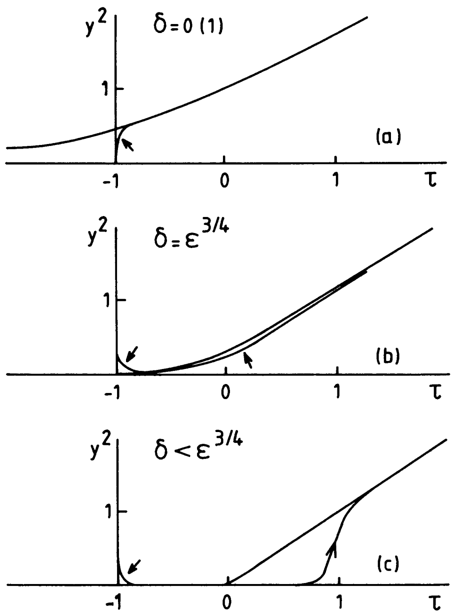

FIG. 5. *We represent y^2 as a function of $\tau = \lambda_0 + \varepsilon t$. $y(t)$ satisfies the equation $y_t = -y^3 + \tau y + \delta$, where $\tau \equiv \lambda_0 + \varepsilon t$. We also represent the branch of steady state solutions when $\varepsilon = 0$.* (a) $\delta = 1$, $\varepsilon = 0.1$, $y(0) = 0.5$, $\lambda_0 = -1$; (b) $\delta = (0.1)^{3/4}$, $\varepsilon = 0.1$, $y(0) = 0.5$, $\lambda_0 = -1$; (c) $\delta = 10^{-3}$, $\varepsilon = 10^{-1}$, $y(0) = 0.5$, $\lambda_0 = -1$.

3.2. $k = 1$. In this case, we observe two distinct branches of positive steady states connected by a limit point given by

$$(y, \lambda) = \left(\left(\frac{\delta}{2}\right)^{1/3}, -3\left(\frac{\delta}{2}\right)^{2/3}\right). \tag{3.11}$$

When $\varepsilon \neq 0$, the analysis of the slowly-varying solution when $\delta = O(1)$, as $\delta \to 0$ and when $\delta = 0$, is similar to our previous studies. So that we summarize the principal results:

(i) As $\varepsilon \to 0$ and when $\delta = O(1)$, the asymptotic analysis proposed by Haberman [8] and Kapila [1] can be used and leads to the same conclusions as in § 2, namely that the jump appears at an $O(\varepsilon^{2/3})$ distance from the limit point.

(ii) As $\delta \to 0$, the expansion of the slowly-varying solution is singular when (3.4) and (3.5) are satisfied. Thus, as for the case $k = -1$, we must explore numerically the behavior of the solution as δ progressively decreases.

(iii) The case $\delta = 0$ can be solved exactly. We find that the rapid jump from $y \sim 0$ appears at $\tau \simeq -\lambda_0$ (as in §§ 2 and 3.1).

4. Laser equations. In this section, we briefly analyze the semiclassical laser equations and show how the amplitude equation (1.1) can be obtained. We shall consider the case of a simple laser as well as the laser with a saturable absorber.

4.1. The laser with injected signal. The laser problem is described by the following equations [14]:

$$\begin{aligned} E_t &= -E + Av + e, \\ v_t &= d(-v + EF), \\ F_t &= d_{\|}(-F + 1 - Ev); \end{aligned} \tag{4.1}$$

$$E(0) = E_0, \qquad v(0) = F(0) - 1 = 0, \tag{4.2}$$

where $e > 0$ and E are the normalized incident and emitted field amplitudes; v and F are the normalized atomic polarization and population difference, respectively The parameters d and $d_{\|}$ correspond to the transverse and longitudinal atomic decay rates normalized by the cavity damping constant. The control parameter A is proportional to the population inversion per atom created by the pump. If $e = 0$, Equations (4.1) and (4.2) describe the evolution of the simple laser. If $e \neq 0$ but is small, the laser is perturbed by a small imperfection. To analyze the effect of this imperfection, we first seek as $e \to 0$ an expansion of the solution of the form [9]:

$$\begin{pmatrix} E(t,e) \\ v(t,e) \\ F(t,e)-1 \end{pmatrix} = \sum_{j=0}^{\infty} e^j \begin{pmatrix} E_j(t) \\ v_j(t) \\ F_j(t) \end{pmatrix}. \tag{4.3}$$

