Proceedings of Symposia in Applied Mathematics
Volume 56, 1999

An Introduction to Geometric Methods and Dynamical Systems Theory for Singular Perturbation Problems

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ABSTRACT. This expository paper provides an introduction to geometric singular perturbation theory for systems of ordinary differential equations with two time (or length) scales. We focus on systems possessing normally hyperbolic invariant manifolds and, hence, exhibiting a rich and robust geometric structure. These systems arise in a tremendously wide variety of scientific problems, including reaction-diffusion equations, enzyme kinetics, neurophysiology, and coupled mechanical oscillators, to name but a few. First, some low-dimensional (2 and 3) systems are discussed to illustrate the main geometric constructs, such as slow invariant manifolds and their stable and unstable manifolds, in elementary contexts. Then, the Fenichel theory for the persistence of normally hyperbolic invariant manifolds in singular perturbation problems is stated following *Geometric singular perturbation theory* by C.K.R.T. Jones, in *Dynamical systems, Montecatini Terme, 1994*, R. Johnson, ed., Springer Lecture Notes Math., vol. 1609, pp. 44-118. Finally, two further examples are discussed, illustrating how to apply the theory to singularly-perturbed two point boundary value problems and to a strongly nonlinear system arising in mechanics. A brief review of the basic notions of invariant sets, invariant manifolds, and normal hyperbolicity in differential equations is also included.

1. Introduction

Geometric methods and dynamical systems theory play prominent roles in the study of differential equations at all levels. Phase plane methods, for example, are taught in undergraduate courses, and topics such as invariant manifold theory, bifurcations, and attractors, to name a few, are taught at the graduate level. There is also a plethora of current research involving questions about systems modeled by ordinary differential equations and about the mathematical theory for them.

This expository paper focuses on some fundamental geometric constructs and dynamical systems theory for systems of differential equations of the form:

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(\mathbf{x},\mathbf{z},\varepsilon) \\ \mathbf{z}' &= \varepsilon\mathbf{g}(\mathbf{x},\mathbf{z},\varepsilon). \end{aligned} \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 34E15, 58F30; Secondary 34C35, 34A26.
The author thanks A. Doelman, R. Gardner, C. Jones, N. Kopell, G. Kovačič, and C. Soto-Treviño for useful conversations on problems in geometric singular perturbation theory. The author also thanks G. Hall and M. Hayes for commenting on the manuscript and A. Harkin for producing the figures. This work was partially supported by N.S.F. grant DMS-9624471.

Here $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}$, and the prime denotes differentiation with respect to the independent variable t. The functions $\mathbf{f}$ and $\mathbf{g}$ are assumed to be C^∞ functions of $\mathbf{x}$, $\mathbf{z}$, and ε in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ and I is an open interval containing $\varepsilon = 0$; although we remark that the theory we will use holds for the case in which these functions are C^r, $r \geq 1$.

Problems that involve two time (or length) scales naturally give rise to systems of the form (1.1). In particular, the parameter ε explicitly measures the separation of time scales. We are interested in the case when ε is a small positive number. In this case, the variable $\mathbf{x}$ is called a *fast* variable, and $\mathbf{z}$ is called a *slow* variable: $\mathbf{x}$ generally evolves at an $\mathcal{O}(1)$ rate given by the function f; whereas $\mathbf{z}$ evolves more slowly, at a rate proportional to ε.

It is useful at various stages to study a reformulation of the system (1.1) in terms of the rescaled variable $\tau = \varepsilon t$:

$$\begin{aligned} \varepsilon\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \varepsilon) \\ \dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \varepsilon). \end{aligned} \tag{1.2}$$

Here the dot denotes differentiation with respect to the new independent variable τ. The independent variables t and τ are referred to as the *fast* and *slow times*, respectively, and (1.1) and (1.2) are called the *fast* and *slow systems*, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labeled *singular perturbation* problems when ε is a small parameter.

The label 'singular' stems in part from the discontinuous limiting behavior in the systems as $\varepsilon \to 0$. On the one hand, the fast equations (1.1) with $\varepsilon = 0$ reduce to an m-dimensional system with the variable $\mathbf{z}$ as a constant parameter:

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) \\ \mathbf{z}' &= \mathbf{0}. \end{aligned} \tag{1.3}$$

We refer to (1.3) as the reduced fast system. On the other hand, the slow system (1.2) with $\varepsilon = 0$ is differential-algebraic:

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) \\ \dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, 0). \end{aligned} \tag{1.4}$$

Here, the dimension of the system of differential equations decreases from $m+n$ to n, and we refer to (1.4) as the reduced slow system.

Singularly perturbed equations of the form (1.1) arise in a variety of applications: two-point boundary value problems, traveling wave problems in reaction-diffusion equations, chemical pattern formation, the propagation of action potentials in neurophysiology, coupled mechanical oscillators, perturbed Hamiltonian systems, adiabatic Hamiltonian systems, combustion, friction modeling, optics, fluid particle motion in the Lagrangian framework for fluid mechanics, ray theory for wave propagation, celestial mechanics, capture into resonance, control theory, and material science, to name but a few.

There is a long and rich history of work on singularly perturbed systems (1.1). It is far beyond the scope of this article to review, or even to mention, all of the methods that have been developed, as may be evidenced by the following partial list of monographs on singularly perturbed ordinary and partial differential equations: Bogoliubov and Mitropolskii [1961], Chang and Howes [1984], de Jager and Jiang [1996], Diener and Diener [1995], Eckhaus [1979], Fife [1979], Grasman [1987], Holmes [1995], Howes [1979], Il'in [1992], Kevorkian and Cole [1996], Krylov and

Bogoliubov [1947], Lagerstrom [1988], Mischenko and Rozov [1980], Mitropolskii and Lykova [1973], O'Malley [1974], O'Malley [1991], Smith [1985], Stoker [1950], Van Dyke [1979], Vasil'eva, *et al.* [1995], and Wasow [1965]. The reader is also referred to the Appendix of O'Malley [1991] for an overview of the history of the general field of singular perturbations.

The focus on this work will be exclusively on a particular class of methods that have been labeled 'geometric.' The so-called geometric approach to (1.1) endeavors to identify the central dynamical structures, such as invariant sets and invariant manifolds, present in the phase spaces of the systems (1.1). It also seeks to exploit their properties, such as their fast-slow decompositions and the intersections and foliations of the various manifolds, in order to establish the existence of various desired periodic orbits, solutions to boundary value problems, heteroclinic orbits, and Smale horseshoes.

The two reduced, or unperturbed, systems (1.3) and (1.4) offer different insights into the limiting behavior of orbits and geometric structures in the fast and slow regimes, respectively. Certain geometric structures in the phase space of (1.1) with $0 < \varepsilon \ll 1$ can be found close to counterparts from (1.3), and these latter structures are usually easier to locate. In addition, certain other geometric constructs of (1.1) (or equivalently (1.2)) with $0 < \varepsilon \ll 1$ lie close to objects in the phase space of (1.4). By exploiting this decomposition into fast and slow systems, the geometric approach reduces the full singularly perturbed system to separate lower-dimensional regular perturbation problems in the fast and slow regimes, respectively.

One of the earliest geometric approaches to singularly perturbed problems is the Tikhonov-Levinson theory (see Tikhonov [1948], Levinson [1949], Levinson [1950], Levin and Levinson [1954], Levin [1957]). We also refer the reader to Mischenko and Rozov [1980], Nipp [1985], Chapter 2 of O'Malley [1991], and O'Malley [1998], and to the reprinting of Levinson's papers in Nohel and Sattinger [1997]. This theory was originally developed for singularly perturbed systems (1.1) that possess invariant manifolds on which trajectories evolve slowly and toward which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. These manifolds have been labeled asymptotically stable (or unstable) slow manifolds. The later works of Takens [1971], Hirsch, *et al.*, [1977], and Fenichel [1979] establish geometric theory for slow invariant manifolds that possess both expanding and contracting normal directions. Here, we present the theory due to Fenichel [1979], and we follow closely the presentation in Jones [1994]. This theory was established independently in Hirsch, *et al.* [1977], and their monograph develops a comprehensive theory for normally hyperbolic invariant manifolds for general ordinary differential equations; see also Fenichel [1971, 1974, 1977]. Sakamoto [1991] and Wiggins [1994] are also useful works on the theory of normally hyperbolic invariant manifolds.

A rich trove of examples of systems (1.1) that possess the geometric structure studied here may be found in Jones [1994]. Examples include phase transitions modeled by bistable systems, semiconductor theory, solitary waves in a generalized Korteweg-deVries-Kuramoto-Sivashinsky equation, particle settling in a cellular flow field, and the FitzHugh-Nagumo equation.

The work of Levinson, as well as that of other pioneers, see for example Flanders and Stoker [1946], LaSalle [1949], and further references cited in Cronin [1998], also establish fundamental geometric theory for periodic orbits in relaxation oscillators of the form (1.1). In these (and other) problems, it is necessary to analyze the dynamics of initial conditions near 'knees,' *i.e.*, points on certain invariant manifolds where the rate of exponential contraction toward the manifold in the normal directions vanishes, and they lose asymptotic stability. This subject will not be treated here due to space constraints, and the reader is referred to the presentation in Mischenko and Rozov [1980]. See also Remark 5.5 here.

Topological methods, including topological shooting, isolating blocks, and Conley index theory, have also been developed for singular perturbation problems of the form (1.1), and some of these have been developed concurrently or prior to the works of Hirsch, *et al.* [1977] and Fenichel [1979]. See, for example, Bonet [1987], Carpenter [1977], Conley [1978], Hastings [1974], Hastings and McLeod [1991], Kokubu, *et al.* [1996], Kurland [1983], and Levinson [1949].

Finally, we make a technical comment on the nature of the results to be given here. As stated above, we are interested in the case when the parameter ε is a small positive number. The results given will be of the type 'for all sufficiently small ε' or 'in the limit as $\varepsilon \to 0$'. We will often write $0 < \varepsilon \ll 1$, where the relation $\ll$ introduces an ordering; for example: $\varepsilon^2 \ll \varepsilon$ as $\varepsilon \to 0$, and $e^{-1/\varepsilon} \ll \varepsilon^n$ for all $n > 0$, indicating the relative rates of decay (or growth) of the functions as $\varepsilon \to 0^+$, see Chapter 1 of Lagerstrom [1988] or Kevorkian and Cole [1996].

2. Fundamental concepts, background, and an overview

2.1. Some fundamental concepts for ordinary differential equations. Sets of points that have special properties relative to an ordinary differential equation are important for studying the system dynamics. The simplest such sets are fixed points. For the equation $\mathbf{y}' = \mathbf{h}(\mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^k$, a fixed point is a point $\mathbf{y}$ at which the function $\mathbf{h}$ vanishes. They correspond to equilibrium states of the system that is being modeled. One also searches more generally for sets of points that remain invariant relative to the governing equations, where a set V is said to be an *invariant set* of the equation $\mathbf{y}' = \mathbf{h}(\mathbf{y})$ if $\mathbf{y}(t_0) \in V$ for some $t_0 \in \mathbb{R}$ implies that $\mathbf{y}(t) \in V$ for all $t \in \mathbb{R}$, see for example Section I.8 of Hale [1980] and Chapter IX of Hartman [1982]. Simple examples of invariant sets include fixed points and periodic orbits. Finally, a less restrictive notion of invariance, namely local invariance, will be crucial to the Fenichel theory. An open set V is said to be *locally invariant* with respect to an open set W under the system $\mathbf{y}' = \mathbf{h}(\mathbf{y})$ if V is a subset of W and if any trajectory leaving V simultaneously leaves W.

The invariant (and locally-invariant) sets we study have a special property, namely they will be manifolds. Let $\mathbb{R}^p$ denote the $p-$dimensional Euclidean space. A set of points in $\mathbb{R}^p$ is said to be a *smooth manifold of dimension* q, where $q \leq p$, if each point in the set has a neighborhood that is locally C^∞ diffeomorphic to an open subset of $\mathbb{R}^q$, see Milnor [1997]. The simplest examples of manifolds, in $\mathbb{R}^3$ for instance, are points (zero-dimensional), nonintersecting smooth curves (one-dimensional), and nonintersecting smooth surfaces (two-dimensional), such as planes, paraboloids, and two-tori. See Section 1 of Milnor [1997] and Chapter 3 of Abraham, *et al.* [1983] for expositions of the theory of manifolds.

All of the manifolds we study are compact. Standard examples of compact manifolds include a two-torus in $\mathbb{R}^3$, which is boundaryless, and a closed interval in $\mathbb{R}$, which has a boundary. Let H^q denoted the closed half space

$$H^q = \{(x_1, \ldots, x_q) \in \mathbb{R}^q | x_q \geq 0\},$$

and let its boundary, defined as the hyperplane $\mathbb{R}^{q-1} \times 0 \subset \mathbb{R}^q$, be denoted ∂H^q. Then, a subset $X \subset \mathbb{R}^p$ is said to be a *smooth, q-dimensional manifold with boundary* if each point in X has a neighborhood $U \cap X$ that is C^∞ diffeomorphic to an open subset $V \cap H^q$ of H^q. Moreover, the boundary of X, denoted ∂X, is the set of all points in X which correspond to points of ∂H^q under such a diffeomorphism. See section 2 of Milnor [1997], where it is also shown that the definition implies that ∂X is a well-defined, smooth manifold of dimension $q-1$.

Finally, a manifold is said to be a *locally invariant manifold* of the system $\mathbf{y}' = \mathbf{h}(\mathbf{y})$ if the set of points defining it is a locally invariant set.

A second property that plays an equally important role in these notes is that of hyperbolicity. Consider again the general system $\mathbf{y}' = \mathbf{h}(\mathbf{y})$, along with its linearization $\mathbf{w}' = J\mathbf{w}$, where J denotes the $k \times k$ Jacobian matrix of the vector field $\mathbf{h}$ with respect to $\mathbf{y}$. A fixed point of this system is said to be *hyperbolic* if all of the eigenvalues of J evaluated at the fixed point have non-zero real parts. See Section 1.3 of Guckenheimer and Holmes [1983] or Chapter 5.5 of Robinson [1995]. A familiar example of a hyperbolic fixed point is the inverted equilibrium of the classical nonlinear pendulum

$$q' = p, \qquad p' = -\sin q. \tag{2.1}$$

Here, the eigenvalues of the Jacobian at $(-\pi, 0)$ are ± 1, and hence the fixed point is hyperbolic by definition. Solutions approaching (or departing from) the fixed point $(-\pi, 0)$ do so at an exponential rate due to its hyperbolicity.

A second example, of more direct interest in these notes and to be studied in Section 3, is provided by the model system:

$$x' = -x, \qquad z' = -\varepsilon z. \tag{2.2}$$

This example is a special case of (1.1) with $m, n = 1$. The Jacobian of this system is the 2×2 diagonal matrix with entries, and hence also eigenvalues, -1 and $-\varepsilon$. Thus, the origin is a hyperbolic fixed point for all $\varepsilon \neq 0$. In fact, solutions decay exponentially in both directions toward the origin, with small decay rate in the z direction and a (relatively speaking) large decay rate in the x direction. However, when $\varepsilon = 0$, the second eigenvalue is identically zero. Hence, this example also illustrates a situation in which hyperbolicity of the fixed point in the full system is lost when $\varepsilon = 0$, which is a common feature among singularly-perturbed systems of the form (1.1) that will receive further discussion in these notes.

Linearized at a hyperbolic fixed point $\mathbf{y}_0$, a general system $\mathbf{y}' = \mathbf{h}(\mathbf{y})$ has stable and unstable subspaces whose dimensions equal the number of eigenvalues with negative and positive real parts, respectively. Orbits of the linearized system in the stable and unstable subspaces approach the fixed point at an exponential rate in forward and backward time, respectively. Dynamical systems theory asserts that there are nonlinear analogs of these subspaces, known as local stable and unstable manifolds of the hyperbolic fixed point. Let V denote a neighborhood of $\mathbf{y}_0$. The *local stable manifold* of a hyperbolic fixed point $\mathbf{y}_0$ is defined to be:

$$W^S_{\mathrm{loc}}(\mathbf{y}_0) \equiv \{\mathbf{y} | \mathbf{y}(t) \in V \;\; \forall t \geq 0, \mathbf{y}(t) \to \mathbf{y}_0 \text{ exponentially as } t \to \infty\}. \tag{2.3}$$

Similarly, the *local unstable manifold* of a hyperbolic fixed point $\mathbf{y}_0$ is defined to be:

$$W^U_{\mathrm{loc}}(\mathbf{y}_0) \equiv \{\mathbf{y} | \mathbf{y}(t) \in V \;\; \forall t \leq 0, \mathbf{y}(t) \to \mathbf{y}_0 \text{ exponentially as } t \to -\infty\}. \tag{2.4}$$

Specifically, for a system $\mathbf{y}' = \mathbf{h}(\mathbf{y})$, where $\mathbf{h}$ is C^r $(r \geq 1)$ and for which $\mathbf{y}_0$ is a hyperbolic fixed point, the Stable Manifold Theorem establishes the existence of C^r smooth, local stable and unstable manifolds of $\mathbf{y}_0$. Moreover, it states that these local manifolds are tangent to the stable and unstable subspaces, respectively, at $\mathbf{y}_0$. The *global* stable and unstable manifolds, $W^S(\mathcal{M}_0)$ and $W^U(\mathcal{M}_0)$, are then obtained by flowing points on the local manifolds backward and forward in time, respectively. We refer the reader to Chapter 1 of Guckenheimer and Holmes [1983], Chapter IX of Hartman [1982], or Chapter 5 of Robinson [1995] for this theory.

In the example (2.1) of the classical nonlinear pendulum, the hyperbolic fixed points at $(\pm\pi, 0)$ have C^∞, one-dimensional, local stable and unstable manifolds that consist of all the points on the separatrix solution (the $H = 2$ level set of the Hamiltonian $H = p^2/2 + (1 - \cos q)$) that are also in a neighborhood of the equilibrium. Moreover, these local manifolds are tangent to the corresponding subspaces at the equilibria. For example, the points on $W^S_{\text{loc}}(\pi, 0)$ approach $(\pi, 0)$ exponentially in forward time, and use of l'Hopital's rule shows, *e.g.*, that the separatrices are tangent to the stable subspace spanned by $(1, -1)^T$ at the point $(\pi, 0)$.

2.2. Some fundamental concepts for systems (1.1). Looking at the form of the reduced slow system (1.4), one is naturally led to study the zero set of $\mathbf{f}$. Locally, this set is expected to be n−dimensional, and for the problems studied here, it is natural to consider those problems for which (1.4) has an n−dimensional C^∞ smooth manifold $\mathcal{M}_0$ of points $(\mathbf{x}, \mathbf{z})$ inside the zero set of $\mathbf{f}$. Furthermore, we will for convenience take such a manifold $\mathcal{M}_0$ to be given by a C^∞ function $\mathbf{x} = \mathbf{X}_0(\mathbf{z})$. This last assumption is not necessary, since one can patch together in a smooth fashion the results from applying the theory to individual charts for those examples in which the manifold cannot be given by the graph of a single function.

The model system (2.2) illustrates the above situation. The model rewritten in slow time is: $\varepsilon\dot{x} = -x, \dot{z} = -z$, and hence the set of points along which the function f vanishes is the z−axis, *i.e.*, $\mathcal{M}_0$ is the z−axis, and the desired function is the constant function $X_0 = 0$, independent of z.

For general systems (1.1) with $\varepsilon = 0$, initial conditions on $\mathcal{M}_0$ evolve on the slow time scale, according to the reduced slow system (1.4) with $\mathbf{x} = \mathbf{X}_0(\mathbf{z})$:

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0).$$

Conversely, when viewed from the perspective of the fast system (1.1) with $\varepsilon = 0$, $\mathcal{M}_0$ is a manifold of fixed points, since by definition $\mathbf{x}' = \mathbf{0}$ at all points $(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0)$ at which $\mathbf{f}$ vanishes. Hence, $\mathcal{M}_0$ is trivially invariant. The fact that $\mathcal{M}_0$ consists of fixed points in the reduced fast system contrasts markedly with the result that there is dynamics on $\mathcal{M}_0$ when $\mathcal{M}_0$ is analyzed in the reduced slow system. In the examples and theory of Sections 3-7, we will see that both perspectives are needed.

Finally, among all of the systems (1.1) with locally invariant manifolds, we concentrate exclusively on those in which, when $\varepsilon = 0$, each point $(\mathbf{X}_0(\mathbf{z}), \mathbf{z})$ on $\mathcal{M}_0$ is a hyperbolic fixed point of the reduced fast system (1.3). In these systems, the invariant manifold $\mathcal{M}_0$ is said to be *normally hyperbolic,* and it is these manifolds that occupy a central place in geometric singular perturbation theory.

In the example (2.2), the z−axis is a normally hyperbolic invariant manifold. Another example is provided by a slowly, parametrically-excited pendulum, which may be written in system form as:

$$q' = p, \qquad p' = -(1 + \alpha \sin z) \sin q, \qquad z' = \varepsilon, \tag{2.5}$$

where $0 < \alpha < 1$. This model arises for example when the pendulum's length varies in a slow periodic fashion. For each $z \in [0, 2\pi)$, the points $(q = \pm\pi, p = 0)$ are hyperbolic fixed points of the reduced fast system $q' = p, p' = -(1 + \alpha \sin z) \sin q$, in which z is a fixed parameter. In the extended (q, p, z) phase space, the unions

$$\mathcal{M}_0^{\pm} \equiv \bigcup_{z \in [0,2\pi)} (\pm\pi, 0, z)$$

are normally hyperbolic invariant manifolds of (2.5) when $\varepsilon = 0$. See also Kaper and Wiggins [1992].

Finally, for a C^r vector field, a normally hyperbolic invariant manifold has local, C^r smooth, stable and unstable manifolds: $W^S_{\text{loc}}(\mathcal{M}_0)$ and $W^U_{\text{loc}}(\mathcal{M}_0)$. These are the unions over all $\mathbf{z}$ in $\mathcal{M}_0$ of the local stable and unstable manifolds of the reduced fast system's hyperbolic fixed points $(\mathbf{X}_0(\mathbf{z}), \mathbf{z})$. Moreover, these unions are C^r smooth by standard smooth dependence theory.

REMARK 2.1. The definition of normal hyperbolicity given here is specifically adapted to the context of the singularly perturbed system (1.1) when $\varepsilon = 0$. We therefore emphasize that the notion of normal hyperbolicity is relevant to a much broader class of equations, to general systems $\mathbf{y}' = \mathbf{h}(\mathbf{y})$. In broad terms, the definition of normal hyperbolicity for general systems involves verifying that the exponential growth rates of orbits in the directions normal to $\mathcal{M}_0$ are stronger than the growth rates in the tangential directions, as may be measured, for example, using Lyapunov type numbers. We refer the reader to Fenichel [1971] and Hirsch, *et al.* [1977], as well as to the exposition in Wiggins [1994].

2.3. An overview of the examples treated in these notes. In the following sections, we first discuss elementary, low-dimensional examples of the general class of systems (1.1). In studying these examples before stating the general theorem, we hope to provide a clear context in which to introduce the various geometric objects and dynamical systems terminology that will be encountered in the general theory. We begin in Section 3 with example (2.2). Precisely because straightforward, explicit solutions are available for this problem, it provides a good template for our initial study of the geometry both when $\varepsilon = 0$ and when $0 < \varepsilon \ll 1$.

Lest it be thought that planar examples are all trivial, however, a nonlinear planar example is also given in Section 3 from enzyme kinetics (see Chapter 2 of O'Malley [1991]):

$$x' = z - (z + \kappa)x, \qquad z' = \varepsilon\left[-z + (z + \kappa - \lambda)x\right].$$

The dynamics of this example, as well as those of other nonlinear, planar, singularly perturbed systems, can be decomposed in terms of the same types of geometric structures as identified in the first example.

The third example, given in Section 4, is again linear:

$$x_1' = x_1, \qquad x_2' = -x_2, \qquad z' = -\varepsilon z.$$

Its dimension is one larger than that of the first example, and it enables us to show that results analogous to those we obtain for the first and second examples (which are systems with only a fast exponentially decaying direction) hold also when the system possesses both exponentially decaying and growing directions.

Also, in Section 4, an application from resonant sloshing of shallow water in a tank is given:

$$q' = p, \quad p' = q^2 - F(z), \quad z' = \varepsilon,$$

where $F(z) \equiv 1 + c + \cos z$ and $c > 0$. Analyzed in Hastings and McLeod [1991], this fourth example illustrates the general geometric features in nonlinear, third-order, singularly perturbed systems. It is one of a class of Hamiltonian systems $H = H(q, p, z = \varepsilon t)$, that also includes the slowly-varying pendulum (2.5).

Once finished with introducing the main geometric constructs and concepts in the context of these successively more complex examples, we turn our attention in Section 5 to general systems (1.1) that have a normally hyperbolic invariant manifold $\mathcal{M}_0$ when $\varepsilon = 0$. A theorem due to Fenichel (see Fenichel [1979]) is stated giving conditions under which normally hyperbolic invariant manifolds in (1.1) persist when the perturbation is turned on, *i.e.,* when $0 < \varepsilon \ll 1$. This theorem was also established independently by Hirsch, Pugh, and Shub, see Hirsch, *et al.* [1977]. Persistent manifolds are labeled *slow manifolds*, and the proof of their persistence is carried out by demonstrating that the local stable and unstable manifolds of $\mathcal{M}_0$ also persist as locally invariant manifolds in the perturbed system. In other words, one shows that the local hyperbolic structure persists, and then the slow manifold is immediately at hand as a locally invariant manifold in the transverse intersection of these persistent local stable and unstable manifolds. The complete local picture that emerges is crucial to establishing a normal form for the vector field in the neighborhood of the slow manifold that contains all of the information about the slow dynamics on the slow manifolds and the fast dynamics in the normal directions to the slow manifolds.

The Fenichel persistence theory for singular perturbed systems relies on many results developed for general systems of ordinary differential equations in Fenichel [1971,1974,1977]. The proof will not be given here. It is long and given in Fenichel [1979]. Complete presentations and new proofs of the main results are given in Jones [1994] using the Wazewski Principle and cone estimates and in Sakamoto [1991] using exponential dichotomies.