To leading order, we obtain the unperturbed laser equations ($e = 0$) which admit a bifurcation from the zero-intensity solution to nonzero intensity steady states. The bifurcation point is defined by

$$E = v = F - 1 = 0 \quad \text{and} \quad A = 1. \tag{4.4}$$

The study of the higher order corrections, however, indicates that (4.3) is singular if

$$|A - 1| = O(e^{2/3}). \tag{4.5}$$

Therefore, we must analyze the solution of (4.1), (4.2) in the vicinity of the bifurcation point (4.4). To this end, we first assume an expansion of A and E_0 of the form

$$A(e^{1/3}) - 1 = \sum_{2}^{\infty} e^{j/3} A_j, \qquad E_0(e^{1/3}) = \sum_{1}^{\infty} e^{j/3} E_{0j}. \tag{4.6}$$

Then, we seek as $e \to 0$ a solution of (4.1), (4.2) of the form

$$\begin{pmatrix} E(t,\tau,e^{1/3}) \\ v(t,\tau,e^{1/3}) \\ F(t,\tau,e^{1/3})-1 \end{pmatrix} = \sum_{1}^{\infty} e^{j/3} \begin{pmatrix} E_j(t,\tau) \\ v_j(t,\tau) \\ F_j(t,\tau) \end{pmatrix} \tag{4.7}$$

where $\tau \equiv e^{2/3} t$.

This expansion is suggested by the behavior of (4.3) when (4.5) is satisfied. After introducing (4.6) and (4.7) into (4.1), (4.2), we obtain a sequence of linear problems for E_1, v_1, F_1; E_2, v_2, F_2; $\cdots$. We find the following results as $t \to \infty$:

$$\begin{pmatrix} E \\ v \\ F-1 \end{pmatrix} = e^{1/3} \alpha(\tau) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + O(e^{2/3}) \tag{4.8}$$

where the amplitude α satisfies a solvability condition which is given by

$$\alpha_{\tau'} = 1 + A_2 \alpha - \alpha^3, \qquad \alpha(0) = E_{01} \tag{4.9}$$

where $\tau' \equiv \tau/(1 + d^{-1})$.

We now consider the bifurcation problem with a time-dependent control parameter. Specifically, we assume that A is a function of the slow time τ and is given by

$$A(\tau, e^{1/3}) - 1 = e^{2/3}(A_0 + \gamma\tau) + O(e) \tag{4.10}$$

where A_0 and γ are $O(1)$ quantities. Then from a similar perturbation analysis, we find that the asymptotic solution of (4.1), (4.2) is given by (4.8), where α now satisfies

(4.9) with

$$A_2(\tau') = A_0 + \gamma(1+d^{-1})\tau'. \tag{4.11}$$

In conclusion, the bifurcation analysis of the laser equations (4.1), (4.2) leads to the amplitude equation (4.9) with (4.11), which is of the form (1.1) with $p=3$ and $k=-1$.

4.2. The laser with a saturable absorber and injected signal. The laser with a saturable absorber (LSA) presents several advantages compared to the usual laser and has recently been the subject of active experimental and theoretical investigations [2]–[5], [7], [12]. The semiclassical LSA equations for a particular class of solutions and with the injected signal are given by

$$\begin{aligned} E_t &= -E + Av + \bar{A}\bar{v} + e,\\ v_t &= d(-v+FE),\\ \bar{v}_t &= \bar{d}(-\bar{v}+\bar{F}E),\\ F_t &= d_{\|}(-F+1-vE),\\ \bar{F}_t &= \bar{d}_{\|}(-\bar{F}+1-a\bar{v}E); \end{aligned} \tag{4.12}$$