Finally, in Sections 6 and 7, we present applications of the Fenichel theory. The first is a textbook example of an elementary linear equation with boundary conditions:

$$\varepsilon\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \quad \text{with} \quad 0 < \varepsilon \ll 1, \quad y(0) = 1 \quad \text{and} \quad y(1) = -1.$$

This example can be solved using several different techniques, including the method of matched asymptotic expansions. We use it to illustrate the application of the geometric theory, and remark that this same geometric approach has been applied to more general singularly perturbed two-point boundary value problems. We follow Hayes, *et al.,* [1998]. In the second application, we show how Fenichel theory may be applied to study perturbed two (or more) degree of freedom Hamiltonian systems, using the example of a nonlinear Duffing oscillator coupled strongly to an anharmonic oscillator, see Kaper and Kovačič [1996].

3. Linear and nonlinear planar systems

In this section, we study two planar examples of systems (1.1). We begin with the decoupled pair of linear ordinary differential equations:

$$\begin{aligned} x' &= -x \\ z' &= -\varepsilon z. \end{aligned} \tag{3.1}$$

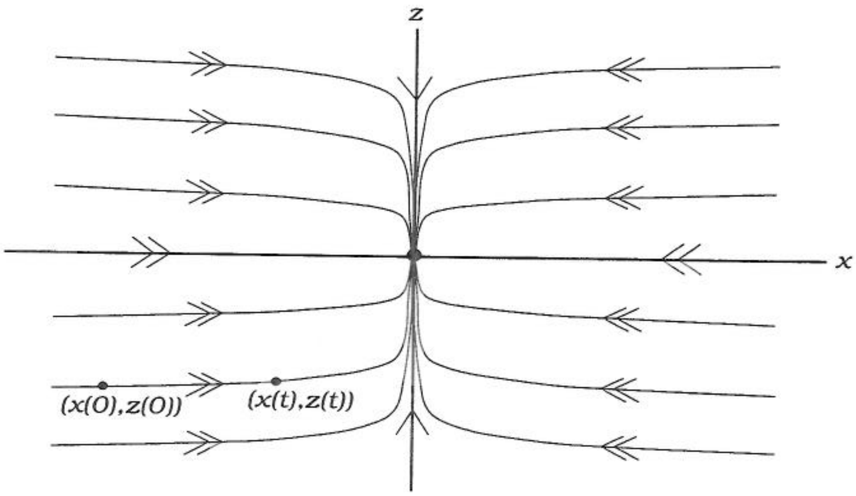

FIGURE 1. Sketch of the phase plane of (3.1) with $0 < \varepsilon \ll 1$. In this figure, as well as throughout this work, a single arrow indicates a slow direction and a double arrow signifies a fast direction.

Then, we turn to an example from enzyme kinetics:

$$\begin{aligned} x' &= z - (z+\kappa)x \\ z' &= \varepsilon\left[-z + (z+\kappa-\lambda)x\right], \end{aligned} \tag{3.2}$$

where κ and λ are bounded, positive constants, and the variables represent scaled, nondimensionalized concentrations. Both systems are special cases of (1.1) in which $m, n = 1$, and they are singularly perturbed when $0 < \varepsilon \ll 1$. System (3.1), which appears in Kovačič [1990], has been chosen because it facilitates our initial exploration of the Fenichel theory. System (3.2), taken from Chapter 2 of O'Malley [1991], exhibits general phenomena common to planar singularly perturbed systems, and it illustrates the fact that nonlinear problems can be decomposed using the same geometric methods.

3.1. Exact solutions of (3.1). Due to the fact that the linear system (3.1) decouples, the general solution with initial data $(x(0), z(0), \varepsilon)$ is readily seen to be:

$$x(t) = x(0)e^{-t} \qquad \text{and} \qquad z(t) = z(0)e^{-\varepsilon t}.$$

In the (x, z) phase plane of (3.1), see Figure 1, trajectories are given explicitly in terms of the one-parameter family of functions: $x = Cz^{1/\varepsilon}$. Moreover, the origin is the only fixed point when $\varepsilon \neq 0$. All trajectories approach it in forward time, with initial conditions on the x−axis approaching it horizontally, while all others (not already on the z−axis) approach tangentially to the z−axis in a $C^{r(\varepsilon)}$ tangency, where $r(\varepsilon)$ is less than or equal to the greatest integer that is less than or equal to $\frac{1}{\varepsilon}$. This tangency has infinite order in the limit as $\varepsilon \to 0$. Having briefly reviewed the straightforward exact solutions, we now turn our attention to discussing the relevant geometric objects from a dynamical systems viewpoint, beginning with the simpler $\varepsilon = 0$ system and then considering the full system.

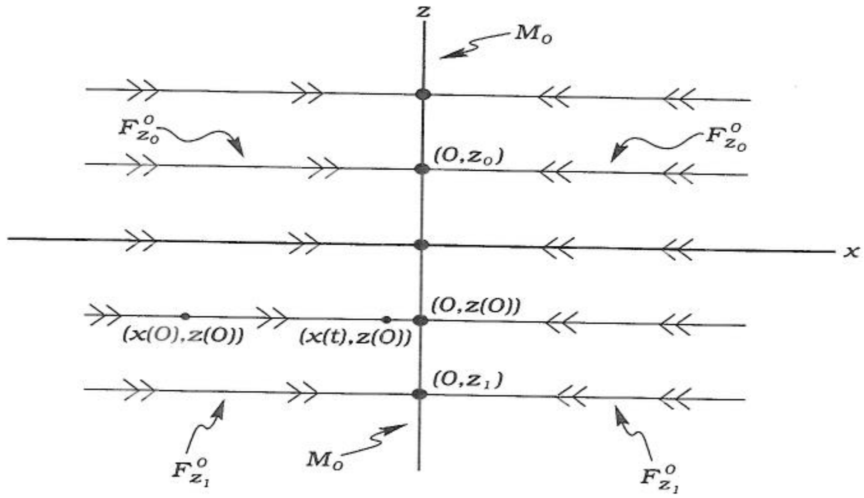

FIGURE 2. The geometry of (3.1) when $\varepsilon = 0$.

3.2. On the geometric structure of the system (3.1) when $\varepsilon = 0$. The reduced fast system associated to equations (3.1) is:

$$x' = -x, \qquad \text{and} \qquad z' = 0.$$

In the (x, z) phase plane (see Figure 2 for an illustration), the $z-$axis is a line of fixed points, and hence an invariant set. In addition, since a line in $\mathbb{R}^2$ is a manifold, we know that the $z-$axis is an invariant manifold of (3.1) with $\varepsilon = 0$. Moreover, for each z, the fast component of the vector field (3.1), $f(x, z, 0) = -x$, linearized at the fixed point $x = 0$ has only one eigenvalue, and it is negative, namely -1. Hence, the $z-$axis is a normally hyperbolic invariant manifold of (3.1) with $\varepsilon = 0$. We label it as $\mathcal{M}_0$. Finally, due to the fact that all orbits approach $\mathcal{M}_0$ at an exponential rate in forward time, the stable manifold of $\mathcal{M}_0$ is the entire (x, z) plane. We denote the stable manifold of $\mathcal{M}_0$ by $W^S(\mathcal{M}_0)$, and we observe that there are no finite fixed points with unstable directions.

The lines of constant z are invariant sets for each $z \in \mathbb{R}$ when $\varepsilon = 0$. These lines are also called *one-dimensional fibers*, and we label them by F^0_z, where the superscript zero indicates $\varepsilon = 0$, and the subscript z denotes the constant value of z. See Figure 2. These lines may be further classified as *fast* fibers, because the exponential rate of contraction along them toward $x = 0$ is $\mathcal{O}(1)$ in the t time variable, and the points $(0, z)$ at which these lines are attached to the manifold $\mathcal{M}_0$ are called the *basepoints* of the fibers. Finally, taking the union over all z of the fast fibers yields an invariant family of fibers (lines) that we denote $\mathcal{F}^0$.

REMARK 3.1. In this example, the family of fast fibers $\mathcal{F}^0$ is said to *foliate* the entire (x, z) plane (and, hence, also $W^S(\mathcal{M}_0)$), since there is one line for every z and the lines completely fill out the plane. See Section 4.3 of Abraham, *et al.* [1983] for the definition of a foliation.

3.3. On the geometric structures present in (3.1) when $0 < \varepsilon \ll 1$. By the construction of this example, all of the geometric structures just identified for the $\varepsilon = 0$ system are also present, but with modified properties, in the phase

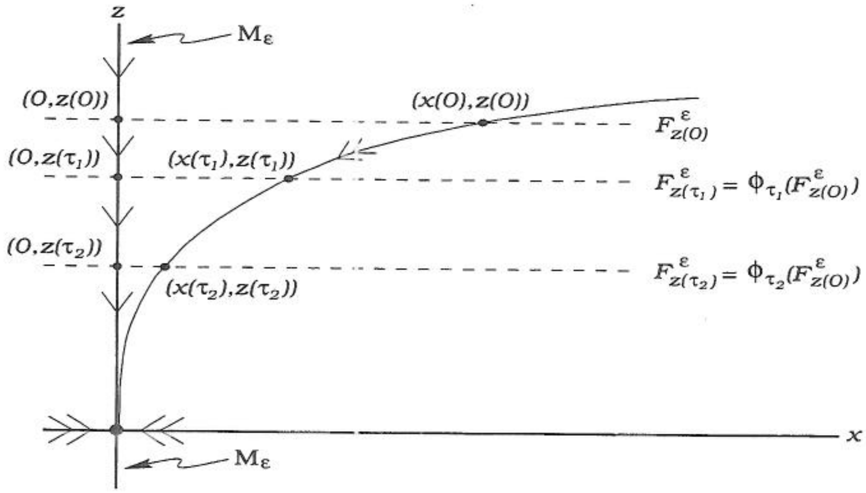

FIGURE 3. $\mathcal{M}_\varepsilon$ and the invariant family $\mathcal{F}^\varepsilon$ of fast stable fibers (dashed lines) for (3.1) with $0 < \varepsilon \ll 1$. The evolution of a trajectory can be decomposed into two pieces: one component along the manifold $\mathcal{M}_\varepsilon$, which is determined by the evolution of the base point $(0, z(0))$ of the fiber $F^\varepsilon_{z(0)}$, and a component in the orthogonal direction, along the fiber $F^\varepsilon_{z(0)}$ and its images $\phi_\tau F^\varepsilon_{z(0)} = F^\varepsilon_{\phi_\tau(z(0))}$ under the flow ϕ_τ of (3.3). $\tau_i = \varepsilon t_i$, where $i = 1, 2$.

space of (3.1) with $0 < \varepsilon \ll 1$; *i.e.*, they persist when the $\varepsilon = 0$ system is perturbed and includes the $\mathcal{O}(\varepsilon)$ term. Moreover, it turns out that this is but one of the simplest examples of the general persistence theory for normally hyperbolic invariant manifolds for the general systems (1.1), as we shall see in Section 5.

The z−axis is again an invariant manifold when $0 < \varepsilon \ll 1$. See Figure 3. We now denote it $\mathcal{M}_\varepsilon$. The flow on $\mathcal{M}_\varepsilon$ is slow, governed by the slow (second) component of the system:

$$\varepsilon \frac{dx}{d\tau} = -x \qquad \text{and} \qquad \frac{dz}{d\tau} = -z. \tag{3.3}$$

In addition, the family of lines $z =$ constant still constitutes a family of fast fibers, and the basepoints of these fibers are precisely the points on $\mathcal{M}_\varepsilon$. We label the individual fibers by F^ε_z, where z denotes the z−coordinate of the fiber's basepoint, and the family by $\mathcal{F}^\varepsilon$. It must be noted, however, that the fibers are now invariant relative to (3.3) only as a family, not individually, as was the case when $\varepsilon = 0$. Fix an arbitrary $z(0)$. For every $x(0)$, an initial condition $(x(0), z(0))$ that starts out on the fiber $F^\varepsilon_{z(0)}$ evolves in such a way that its image, $(x(\tau), z(\tau))$, at time τ lies on the image fiber $F^\varepsilon_{\phi_\tau(z(0))}$, where ϕ denotes the flow of the full system.

In the above sense, the evolution of any trajectory with $(x(0), z(0)) \neq (0, 0)$ can be decomposed into two pieces: one component along the manifold $\mathcal{M}_\varepsilon$, which is governed by the evolution $(0, z(\tau))$ of the basepoint of the fiber it is on, and a second component in the normal direction, which is governed by the exponential rate of contraction along the fibers.

$\mathcal{M}_\varepsilon$ is not a unique slow manifold. As we show now, the system (3.1) possesses a family of invariant slow manifolds along which solutions approach the origin in a C^∞ tangency and at an $\mathcal{O}(\varepsilon)$ rate. In fact, the nonuniqueness and closeness of slow manifolds are general phenomena, and they are intimately connected to the notion of a center manifold. Hence, before showing this for example (3.1), we briefly review the definition of a center manifold. Consider again a general system $\mathbf{y}' = \mathbf{h}(\mathbf{y})$, where $\mathbf{h}$ is C^r. Let this system have a fixed point at $\mathbf{y}_0$ at which the Jacobian of the vector field has at least one eigenvalue with zero real part. The linearized system will then have a center subspace that is precisely the span of the generalized eigenvectors associated to the eigenvalues with zero real parts. Any locally invariant manifold that is tangent to the center subspace at $\mathbf{y}_0$ is said to be a *center manifold of* $\mathbf{y}_0$. See Kelley [1967], Carr [1981], Chapter 3 of Guckenheimer and Holmes [1983], or Chapter 5 of Robinson [1995] for the theory of center manifolds in ordinary differential equations.

Kelley [1967] gives an interesting and illustrative planar example: $y_1' = y_1^2$ and $y_2' = -y_2$, and a sketch of the phase plane is given in Section 3.2 of Guckenheimer and Holmes [1983]. In this example, the origin is not a hyperbolic fixed point, since the linearized system has a simple zero eigenvalue with eigenvector $(1,0)^T$, and the center subspace is the y_1-axis. Hence, by definition, the y_1-axis is a center manifold. However, it is not the only one in this system. One observes that the trajectories through any initial condition with $y_1(0) < 0$ approach the origin with a C^∞ tangency to the y_1-axis, while in the right half plane, only trajectories on the positive y_1-axis asymptote to the origin (in backward time). Hence, by the definition of a center manifold, any set obtained by concatenating a trajectory with initial condition in the left half plane and the positive y_1-axis is also a center manifold. In this way, we see that there is a family of center manifolds of the origin in this example, and each center manifold is one-dimensional.

For system (3.1), the fixed point $(0,0)$ is nonhyperbolic in the limit as $\varepsilon \to 0$. The linearized system has a one-dimensional stable subspace, spanned by $(1,0)^T$, and a one-dimensional center subspace, spanned by $(0,1)^T$. The z-axis is a center manifold of the origin, by the definition. Moreover, just as was the case for the y_1-axis in Kelley's example, one of the axes here, namely the z-axis, is a nonunique center manifold. Both in the lower and upper half planes, there are families of trajectories that approach the origin tangent to the $z-$axis. Hence, there is a family of center manifolds, each one obtained by concatenating either the negative z-axis or a trajectory in the lower half plane to either the positive z-axis or a trajectory in the upper half plane.

Finally, each center manifold constructed in this fashion is C^∞ in the limit as $\varepsilon \to 0$ (only the z-axis itself is an analytic center manifold), and each center manifold is exponentially close to every other one on compact intervals. Pictorially, if we restrict to an open interval $J \subset \{z - \text{axis}\}$ that contains the origin, then these center manifolds can only be distinguished by the location outside J at which they 'peel away' from (exit a fixed tubular neighborhood of) the $z-$axis.

3.4. The method of matched asymptotic expansions applied to solutions of (3.1). We fix an initial condition $(x(0), z(0))$ with both components positive and $\mathcal{O}(1)$, *i.e.*, independent of ε. Using the method of matched asymptotic expansions, one tries to write the solution through this initial condition as a power series in ε: $x^{\text{inner}}(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \mathcal{O}(\varepsilon^3)$ and $z^{\text{inner}}(t,\varepsilon) = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + \mathcal{O}(\varepsilon^3)$. Plugging these expansions order-by-order into the governing equation (3.1) yields the following expansion as $\varepsilon \to 0$:

$$x^{\text{inner}}(t,\varepsilon) = x(0)e^{-t}, \qquad z^{\text{inner}}(t,\varepsilon) = z(0)\left(1 - \varepsilon t + \varepsilon^2 \frac{t^2}{2} + \mathcal{O}(\varepsilon^3)\right). \tag{3.4}$$

Clearly, the x−component changes by $\mathcal{O}(1)$ amounts over $\mathcal{O}(1)$ t−time intervals, whereas the z−component changes slowly. Hence, in the language of matched asymptotic expansions this expansion is an *inner* (or fast) expansion, and (3.1) is called the inner equation.

There is a natural restriction on the time interval of validity of the inner expansion. Based on only the leading order term, one may say that *a priori*, the inner regime is an interval $[0, t_1)$ of t−time, where $t_1 > 0$ is any $\mathcal{O}(1)$ quantity. However, with the more precise knowledge of the inner expansion that is available from the equation (or from (3.4)), namely the terms in the expansion z^{inner} remain ordered only as long as $t \ll \frac{1}{\varepsilon}$, one sees that the interval is much larger: $[0, t_1(\varepsilon))$ for any function $t_1(\varepsilon) \ll \frac{1}{\varepsilon}$, since z can only change by an $\mathcal{O}(1)$ amount after times of more than $\mathcal{O}(1/\varepsilon)$.

It is also natural to look at the governing equations (3.3) written in terms of the slow variable. Again one looks for a power series expansion of the solution, but now with the coefficients being functions of τ: $x^{\text{outer}}(\tau,\varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \mathcal{O}(\varepsilon^3)$ and $z^{\text{outer}}(\tau,\varepsilon) = z_0(\tau) + \text{h.o.t}$, where h.o.t. denotes higher order terms. Plugging these order-by-order into the slow system yields:

$$\begin{aligned} x^{\text{outer}}(\tau,\varepsilon) &= 0 + \varepsilon \cdot 0 + \varepsilon^2 \cdot 0 + \cdots + x(0)e^{-\frac{\tau}{\varepsilon}} \\ z^{\text{outer}}(\tau,\varepsilon) &= z(0)e^{-\tau}, \end{aligned} \tag{3.5}$$

as $\varepsilon \to 0$. The slow variable z evolves an $\mathcal{O}(1)$ amount during $\mathcal{O}(1)$ intervals of slow time, while the x−component remains exponentially small. Note that we sought (and it was necessary to seek) the exponentially small term here only because all of the coefficients on the powers of ε vanish in the outer expansion.

The equations (3.3) are labeled the outer equations, and (3.5) is called the outer expansion. It is clearly a valid asymptotic expansion of the full solution on the slow time interval (τ, ∞) for any positive, $\mathcal{O}(1)$ value of τ. Moreover, from the full expansion (3.5) one sees also that this time interval can be enlarged to be $(\tau_1(\varepsilon), \infty)$, where $\tau_1(\varepsilon)$ is any positive function that vanishes as $\varepsilon \to 0$ and satisfies $\varepsilon \ll \tau_1(\varepsilon)$. We also note that these types of extensions of the domains of validity are often possible in the general case via the Kaplun and Eckhaus Extension Theorems, see Lagerstrom [1988] and Eckhaus [1979].

The enlarged outer domain overlaps with the inner domain. The inner and outer solutions (3.4) and (3.5) are identical up to a change of coordinates for all $\varepsilon \neq 0$, due to the simplicity of the system. Hence, they trivially have the same functional form, and therefore match, in the overlap domain. This is not typically the case for general systems (1.1).

3.5. Relating the geometric and matching approaches for (3.1). In this subsection, we finish our discussion of the example (3.1) by describing some aspects of the relation between the geometric theory, given in subsections 3.2-3.3, and the method of matched asymptotic expansions, reviewed in subsection 3.4.

Geometrically, the natural decomposition of the evolution of an arbitrary initial condition into (i) a fast component governed by the exponential contraction along the fast stable fibers and (ii) a slow component given by the slow motion of the fibers' basepoints along $\mathcal{M}_\varepsilon$ makes this connection between the fast and slow regimes explicitly visible. The leading order term in the inner expansion $(x(t) = x(0)e^{-t}, z(t) = z(0))$ is precisely the solution along the unperturbed fiber $F^0_{z(0)}$ that the initial condition lies on. In addition, the leading order term in the outer expansion $(x(\tau) = 0, z(\tau) = z(0)e^{-\tau})$ corresponds to the evolution of that fiber's basepoint along the manifold $\mathcal{M}_0$. Then, higher order terms in each of the expansions capture the higher order terms in the expansions of the functions whose graphs are the invariant fibers and slow manifolds. The extended inner expansion describes not only the initial evolution of points on the fiber $F^\varepsilon_{z(0)}$, but also the motion in the corner when the points on the fibers are already very close to the manifold $\mathcal{M}_\varepsilon$. Similarly, the extended outer expansion now describes the evolution of the basepoints on $\mathcal{M}_\varepsilon$ and other points in a neighborhood of $\mathcal{M}_\varepsilon$.

3.6. A nonlinear planar example from enzyme kinetics. In example (3.1), the unperturbed manifold $\mathcal{M}_0$ and the specific perturbed manifold we labeled $\mathcal{M}_\varepsilon$ coincide by construction. This was convenient for learning the geometry, but it is not typical. Generally, in systems of the form (1.1), even in planar ones, the unperturbed and perturbed manifolds do not coincide. We illustrate the more general phenomena and a technique to locate slow manifolds on the nonlinear model (3.2) from enzyme kinetics.

This model arises by applying the law of mass action to the system of two reactions in which a substrate and enzyme react reversibly to produce a substrate-enzyme complex, and in turn this complex gets converted irreversibly into a product, releasing the enzyme in the process. We have followed the exposition in Chapter 2 of O'Malley [1991], and we refer the reader there for the derivation and for a complete analysis of how the method of matched asymptotic expansions is used to solve the initial value problem with $z(0) = 1$ and $x(0) = 0$.

The example (3.2) is a special case of the general system (1.1) with $f(x, z, \varepsilon) = z - (z + \kappa)x$, and $g(x, z, \varepsilon) = -z + (z + \kappa - \lambda)x$. When $\varepsilon = 0$, the system has a normally hyperbolic invariant manifold

$$\mathcal{M}_0 \equiv \left\{ (x, z, \varepsilon) | x = X_0(z) = \frac{z}{z + \kappa} \right\}.$$

(We are recycling the notation $\mathcal{M}_0$ to label the $\varepsilon = 0$ invariant manifolds in each example.) In terms of the slow time variable, the dynamics on $\mathcal{M}_0$ is governed by the reduced equation $\dot{z} = -\lambda z/(z + \kappa)$. Moreover, locally, the fast stable fibers F^0_z are horizontal lines. For every z, there is exponential contraction along the fiber F^0_z toward $x = X_0(z)$ as $t \to \infty$.

Next, we turn to the geometry when $0 < \varepsilon \ll 1$. The phase space with $\varepsilon = 0.2$, $\kappa = 1.0$, and $\lambda = 0.5$, obtained from numerical simulations, is shown in Figure 4. By calculating the rate of change of the function $X_0(z) = z/(z + \kappa)$ in $t-$time using the chain rule and the second component of the vector field for the rate of change of z, one sees that $x' = \mathcal{O}(\varepsilon)$, and not zero, at points on $\mathcal{M}_0$. Hence, the set $\mathcal{M}_0$ is no longer invariant in the system when $0 < \varepsilon \ll 1$. Nevertheless, we know from our first example (3.1) that a normally hyperbolic invariant manifold is a robust structure that should persist under perturbation. After all, trajectories still experience fast exponential contraction in the normal directions, while the growth rate in the tangential direction is weak. In fact, we expect there to be an entire family of slow manifolds in this example when $0 < \varepsilon \ll 1$, just as we saw for the linear system (3.1) in subsection 3.3.

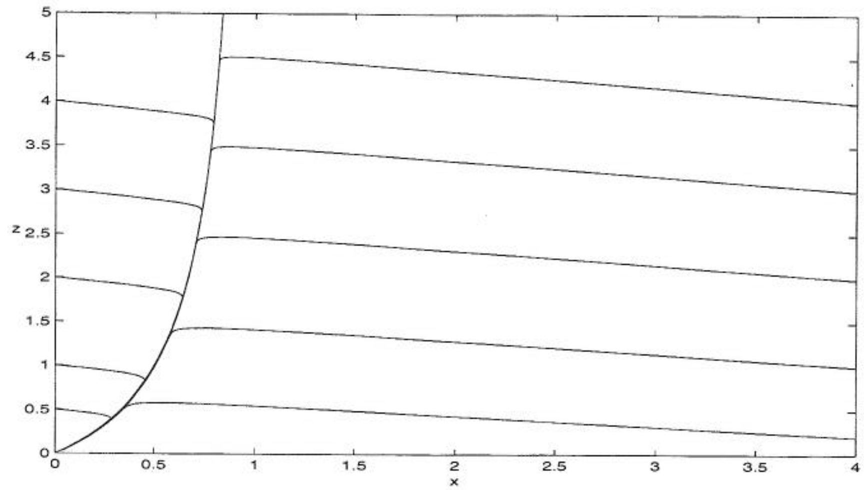

FIGURE 4. Phase plane of the system (3.2) with $\varepsilon = 0.2$, $\kappa = 1.0$, and $\lambda = 0.5$, with the manifold $\mathcal{M}_\varepsilon$ given by $X(z,\varepsilon)$ superimposed.