$$E(0)=E_0, \qquad v(0)=\bar{v}(0)=F(0)-1=\bar{F}(0)-1=0. \tag{4.13}$$

The constant $e>0$ and E correspond to the normalized incident and emitted field amplitudes; $v(\bar{v})$ and $F(\bar{F})$ represent the normalized atomic polarization and population difference of the emitting (absorbing) atoms, respectively. The solution of the LSA equations (4.12), (4.13) depends on the values of the fixed parameters d, $\bar{d}$, $d_{\|}$, $\bar{d}_{\|}$, a and the two control parameters A and $\bar{A}$. $d(\bar{d})$, $d_{\|}(\bar{d}_{\|})$ correspond to the normalized transverse and longitudinal atomic decay rates of the emitting (absorbing) atoms. a represents the ratio of the saturation intensity of the absorbing to the amplifying atoms. A and $\bar{A}$ are the pump parameters for the amplifying and absorbing atoms, respectively.

Our purpose is to analyze the solution of the LSA equations near the steady bifurcation point of the zero-intensity solution. This bifurcation point is defined by

$$E=v=\bar{v}=F-1=\bar{F}-1=0 \quad \text{and} \quad A=1-\bar{A}. \tag{4.14}$$

Furthermore, the linear stability analysis of the zero-intensity solution indicates that if

$$\bar{A} > \bar{A}_c \equiv \frac{\bar{d}(d+1)}{\bar{d}-d} \qquad (d>\bar{d}), \tag{4.15}$$

(4.14) also corresponds to the first bifurcation point.

Our purpose is to analyze the solution of the LSA equations in the vicinity of (4.14). The analysis is similar to the case of a simple laser with injected signal; therefore, we summarize the principal results. First, we assume the following expansion of the bifurcation parameter A and the initial conditions:

$$\begin{aligned} A-(1-\bar{A}) &= e^{2/3}A_2(\tau)+O(e),\\ E(0)=E_0 &= e^{1/3}E_{01}+O(e^{2/3}) \end{aligned} \tag{4.16}$$

where $A_2(\tau) \equiv A_0 + \gamma\tau$ is an $O(1)$ quantity and $\tau \equiv e^{2/3}t$ is a slow time variable. Then, we seek an asymptotic solution of (4.12), (4.13) as $e \to 0$ of the form (4.7). The perturbation analysis leads to the following conclusions. As $t\to\infty$, the solution is given by

$$\begin{pmatrix} E \\ v \\ \bar{v} \\ F-1 \\ \bar{F}-1 \end{pmatrix} = e^{1/3}\alpha(\tau)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + O(e^{2/3}) \tag{4.17}$$

where the amplitude α satisfies

$$\begin{aligned} &\left(\frac{d-\bar{d}}{d\bar{d}}\right)(\bar{A}-\bar{A}_c)\alpha_\tau = 1 + A_2(\tau)\alpha - \alpha^3[1+\bar{A}(a-1)], \\ &\alpha(0) = E_{01}/[1+(1-\bar{A})/d + \bar{A}/\bar{d}]. \end{aligned} \tag{4.18}$$

From (4.15), we find that the coefficient of α_τ is always positive. By redefining the time and amplitude variables, equation (4.18) can be reduced to an equation of the form (1.1) with $p=3$ and $k=1$ or -1 if $1+\bar{A}(a-1)<0$ or >0. Thus, we have shown that equation (1.1) may also be obtained for the more complex LSA equations.

5. Role of the initial condition. In §§ 2 and 3, we have analyzed equation (1.1) when the system is initially at a stable state. However, if the steady state is initially perturbed, it is possible to observe a different long-time behavior. We investigate this problem by studying the possible solution of equation (1.1) with $p=2$ in the phase plane (y, τ). Figures 6 and 7 correspond to the cases $k=-1$ and $k=1$, respectively.