So, we ask if, near the origin, there are functions $X(z,\varepsilon)$ whose graphs are locally invariant slow manifolds of the full system. Also, since it will turn out that such manifolds $\mathcal{M}_\varepsilon$ will lie close to $\mathcal{M}_0$, their $\varepsilon = 0$ counterpart, we expand these functions in power series in ε about the unperturbed function X_0:

$$X(z,\varepsilon) = X_0(z) + \varepsilon X_1(z) + \mathcal{O}(\varepsilon^2). \tag{3.6}$$

Local invariance of the manifold implies that the rate of change in time of our functions $X(z,\varepsilon)$ is determined self-consistently by the system (3.2). First, direct substitution into the first component of the system (3.2) yields:

$$\frac{d}{dt}X(z,\varepsilon) = f(X(z,\varepsilon), z, \varepsilon).$$

Second, application of the chain rule and a substitution involving the second component of the system show:

$$\frac{d}{dt}X(z,\varepsilon) = \frac{\partial X}{\partial z}(z,\varepsilon)\cdot z' = \varepsilon\frac{\partial X}{\partial z}(z,\varepsilon)\cdot g(X(z,\varepsilon), z, \varepsilon).$$

Then, invariance of the manifold means that these two expressions are equal:

$$\varepsilon\frac{\partial X}{\partial z}(z,\varepsilon)\cdot g(X(z,\varepsilon), z, \varepsilon) = f(X(z,\varepsilon), z, \varepsilon), \tag{3.7}$$

and hence we have the equation that determines $X(z,\varepsilon)$.

Plugging the expansion (3.6) into (3.7) and equating terms with like powers of ε yields the following results. At $\mathcal{O}(1)$, one finds $X_0(z) = z/(z+\kappa)$, as was already known. Then, at $\mathcal{O}(\varepsilon)$, the equation is:

$$\frac{\partial X_0}{\partial z}(z)g(X_0(z), z, 0) = \frac{\partial f}{\partial x}(X_0(z), z, 0)X_1(z) + \frac{\partial f}{\partial \varepsilon}(X_0(z), z, 0).$$

Hence, we find $X_1(z) = \lambda\kappa z/(z+\kappa)^4$. One may continue to obtain ever more higher order terms. We stop here with:

$$X(z,\varepsilon) = \frac{z}{z+\kappa} + \varepsilon\frac{\lambda\kappa z}{(z+\kappa)^4} + \mathcal{O}(\varepsilon^2).$$

All of the slow manifolds $\mathcal{M}_\varepsilon$ are given by the same expansion up to order ε^n for all n. These expansions, and hence the slow manifolds, differ only in the exponentially small terms.

In addition, each fiber in the family of fast stable fibers persists. Locally near $\mathcal{M}_\varepsilon$, the individual fibers F_z^ε are C^r $\mathcal{O}(\varepsilon)$ close for any $r < \infty$ to their unperturbed counterparts F_z^0. The solution along these fibers is given to leading order by $x(t) = x(0)e^{-(z+\kappa)t} + \frac{z}{z+\kappa}\left(1 - e^{-(z+\kappa)t}\right)$. This agrees with the leading order inner (fast layer) solution.

We conclude this section by observing that both Theorem 1 (the Fenichel Theorem for compact manifolds with boundary) stated in Section 5 and Tikhonov-Levinson theory cited in the Introduction guarantee the existence of perturbed invariant slow manifolds $\mathcal{M}_\varepsilon$, and hence justify the formal results stated for this example. See also Nipp [1985]. Finally, we note that Fraser [1987] develops iterative methods for finding lower dimensional invariant manifolds in enzyme reaction problems, and that Morgan [1997] establishes the connection between this iterative method and the singular perturbation theory.

4. Examples with both fast contracting and fast expanding directions

4.1. Geometry of a simple linear system when $\varepsilon = 0$. Let us augment the example (3.1) to include a linear equation for an exponentially growing variable:

$$\begin{aligned} x_1{}' &= x_1 \\ x_2{}' &= -x_2 \\ z' &= -\varepsilon z. \end{aligned} \tag{4.1}$$

Here $m = 2$ and $n = 1$, and this vector field is also decoupled by design. In fact, the phase space of this augmented system is obtained by crossing that of the system (3.1) in the (x_2, z) plane with the one-dimensional phase line of the appended equation $x_1' = x_1$. Let ϕ_t denote the flow of (4.1); it maps points $(x_1(t_0), x_2(t_0), z(t_0))$ to their images $(x_1(t_0+t), x_2(t_0+t), z(t_0+t))$ after time t.

When $\varepsilon = 0$, the $z-$axis is an invariant manifold of (4.1): it consists purely of fixed points. See Figure 5. Moreover, the fast subsystem is $f_1(x_1, x_2, z, 0) = x_1$ and $f_2(x_1, x_2, z, 0) = -x_2$; and, for each z, the Jacobian of $\mathbf{f}$ at the fixed point $(x_1 = 0, x_2 = 0)$ has eigenvalues -1 and $+1$; *i.e.*, neither eigenvalue has zero real part. Hence, $\mathcal{M}_0$ is also normally hyperbolic by construction.

The (x_1, z) plane is the set of points that approach $\mathcal{M}_0$ backward in time at an exponential rate. Similarly, the (x_2, z) plane is precisely the set of points that approach $\mathcal{M}_0$ forward in time at an exponential rate. Therefore, the (x_1, z) plane is the unstable manifold of $\mathcal{M}_0$, and the (x_2, z) plane is the stable manifold of $\mathcal{M}_0$. They are labeled by $W^U(\mathcal{M}_0)$ and $W^S(\mathcal{M}_0)$, respectively. Let $F_z^{0,U}$ and $F_z^{0,S}$ denote the lines $\{(x_1, x_2, z) | x_2 = 0\}$ (*i.e.*, the x_1-axes for all z) and $\{(x_1, x_2, z) | x_1 = 0\}$ (*i.e.*, the x_2-axes for all z), respectively. These lines $F_z^{0,U}$ and $F_z^{0,S}$ are fast unstable and stable fibers over $\mathcal{M}_0$, and the unions of these fibers over all z are the manifolds $W^U(\mathcal{M}_0)$ and $W^S(\mathcal{M}_0)$, respectively. See Figure 5.

The remainder of this subsection is devoted to analyzing the geometry relevant to all orbits off of the invariant manifold $\mathcal{M}_0$ and its stable and unstable manifolds $W^S(\mathcal{M}_0)$ and $W^U(\mathcal{M}_0)$. In the three-dimensional (x_1, x_2, z) phase space, each plane $z =$ constant is an invariant set of (4.1) with $\varepsilon = 0$. We label these planes by Π_z. The dynamics on Π_z is given for every z by the fast system: $x_1' = x_1$, $x_2' = -x_2$, see Figure 6. In the two-dimensional (x_1, x_2) plane, the family of vertical lines $x_1 =$

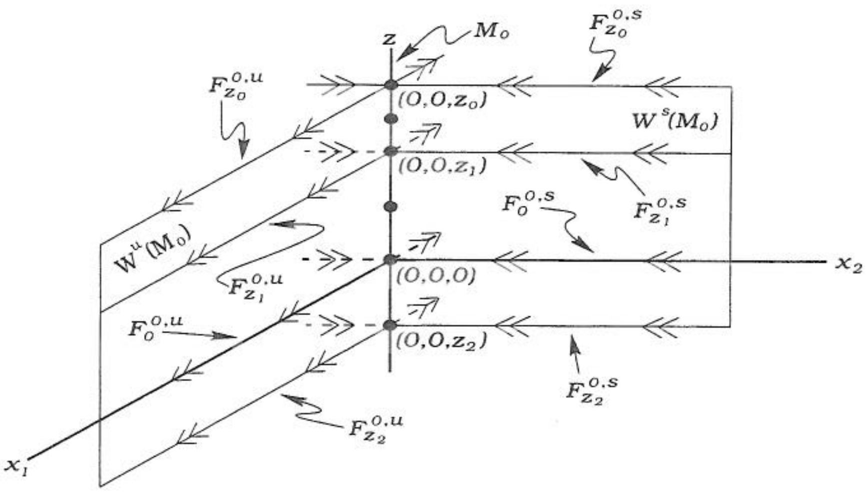

FIGURE 5. The geometry of the example (4.1) with $\varepsilon = 0$.

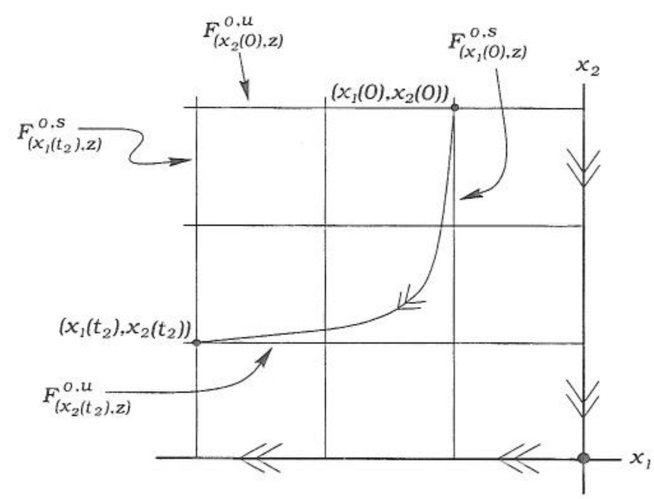

FIGURE 6. Sketch of the invariant fibers on Π_z for the system (4.1) with $\varepsilon = 0$ and an illustration of the decomposition of a representative trajectory.

constant and the family of horizontal lines $x_2 =$ constant share a special property in this example. In a given amount of fast time t_2, all of the points on the vertical line $x_1 = x_1(0)$ flow to the vertical line $x_1 = x_1(t_2)$; and, similarly all of the points on the horizontal line $x_2 = x_2(0)$ flow to the horizontal line $x_2 = x_2(t_2)$. Thus, both are invariant families of lines in (4.1).

As in Section 3, we are now in a good position to introduce the appropriate geometric terminology for these lines. The jump from two dimensions (example 1) to three dimensions (here) reveals new features important for understanding general systems. On Π_z, each horizontal line of constant x_2 is a fast unstable fiber, denoted $F^{0,u}_{(x_2,z)}$. The basepoint of $F^{0,u}_{(x_2,z)}$ is the point $(x_1 = 0, x_2)$ on the x_2-axis.

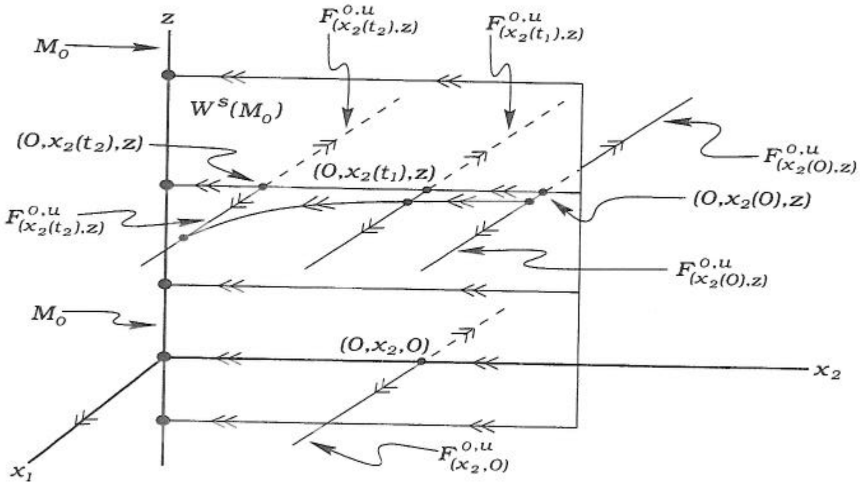

FIGURE 7. Sketch of the invariant fibers, manifolds, and a representative trajectory for the system (4.1) with $\varepsilon = 0$ in the full three-dimensional phase space.

This basepoint evolves according to the contracting fast component $x_2' = -x_2$. For example, for the trajectory sketched in Figures 6 and 7, the point $(0, x_2(0))$ is the basepoint of the fiber $F^{0,u}_{(x_2(0),z)}$ on which our initial condition lies, and the image of this point after fast time t_2 is $(0, x_2(t_2))$. Moreover, the fiber $F^{0,u}_{(x_2(t_2),z)}$ is precisely the image of the initial fiber $F^{0,u}_{(x_2(0),z)}$, and $(0, x_2(t_2))$ is its basepoint; *i.e.*,

$$F^{0,u}_{(x_2(t_2),z)} = \phi_{t_2} F^{0,u}_{(x_2(0),z)}.$$

In this way, the evolution of any initial condition in Π_z is decomposed into two components: one corresponding to exponential expansion along the unstable fibers and the second corresponding to the motion - exponential contraction in the x_2−direction - of the basepoints of the fibers. See Figures 6 and 7. Moreover, taking the union over all z of the fibers $F^{0,u}_{(x_2,z)}$, one obtains a family $\mathcal{F}^{0,u}$ that is invariant with respect to (4.1) and that is transverse (even normal) to $W^S(\mathcal{M}_0)$. Each point on $W^S(\mathcal{M}_0)$ is the basepoint of a fiber from this family, and conversely all basepoints lie on $W^S(\mathcal{M}_0)$. This family foliates the three-dimensional space.

An analogous geometric interpretation may be given for the invariant vertical lines (stable fibers). Briefly, each vertical line defined by constant x_1 is a fast stable fiber $F^{0,s}_{(x_1,z)}$. Its basepoint $(x_1, 0, z)$ evolves according to the fast expanding component $x_1' = x_1$, and one has the desired invariance property

$$F^{0,s}_{(x_1(t_2),z)} = \phi_{t_2} F^{0,s}_{(x_1(0),z)}.$$

The union of the $F^{0,s}_{(x_1,z)}$ over all x_1 and z is a family $\mathcal{F}^{0,s}$ of stable fibers is invariant with respect to (4.1) when $\varepsilon = 0$. All basepoints lie on $W^U(\mathcal{M}_0)$, and conversely each point on $W^U(\mathcal{M}_0)$ is the basepoint of a fiber from this family.

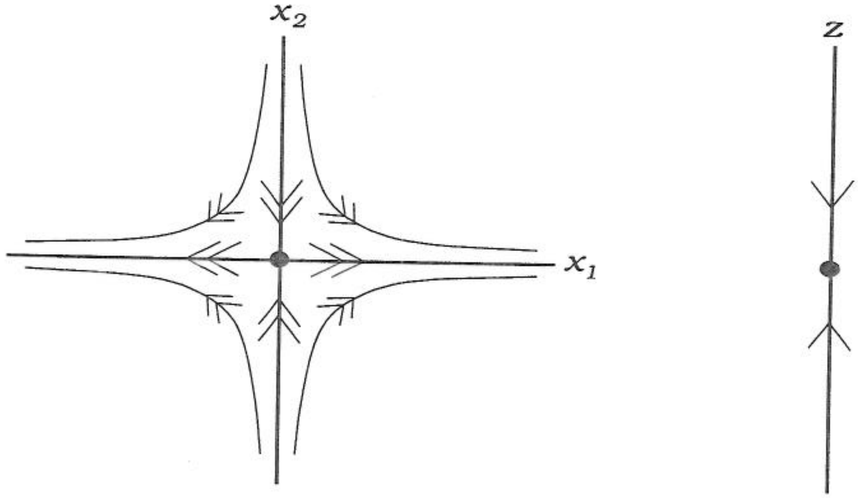

FIGURE 8. The phase space of the full system (4.1) with $0 < \varepsilon \ll 1$ is obtained by crossing those of the fast and slow subsystems.

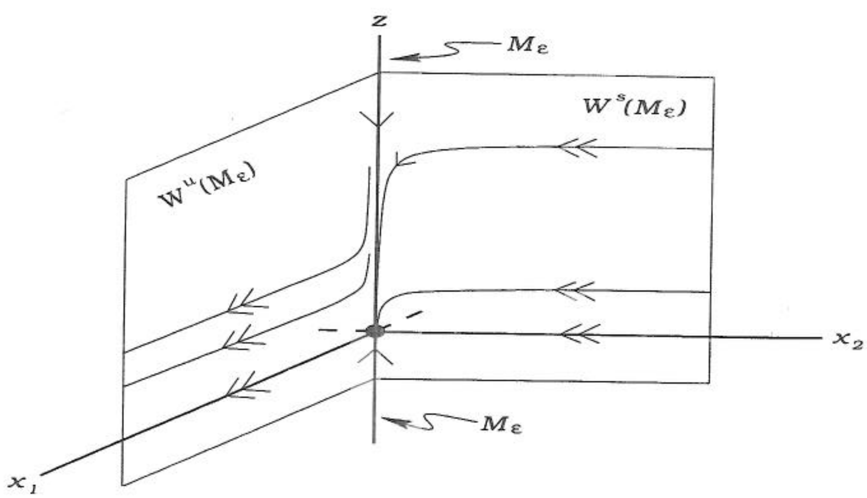

FIGURE 9. A slow manifold $\mathcal{M}_\varepsilon$ and its stable and unstable manifolds of the perturbed system (4.1).

4.2. Geometry of the example (4.1) when $0 < \varepsilon \ll 1$**.** By construction of this example, the phase space may be obtained directly by crossing that of the fast and slow subsystems, see Figure 8.

The geometric structures present when $\varepsilon = 0$ persist in the full system (4.1) when $0 < \varepsilon \ll 1$. The $z-$axis is still an invariant manifold, and we denote it by $\mathcal{M}_\varepsilon$. Initial conditions on $\mathcal{M}_\varepsilon$ remain on it in both forward and backward time, and they contract at a weak exponential rate toward the origin as $t \to \infty$. See Figure 9. Also, $\mathcal{M}_\varepsilon$ is but one member of a continuous family of center manifolds, for the same reason as discussed in subsection 3.3.

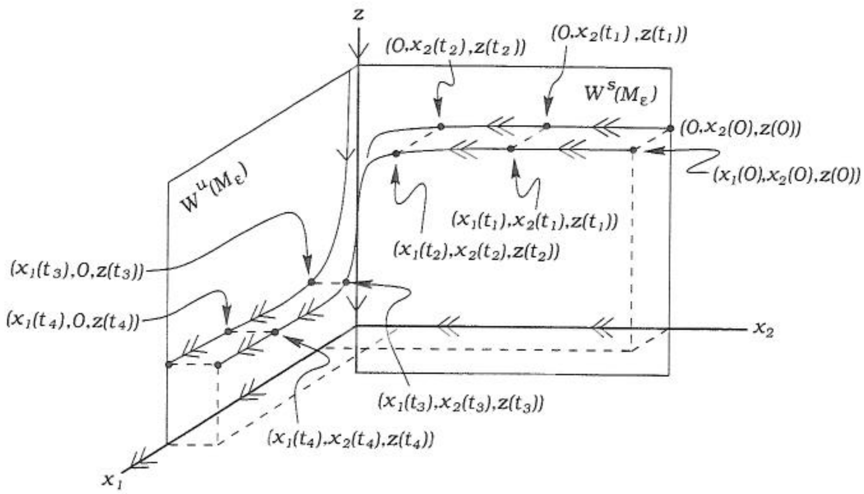

FIGURE 10. An illustration of the decomposition of a general trajectory into components along the fast fibers and slow manifold in the example (4.1). The times t_3 and t_4 are $\mathcal{O}(1/\varepsilon)$.

The (x_2, z) plane and the (x_1, z) plane are now the perturbed stable and unstable manifolds, $W^S(\mathcal{M}_\varepsilon)$ and $W^U(\mathcal{M}_\varepsilon)$, respectively. Each horizontal line of constant z in the (x_2, z) plane (or (x_1, z) plane) is a persistent fast stable (or unstable) fiber $F_z^{\varepsilon,s}$ ($F_z^{\varepsilon,u}$, respectively), and its base point is the point $(0,0,z)$ on $\mathcal{M}_\varepsilon$. The individual stable (or unstable) fibers are no longer invariant, as they were when $\varepsilon = 0$; rather, they are invariant as a family. In addition, because there is one fiber of each type for each z, we say that these families foliate $W^S(\mathcal{M}_\varepsilon)$ and $W^U(\mathcal{M}_\varepsilon)$, respectively. Also, in this example, as in general, the existence of the perturbed slow manifolds $\mathcal{M}_\varepsilon$ may be viewed as a consequence of the persistence and transverse intersection of the local stable and unstable manifolds.

The fibers not in the (x_2, z) and (x_1, z) planes also persist. In the three-dimensional phase space, each horizontal line parallel to the x_2−axis is again a fast stable fiber $F^{\varepsilon,s}_{(x_1,z)}$ with basepoint $(x_1, 0, z)$ on $W^U(\mathcal{M}_\varepsilon)$, and similarly, each horizontal line parallel to the x_1−axis is again a fast unstable fiber $F^{\varepsilon,u}_{(x_2,z)}$ with basepoint $(0, x_2, z)$ on $W^S(\mathcal{M}_\varepsilon)$. See Figure 10. These fibers again form two invariant families:

$$\mathcal{F}^{\varepsilon,u} \equiv \bigcup_{(x_2,z_0)} F^{\varepsilon,u}_{(x_2,z_0)} \quad \text{and} \quad \mathcal{F}^{\varepsilon,s} \equiv \bigcup_{(x_1,z_0)} F^{\varepsilon,s}_{(x_1,z_0)},$$

For completeness, we note that the stable and unstable manifolds of $\mathcal{M}_\varepsilon$ may now also be expressed as: $\mathcal{W}^S(\mathcal{M}_\varepsilon) \equiv \bigcup_{z\in\mathcal{M}_\varepsilon} F^{\varepsilon,s}_{0,z}$ and $\mathcal{W}^U(\mathcal{M}_\varepsilon) \equiv \bigcup_{z\in\mathcal{M}_\varepsilon} F^{\varepsilon,u}_{0,z}$.

The evolution of general initial conditions *not* on $\mathcal{M}_\varepsilon$ or on $W^S(\mathcal{M}_\varepsilon)$ and $W^U(\mathcal{M}_\varepsilon)$ can be decomposed as follows. See Figure 10. Fix arbitrary, nonzero, $\mathcal{O}(1)$ values of $x_1(0)$, $x_2(0)$, and $z(0)$. This initial condition lies on both of the fibers $F^{\varepsilon,u}_{(x_2(0),z(0))}$ and $F^{\varepsilon,s}_{(x_1(0),z(0))}$. The orbit through this point evolves so that at any time τ, the point $(x_1(\tau), x_2(\tau), z(\tau))$ lies on the unstable fiber $F^{\varepsilon,u}_{\phi_\tau(x_2(0),z(0))}$ and on the stable fiber $F^{\varepsilon,s}_{\phi_\tau(x_1(0),z(0))}$. In addition, it moves inward toward $\mathcal{M}_\varepsilon$ at the exponential rate -1 along the fibers of the family $\mathcal{F}^{\varepsilon,s}$ and outward away from

$\mathcal{M}_\varepsilon$ at the exponential rate $+1$ along the fibers of the family $\mathcal{F}^{\varepsilon,u}$, since its x_1 and x_2 components are exponentially increasing and decreasing, respectively. Its slow component evolves according to the motion of the fiber's basepoint $(0,0,z)$. Therefore, within the context of this example, the system dynamics can be naturally decomposed into fast components in which the dynamics is governed by the exponential rates of growth and decay along the fast stable and unstable fibers and into a slow component governed by the motion of the basepoints of fibers along the slow (or center) manifold.

We conclude this subsection by observing that the results from the linear example (4.1) extend in a straightforward fashion to:

$$x_1' = \lambda(z)x_1, \quad x_2' = \gamma(z)x_2, \quad z' = -\varepsilon z, \tag{4.2}$$

where $\lambda(z)$ and $\gamma(z)$ are C^∞, $\lambda(z) > C_1 > 0$ for all z, and $\gamma(z) < -C_2 < 0$ for all z. The geometric structures, including $\mathcal{M}_\varepsilon$ and its stable and unstable manifolds, are the same. However, the rates of exponential contraction and expansion in the directions normal to a slow manifold now vary with z.

4.3. Invariant manifolds in a nonlinear third-order system. In this subsection, we analyze aspects of a third-order system that arises as a model for sloshing of shallow water in a tank shaken at a near-resonant frequency for surface waves, see Hastings and McLeod [1991]. The equations reduce to a planar Hamiltonian system with cubic potential dependent on a slowly-varying parameter:

$$\begin{aligned} q' &= p \\ p' &= q^2 - F(z) \\ z' &= \varepsilon, \end{aligned} \tag{4.3}$$

where $F(z) \equiv 1+c+\cos z$, $c > 0$, and the Hamiltonian is $H(q,p,z) = p^2/2 - q^3/3 + F(z)q - (2/3)F^{3/2}(z)$.

The system (4.3) is an example of the general fast-slow systems (1.1) with $m = 2$, $\mathbf{x} = (q,p)$, and $n = 1$ with $z \in S^1$. When $\varepsilon = 0$, $z \in [0, 2\pi)$ is a fixed parameter, and for every z the system has a hyperbolic fixed point at $(q = \sqrt{F(z)}, p = 0)$ which is of saddle type, since the eigenvalues of the linearized system are a pair of positive and negative real numbers: $\pm\sqrt{2}F^{1/4}(z)$. Also, the factor of $-(2/3)F^{3/2}(z)$ in H implies that $H = 0$ at the hyperbolic fixed point for each z. This hyperbolic fixed point is attached to itself by a homoclinic orbit (an orbit that approaches the fixed point both as $t \to \infty$ and as $t \to -\infty$) that lies in the half plane $q \leq \sqrt{F(z)}$. This orbit is commonly referred to as a separatrix, because it divides the family of periodic orbits inside it from the family of unbounded orbits outside, and we note that $H \equiv 0$ along it. There are also unbounded curves in the half plane $q \geq \sqrt{F(z)}$ that consist of all initial conditions that approach (for $p < 0$) or depart from (for $p > 0$) the saddle fixed point. These curves also lie on the $H = 0$ level set, and they may also be referred to as separatrices.

Let us now consider the manifold $\mathcal{M}_0$ obtained by taking the union over z of these hyperbolic fixed points:

$$\mathcal{M}_0 \equiv \bigcup_{z\in[0,2\pi)} \left(\sqrt{F(z)}, 0\right).$$

By definition, $\mathcal{M}_0$ is a normally hyperbolic invariant manifold. Moreover, $\mathcal{M}_0$ is topologically a circle, and hence compact and boundaryless.