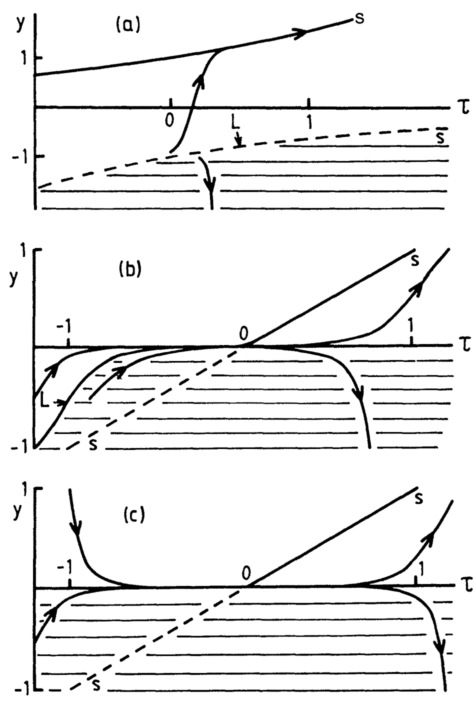

FIG. 6. *Different solutions of* $y_t = -y^2 + \tau y + \delta$, *where* $\tau \equiv \lambda_0 + \varepsilon t$ *and* $\varepsilon = 0.1$ *are represented as functions of* $\tau(t)$ *for* (a) $\delta = 1$, (b) $\delta = 10^{-3}$, *and* (c) $\delta = 0$. *The curves denoted by S represent the branches of steady states when* $\varepsilon = 0$. *The curve denoted by L corresponds to the separatrix curve between the initial points leading to a bounded or an unbounded solution as* $\tau \to \infty$.

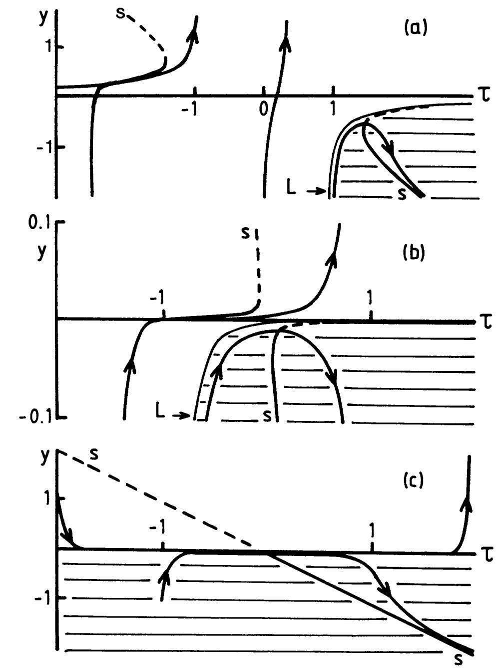

FIG. 7. *We represent different solutions of* $y_t = y^2 + \tau y + \delta$, *where* $\tau = \lambda_0 + \varepsilon t$ *and* $\varepsilon = 10^{-3}$, *as functions of* $\tau(t)$. (a), (b), *and* (c) *correspond to* $\delta = 0.5$, 10^{-3}, *and* 0, *respectively. The branches of steady states when* $\varepsilon = 0$ *are denoted by S. The curve denoted by L represents the separatrix curve between the initial points leading to two different long-time behaviors.*

If $k = -1$ and $\delta = O(1)$, the branch of unstable steady states represents in first approximation the separatrix curve (L) between the trajectories leading to a bounded or an unbounded behavior (Fig. 6a). However, if the ratio δ/ε decreases, the separatrix L deviates from the branch of unstable steady states (Fig. 6b) and if $\delta = 0$, it corresponds exactly to the τ axis (Fig. 6c). Consequently, if $y_i < 0$ and δ/ε is sufficiently small, a perturbation of the stable steady state may lead to an unexpected jump transition. The separatrix curve can be found as the solution of (1.1) which approaches the unstable steady states as $\tau \to \infty$:

$$y(\tau) \to -\frac{\delta}{\tau} \quad \text{as } \tau \to \infty. \tag{5.1}$$

When $k = 1$ a similar behavior is possible. The system may escape from the slowly-varying jump transition (Fig. 7a) if $y_i < 0$ and δ/ε is sufficiently small (Fig. 7b). A trajectory starting below the separatrix L will approach as $\tau \to \infty$ the branch of stable steady states: $y(\tau) \to -\tau$. The separatrix is also defined as the solution of (1.1) which satisfies the condition (5.1). If $\delta = 0$, the separatrix curve becomes the τ axis (Fig. 7c).

Similar conclusions have been obtained for the case $p = 3$.