For each z, the restriction of the one-dimensional separatrix ($H = 0$ level set) to a neighborhood of the saddle fixed point yields a local stable fiber (the local stable manifold of the saddle) and a unstable local fiber (the local unstable manifold of the saddle). For a fixed z, the set of initial conditions on the $H = 0$ level set in the quadrants $q < \sqrt{F(z)}, p > 0$, and $q > \sqrt{F(z)}, p < 0$, together for example, make up the local stable manifold of the saddle. These initial conditions all approach $(\sqrt{F(z)}, 0)$ exponentially in time, and they do so tangentially to the stable subspace of the linearized system, as may be verified using l'Hopital's rule. They constitute a stable fiber, whose base point is the saddle $(\sqrt{F(z)}, 0, z)$. In this way, each point on $\mathcal{M}_0$ is the basepoint of both a one-dimensional stable fiber $F_z^{0,s}$ and a one-dimensional unstable fiber $F_z^{0,u}$.

The unions over $z \in [0, 2\pi)$ of the local stable and unstable fibers (or equivalently of the local stable and unstable manifolds of the saddles) then give precisely the local stable and unstable manifolds of $\mathcal{M}_0$:

$$W^S_{\text{loc}}(\mathcal{M}_0) = \bigcup_{z\in[0,2\pi)} F_z^{0,s} \qquad \text{and} \qquad W^U_{\text{loc}}(\mathcal{M}_0) = \bigcup_{z\in[0,2\pi)} F_z^{0,u}.$$

Finally, off of $\mathcal{M}_0$, $W^S_{\text{loc}}(\mathcal{M}_0)$ and $W^U_{\text{loc}}(\mathcal{M}_0)$, there are two other families of fibers that are relevant. In particular, each point on $W^S_{\text{loc}}(\mathcal{M}_0)$ ($W^U_{\text{loc}}(\mathcal{M}_0)$, resp.) is the basepoint of a one-dimensional fast unstable (stable) fiber. These stable (and unstable) fibers form invariant, two-parameter families of fibers that each foliate a neighborhood of $\mathcal{M}_0$.

The perturbed system, (4.3) with $0 < \varepsilon \ll 1$, has an invariant slow manifold $\mathcal{M}_\varepsilon$ that is C^r $\mathcal{O}(\varepsilon)$ close to $\mathcal{M}_0$ for all $r < \infty$ and for all z. This manifold $\mathcal{M}_\varepsilon$ is the graph of a function $\mathbf{X}(z, \varepsilon)$ that is $\mathcal{O}(\varepsilon)$ close to $(\sqrt{F(z)}, 0)$ for every z. A slight generalization of the procedure used in subsection 3.6 (see also subsection 5.5) leads to the asymptotic expansion for this function:

$$\mathbf{X}(z,\varepsilon) = \begin{pmatrix} \sqrt{F(z)} \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ (1/2)F^{-1/2}(z) \end{pmatrix} + \mathcal{O}(\varepsilon^2).$$

Moreover, $\mathcal{M}_\varepsilon$ is nonunique, since on any given compact z−interval there is a family of slow manifolds, which all have the same expansion in powers of ε, but differ by exponentially small terms. We remark that, since $F(z)$ is periodic, the above results for a fixed ε may also be obtained from the Poincaré map of (4.3).

Each point on $\mathcal{M}_\varepsilon$ is the basepoint of a persistent fast stable fiber $F_z^{\varepsilon,s}$ and of a persistent fast unstable fiber $F_z^{\varepsilon,u}$. The unions of these fibers over $z \in [0, 2\pi)$ are the two-dimensional, local stable and unstable manifolds of $\mathcal{M}_\varepsilon$, $W^S_{\text{loc}}(\mathcal{M}_\varepsilon)$ and $W^U_{\text{loc}}(\mathcal{M}_\varepsilon)$, respectively. In addition, each of the points on the stable (resp. unstable) manifold is a basepoint for a one-dimensional fast unstable (resp. stable) fiber, and these form two parameter invariant families of the type $\mathcal{F}^{\varepsilon,u}$ and $\mathcal{F}^{\varepsilon,s}$, also found for the previous example, except here each fiber is nonlinear. Therefore, the dynamics of all orbits in a neighborhood of $\mathcal{M}_\varepsilon$ can be decomposed into motion along $\mathcal{M}_\varepsilon$ and motion in the normal directions.

REMARK 4.1. The slowly-varying pendulum example (2.5) has essentially the same types of geometric structures as example (4.3). It is also slightly simpler in that there is one distinguished center manifold, in the family of center manifolds that are all exponentially close to each other on compact intervals, for which $\mathbf{X}(z, \varepsilon) = \mathbf{X}_0(z)$ for all z and ε. This is because the orbits $(\pm\pi, 0, z)$ for all real z

are periodic orbits of (2.5). Moreover, they are of saddle type with Floquet exponents equal to $0, \alpha, \beta$, where $\alpha < 0$ and $\beta > 0$. Some further examples of slowly-varying planar Hamiltonian systems may be found, for example, in Angenent, *et al.* [1987], Bruhwiler and Kaper [1995], Hastings and McLeod [1991], Kaper and Kovačič [1994], Kath [1985], and Robinson [1983].

5. Fenichel theory for singularly perturbed systems (1.1)

Having introduced the various geometric constructs and concepts that we study in the sequence of elementary low-dimensional examples in Sections 3 and 4, we are now in a good position to state the Fenichel geometric theory for the general systems (1.1). We state the theorem for compact manifolds with boundary first, and then briefly also state the corresponding theorem for boundaryless compact manifolds. Historically, as one may see for example from the presentation of the persistence theory in Fenichel [1971], the theory was developed in this order also, first for compact manifolds with boundary and then for boundaryless compact manifolds. We follow the presentation in Jones [1994] in this section and strongly encourage the reader to study Jones [1994] for the proof of the theorem below.

5.1. Normally hyperbolic manifolds in the unperturbed system (1.1). We make the following assumptions about (1.1):

(H1) The functions $\mathbf{f}$ and $\mathbf{g}$ are C^∞ in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ and I is an open interval containing $\varepsilon = 0$.

(H2) There exists a set $\mathcal{M}_0$ that is contained in $\{(\mathbf{x}, \mathbf{z}) : \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) = \mathbf{0}\}$ such that $\mathcal{M}_0$ is a compact manifold with boundary and $\mathcal{M}_0$ is given by the graph of a C^∞ function $\mathbf{x} = \mathbf{X}_0(\mathbf{z})$ for $\mathbf{z} \in D$, where $D \subseteq \mathbb{R}^n$ is a compact, simply connected domain and the boundary of D is an $(n-1)$ dimensional C^∞ submanifold. Finally, the set D is overflowing invariant with respect to (1.2) when $\varepsilon = 0$.

Here we note that the definition given in Section 2.1 of a manifold with boundary entails that the boundary is $(n-1)$ dimensional and C^∞ smooth, and we recall that a compact manifold with boundary is said to be *overflowing invariant* when (i) at every point inside the manifold the vector field is tangent to the manifold, and (ii) at every point on the boundary of the manifold the vector field points outward. So, we need $\mathcal{M}_0$ to be overflowing invariant with respect to the reduced slow system: $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0)$.

The final (and central) hypothesis we need is:

(H3) $\mathcal{M}_0$ is normally hyperbolic relative to (1.3), and in particular we require that, for all points $\mathbf{p} \in \mathcal{M}_0$, there are k (resp., ℓ) eigenvalues of $D_\mathbf{x}\mathbf{f}(\mathbf{p}, 0)$ with positive (resp., negative) real parts bounded away from zero, where $k + \ell = m$.

5.2. Statement of the Fenichel persistence theory for (1.1). For compact manifolds with boundary, Fenichel's persistence theory states that, provided the hypotheses (H1)-(H3) are satisfied, (1.1) possesses a slow (or center) manifold, and this slow manifold has fast stable and unstable manifolds. See Figure 11.

THEOREM 1 (Fenichel Thm. for compact manifolds with boundary). *Let system (1.1) satisfy the conditions (H1) - (H3). If $\varepsilon > 0$ is sufficiently small, then there exists a function $\mathbf{X}(\mathbf{z}, \varepsilon)$ defined on D such that the manifold*

$$\mathcal{M}_\varepsilon = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} = \mathbf{X}(\mathbf{z}, \varepsilon)\}$$

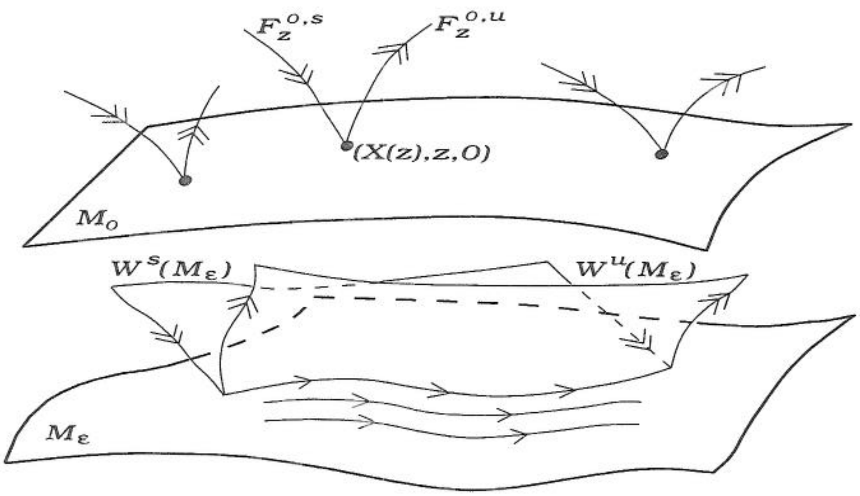

FIGURE 11. Sketches of the invariant manifolds present in the general system (1.1) when $\varepsilon = 0$ (top) and when $0 < \varepsilon \ll 1$ (bottom).

is locally invariant under (1.1). Moreover, $\mathbf{X}(\mathbf{z}, \varepsilon)$ is C^r for any $r < +\infty$, and $\mathcal{M}_\varepsilon$ is C^r $\mathcal{O}(\varepsilon)$ close to $\mathcal{M}_0$. In addition, there exist perturbed local stable and unstable manifolds of $\mathcal{M}_\varepsilon$. They are the unions of invariant families of fast stable and unstable fibers of dimensions ℓ and k, respectively, and they are C^r $\mathcal{O}(\varepsilon)$ close, for all $r < \infty$, to their unperturbed counterparts.

The label slow manifold is attached to $\mathcal{M}_\varepsilon$ because the magnitude of the vector field restricted to $\mathcal{M}_\varepsilon$ is $\mathcal{O}(\varepsilon)$, in terms of the fast independent variable t, as we have seen in the examples. Also, the function $\mathbf{X}(\mathbf{z}, \varepsilon)$ is generally nonunique, as we already noted in subsection 3.3. There generally exists a family of functions $\mathbf{X}(\mathbf{z}, \varepsilon)$ (and hence slow manifolds), and they are exponentially close in ε to each other on compact sets, as was illustrated on the examples in Sections 3 and 4.

In this case, when $\mathcal{M}_0$ has a boundary, $\mathcal{M}_\varepsilon$ is in general only locally invariant because trajectories on it may cross the boundary, and possibly even reenter the domain. In these cases, the local fast stable and unstable fibers, and hence also the local stable and unstable manifolds, are only defined as long as the basepoints stay inside the boundary. See also Remark 5.1.

The individual stable and unstable fibers with basepoints on $\mathcal{M}_\varepsilon$ have dimension ℓ and k. Hence, the stable and unstable manifolds $W^S_{\text{loc}}(\mathcal{M}_\varepsilon)$ and $W^U_{\text{loc}}(\mathcal{M}_\varepsilon)$ have dimension $\ell + n$ and $k + n$, respectively.

REMARK 5.1. In applications for which the compact manifold $\mathcal{M}_0$ with boundary is not overflowing invariant with respect to the reduced slow system $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0)$, a standard technique can be used to make the manifold overflowing invariant and hence facilitate the application of the Fenichel theory. We follow the presentation in Section 2.2 of Jones [1994], introducing a sufficiently smooth bump function as follows. First, one enlarges the set D. In particular, since D is compact, and since $\mathcal{M}_0$ is given by the C^∞ function $\mathbf{X}_0$ on D, a set $\hat{D}$ can be found such that $D \subset \text{int}\hat{D}$ and such that $\mathbf{X}_0$ is also defined and C^∞ on $\hat{D}$. The set $\hat{\mathcal{M}}_0 = \{(\mathbf{x}, \mathbf{z}) | \mathbf{x} = \mathbf{X}_0(\mathbf{z}), \mathbf{z} \in \hat{D}\}$ is a set of critical points for the vector field with $\varepsilon = 0$, and one can choose $\hat{D}$ such that $\hat{\mathcal{M}}_0$ is normally hyperbolic. Furthermore, the set $\hat{D}$ can be chosen so that the boundary of $\hat{D}$ is given by the condition $\hat{\nu}(\mathbf{z}) = 0$, for some function $\hat{\nu}$ that is C^∞, satisfies $\nabla\hat{\nu}(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \hat{D}$, and is normalized so that $\nabla\hat{\nu}(\mathbf{z}) = n_{\mathbf{z}}$ is an outward unit normal for $\partial\hat{D}$. Next, let $\rho(\mathbf{z})$ be a C^∞ function such that

$$(5.1) \qquad \begin{aligned} \rho(\mathbf{z}) &= 1 \text{ if } \mathbf{z} \in \hat{D}^c \\ \rho(\mathbf{z}) &= 0 \text{ if } \mathbf{z} \in D, \end{aligned}$$

where the superscript c denotes complement of the set, and where the function ρ may be specified as needed in the domain $D^c \cap (\text{int}\hat{D})$. As noted in Jones [1994], such a function $\rho(\mathbf{z})$ can always be constructed using a local C^∞ bump function and then a partition of unity to generate the full function ρ. Finally, one appends the function $\delta\rho(\mathbf{z})n_{\mathbf{z}}$ to the $\mathbf{z}$ component of the vector field, where δ is a sufficiently small number chosen so that the vector field points outward on the boundary.