6. Discussion. The principal conclusion of our study of the imperfect bifurcation problem (1.1) is that the transition from the basic state to the slowly-varying bifurcation states may appear at an $O(1)$ distance from the bifurcation point $\lambda = 0$. This is only possible if the size of the imperfections (i.e., δ) is sufficiently small compared to the rate of change of λ (i.e., ε). Then, the switch between the two branches of solutions occurs at a critical value $\lambda = \lambda_1 > 0$ which depends on the initial position of $\lambda = \lambda_0 < 0$.

From the practical point of view, the shift of the critical point may lead to two important applications. First, since the position of the switch can be controlled by the initial value of λ, a time-dependent bifurcation parameter represents a control mechanism to delay the instability. Second, the slowly-varying evolution near the basic state is generally followed by a rapid jump to the slowly-varying bifurcation states. This contrasts with the usually smooth transitions observed near bifurcation points. Thus, the rapid jump transition provides a fast switching mechanism for the system.

If the basic steady state is initially perturbed, the system may escape from its normal slowly-varying behavior and presents a different evolution as $t \to \infty$. We have shown that this response due to the initial perturbation is particularly successful if the ratio δ/ε is small.

The detailed study of equation (1.1) is motivated by laser problems described in § 4. Although equation (1.1) has been obtained by a local analysis of the more complex laser equations, we have found good qualitative agreements between our analytical results and our previous numerical studies [7]. However, different behaviors may be observed if the control parameter starts to increase far from the bifurcation point. In reference [11], we solve exactly the linearized equations for a simple laser. The general solution exhibits the delayed instability of the basic state but also admits two distinct regimes corresponding to a turning point of the original equation.

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REFERENCES

[1] A. K. KAPILA, *Arrhenius systems: dynamics of jump due to slow passage through criticality*, this Journal, 41 (1981), pp. 29–42.

[2] R. RUSCHIN AND S. H. BAUER, *Bistability, hysteresis, and critical behavior of a* CO_2 *laser, with* SF_6 *intracavity as a saturable absorber*, Chem. Phys. Lett., 66 (1979), pp. 100–103.

[3] A. JACQUES AND P. GLORIEUX, *Observation of bistability in a* CO_2 *laser exhibiting passive Q-switching*, Opt. Commun., 40 (1982), pp. 455–460.

[4] E. ARIMONDO, F. CASAGRANDE, L. A. LUGIATO AND P. GLORIEUX, *Repetitive passive Q-switching and bistability in lasers with saturable absorbers*, Appl. Phys., B30 (1983), pp. 57–77.

[5] E. ARIMONDO AND B. M. DINELLI, *Optical bistability of a* CO_2 *laser with intracavity saturable absorber: experiment and model*, Opt. Commun., 44 (1983), pp. 277–282.

[6] N. J. HALAS, S.-N. LIU AND N. B. ABRAHAM, *Route to mode locking in a three-mode* He–Ne 3.39-μm *laser including chaos in the secondary beat frequency*, Phys. Rev., A28 (1983), pp. 2915–2920.

[7] T. ERNEUX AND P. MANDEL, *Stationary, harmonic and pulsed operations of an optically bistable laser with saturable absorber* II, Phys. Rev. A, 30 (1984), pp. 1902–1909.

[8] R. HABERMAN, *Slowly-varying jump and transition phenomena associated with algebraic bifurcation problems*, this Journal, 37 (1979), pp. 69–105.

[9] B. J. MATKOWSKY AND E. L. REISS, *Singular perturbations of bifurcations*, this Journal, 33 (1977), pp. 230–255.

[10] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1964.

[11] P. MANDEL AND T. ERNEUX, *On the laser-Lorentz equations with a time-dependent parameter*, Phys. Rev. Lett., 53 (1984), pp. 1818–1820.

[12] T. ERNEUX, P. MANDEL AND J. F. MAGNAN, *Quasiperiodicity in lasers with saturable absorbers*, Phys. Rev. A, 29 (1984), pp. 2690–2699.

[13] N. R. LEBOVITZ AND R. J. SCHAAR, *Exchange of stabilities in autonomous systems*, Stud. in Appl. Math., 54 (1975), pp. 229–260.

[14] D. K. BANDY, L. M. NARDUCCI AND L. A. LUGIATO, *Coexisting attractors in a laser with an injected signal*, J. Opt. Am., 2B (1985), pp. 148–155.