The addition of the bump function to the vector field leaves the vector field unaltered inside D, which is the domain in which one is interested in. It only modifies the vector field outside of D. Of course, the manifold $\mathcal{M}_\varepsilon$ obtained in the proof will depend on the choice of bump function, but since each persistent slow manifold $\mathcal{M}_\varepsilon$ is exponentially close in ε to every other persistent slow manifold, this is of no consequence in applications where the algebraic terms $\varepsilon, \varepsilon^2, \ldots$ in the expansion of $\mathbf{X}(\mathbf{z}, \varepsilon)$ are needed. This technical modification has been done where needed in all of the references cited here.

We conclude this subsection by stating the Fenichel persistence theory relevant to the case in which the zero set of $\mathbf{f}$ gives rise to a manifold $\mathcal{M}_0$ that is compact and boundaryless. We shall need to modify slightly the second hypothesis:

(H2)' There exists a set $\mathcal{M}_0$ that is contained in $\{(\mathbf{x}, \mathbf{z}) : \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) = \mathbf{0}\}$ such that $\mathcal{M}_0$ is a compact boundaryless manifold.

THEOREM 2 (Fenichel Thm: compact boundaryless manifolds). Let (1.1) satisfy the conditions (H1), (H2)' and (H3). If $\varepsilon > 0$ is sufficiently small, then there exists a manifold $\mathcal{M}_\varepsilon$ that is C^r smooth for any $r < \infty$, locally invariant under (1.1), and C^r $\mathcal{O}(\varepsilon)$ close to $\mathcal{M}_0$. In addition, there exist perturbed local stable and unstable manifolds of $\mathcal{M}_\varepsilon$. They are the unions of invariant families of fast stable and unstable fibers of dimensions ℓ and k, respectively, and they are C^r $\mathcal{O}(\varepsilon)$ close, for all $r < \infty$, to their unperturbed counterparts.

REMARK 5.2. In this case, an atlas with a finite number of smoothly overlapping charts $\mathbf{X}_{0,i}$ can be used if necessary to represent the manifold $\mathcal{M}_0$ in order to carry out calculations.

5.3. The Fenichel normal form for the perturbed system. The persistence theory cited above enables us to give an explicit construction of the *Fenichel normal form* of the system (1.1) that is valid in a neighborhood of a persistent slow manifold $\mathcal{M}_\varepsilon$. See also Chapter 3 of Jones [1994].

We begin by making a preliminary transformation of the vector field in (1.1) so that the linear and nonlinear parts of the fast component become explicitly visible. Also, from the structure of the transformed system, we will see that the examples (3.1), (3.2), (4.1), (4.2), and (4.3) studied in Sections 3 and 4 were chosen because they capture the essential features of exponential growth in the directions normal to invariant manifolds and slow growth in the tangential directions.

Assume that the function $\mathbf{X}_0(\mathbf{z}) = \mathbf{0}$ for all $\mathbf{z}$ in D. This may be done without loss of generality because one can always change variables from $\mathbf{x}$ to $\mathbf{x} - \mathbf{X}_0(\mathbf{z})$. Now, for each $\mathbf{z}$, the system obtained by linearizing on $\mathcal{M}_0$ has stable and unstable subspaces (corresponding to the spans of all of the exponentially contracting and expanding eigendirections, respectively) that vary smoothly with $\mathbf{z}$, and whose dimensions are k and ℓ for all $\mathbf{z}$ by (H3). Hence, we can split the fast variable $\mathbf{x}$ into two components: $(\mathbf{x}_1, \mathbf{x}_2)$ corresponding to the bases for these subspaces, where $\mathbf{x}_1 \in \mathbb{R}^k$ and $\mathbf{x}_2 \in \mathbb{R}^\ell$. The equations are then:

$$\begin{aligned} \mathbf{x}_1' &= \Lambda_1(\mathbf{z})\mathbf{x}_1 + \tilde{\mathbf{f}}_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}, \varepsilon) \\ \mathbf{x}_2' &= \Lambda_2(\mathbf{z})\mathbf{x}_2 + \tilde{\mathbf{f}}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}, \varepsilon) \\ \mathbf{z}' &= \varepsilon\mathbf{g}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}, \varepsilon). \end{aligned} \tag{5.2}$$

Here, the eigenvalues of the $k \times k$ matrix $\Lambda_1(\mathbf{z})$ have positive real parts that are uniformly bounded away from zero, while those of the $\ell \times \ell$ matrix $\Lambda_2(\mathbf{z})$ have negative real parts that are uniformly bounded away from zero. These linear terms are a natural generalization of example (4.2). Moreover, the functions $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$ satisfy the bounds:

$$\|\tilde{\mathbf{f}}_i\| = \mathcal{O}(\|\mathbf{x}\|^2, \varepsilon, \varepsilon\|\mathbf{x}\|),$$

as $\|\mathbf{x}\| \to 0$ and $\varepsilon \to 0$, for both $i = 1$ and 2, *i.e.*, they contain only terms that are nonlinear in $\mathbf{x}_1$ and $\mathbf{x}_2$, as well as terms that are higher order in ε.

In terms of the new variables $\mathbf{x}_1$ and $\mathbf{x}_2$ just introduced above in the preliminary transformation of the vector field to (5.2), the functions whose graphs are the local stable and unstable manifolds of $\mathcal{M}_\varepsilon$ can be expressed as: $\mathbf{m}^s(\mathbf{x}_2, \mathbf{z}, \varepsilon)$ and $\mathbf{m}^u(\mathbf{x}_1, \mathbf{z}, \varepsilon)$. The local manifolds are given by

$$\mathbf{x}_1 = \mathbf{m}^s(\mathbf{x}_2, \mathbf{z}, \varepsilon). \qquad \text{and} \qquad \mathbf{x}_2 = \mathbf{m}^u(\mathbf{x}_1, \mathbf{z}, \varepsilon)$$

In turn, this naturally suggests the invertible change of coordinates to:

$$\mathbf{a} \equiv \mathbf{x}_1 - \mathbf{m}^s(\mathbf{x}_2, \mathbf{z}, \varepsilon), \qquad \text{and} \qquad \mathbf{b} \equiv \mathbf{x}_2 - \mathbf{m}^u(\mathbf{x}_1, \mathbf{z}, \varepsilon), \tag{5.3}$$

where $a \in \mathbb{R}^k$, $b \in \mathbb{R}^\ell$, and $k + \ell = m$. The perturbed local stable and unstable manifolds correspond to the sets $\{\mathbf{a} = \mathbf{0}\}$ and $\{\mathbf{b} = \mathbf{0}\}$, respectively. In addition, since these manifolds are invariant, we must have $\mathbf{a}' = \mathbf{0}$ along the set $\{\mathbf{a} = \mathbf{0}\}$ and $\mathbf{b}' = \mathbf{0}$ at all points in the set $\{\mathbf{b} = \mathbf{0}\}$. Finally, by changing the $\mathbf{z}$ coordinate if necessary, the individual ℓ and k dimensional fibers $F_{\mathbf{z}}^{\varepsilon,s}$ and $F_{\mathbf{z}}^{\varepsilon,u}$ with basepoints $\mathbf{z} \in \mathcal{M}_\varepsilon$ can also be straightened out. The resulting $\mathbf{a}, \mathbf{b}, \mathbf{z}$ coordinates have been dubbed Fenichel coordinates, see Jones and Kopell [1994] and Jones, *et al.* [1996].

In a neighborhood of $\mathcal{M}_\varepsilon$, the equations (1.1) may then be rewritten as:

$$\begin{aligned} \mathbf{a}' &= \Lambda(\mathbf{a}, \mathbf{b}, \mathbf{z}, \varepsilon)\mathbf{a} \\ \mathbf{b}' &= \Gamma(\mathbf{a}, \mathbf{b}, \mathbf{z}, \varepsilon)\mathbf{b} \\ \mathbf{z}' &= \varepsilon\mathbf{h}_1(\mathbf{z}, \varepsilon) + \varepsilon\mathbf{h}_2(\mathbf{a}, \mathbf{b}, \mathbf{z}, \varepsilon)\mathbf{a}\mathbf{b}, \end{aligned} \tag{5.4}$$

where the vector-valued function $\mathbf{h}_1$ and the matrix-valued functions Λ and Γ are as smooth as the original vector field. Also, all of the eigenvalues of $\Lambda(\mathbf{0}, \mathbf{0}, \mathbf{z}, 0)$ have positive real parts bounded away from zero, and the eigenvalues of $\Gamma(\mathbf{0}, \mathbf{0}, \mathbf{z}, 0)$ all have negative real parts bounded away from zero. Note that $\mathbf{h}_2(\mathbf{a}, \mathbf{b}, \mathbf{z}, \varepsilon)$ is a tensor of rank three; and, in component notation, the third equation is $z_i{}' = \varepsilon(\mathbf{h}_1)_i(\mathbf{z}, \varepsilon) + \varepsilon \sum_{j=1}^k \sum_{p=1}^\ell (\mathbf{h}_2)_{ijp}\mathbf{a}_j\mathbf{b}_p$. See Jones, *et al.* [1996] and Tin [1994].

This system has been dubbed the Fenichel normal form of (1.1), and it is extremely useful for analyzing the dynamics of general orbits near $\mathcal{M}_\varepsilon$, as well as for tracking invariant manifolds while orbits on them are near $\mathcal{M}_\varepsilon$.

The normal form's structure reflects the rigid geometry. As stated above, the manifolds $\mathcal{M}_\varepsilon$ correspond to the set $\{\mathbf{a}, \mathbf{b} = \mathbf{0}\}$, and the manifolds $W^S(\mathcal{M}_\varepsilon)$ and $W^U(\mathcal{M}_\varepsilon)$ are given by sets $\{\mathbf{a} = \mathbf{0}\}$ and $\{\mathbf{b} = \mathbf{0}\}$, respectively. Hence, the normal form directly reveals that all three of these manifolds are invariant. Moreover, the evolution of the slow z components is further decomposed into motion strictly due to the slow variables (in terms of the function $\mathbf{h}_1$) and a term which has the product $\mathbf{ab}$ factored out due to the presence of the stable and unstable manifolds and the associated foliation of $\mathbb{R}^{m+n}$ by invariant families of stable and unstable fibers. Finally, this last term is small for orbits near $\mathcal{M}_\varepsilon$, where at least one of the variables $\mathbf{a}$ or $\mathbf{b}$ is small.

REMARK 5.3. A normal form may also be given for the unperturbed system (1.1) with $\varepsilon = 0$. In light of the assumption that $\mathcal{M}_0$ is normally hyperbolic, we can exploit the structure of the unperturbed vector field near $\mathcal{M}_0$ to obtain a useful (and final) form of the unperturbed vector field. For a fixed $\mathbf{z}$, the local stable and unstable manifolds of $\mathcal{M}_0$ give rise to a new coordinate system in which the dynamics near $\mathcal{M}_0$ are best studied. Let $\mathbf{a} \in \mathbb{R}^k$ and $\mathbf{b} \in \mathbb{R}^\ell$, where $k + \ell = m$, be a new coordinate system such that the local stable and unstable manifolds of $\mathcal{M}_0$ are the axes $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, respectively. Then, in a neighborhood $\mathcal{B}$ of $\mathcal{M}_0$, the normal form of the unperturbed system is:

$$\begin{aligned} \mathbf{a}' &= \Lambda(\mathbf{a}, \mathbf{b}, \mathbf{z})\mathbf{a} \\ \mathbf{b}' &= \Gamma(\mathbf{a}, \mathbf{b}, \mathbf{z})\mathbf{b} \\ \mathbf{z}' &= \mathbf{0}, \end{aligned}$$

where Λ and Γ are matrix-valued functions. The normal hyperbolicity of $\mathcal{M}_0$ implies that for any $\Delta > 0$ sufficiently small, there are constants λ_0 and γ_0 such that for any eigenvalue $\tilde{\lambda}_i$ of $\Lambda(\mathbf{0}, \mathbf{0}, \mathbf{z})$ or any eigenvalue $\tilde{\gamma}_i$ of $\Gamma(\mathbf{0}, \mathbf{0}, \mathbf{z})$, we have: $\mathrm{Re}\tilde{\lambda}_i > \lambda_0 > 0$ and $\mathrm{Re}\tilde{\gamma}_i < \gamma_0 < 0$ for all points in the box $\mathcal{B} \equiv \{(\mathbf{a}, \mathbf{b}, \mathbf{z}) | |\mathbf{a}|, |\mathbf{b}| \leq \Delta\}$, where the $\mathbf{z}$ variables lie in a compact subset of $\mathbb{R}^n$. Of course, despite the fact that they are denoted by the same letters, the variables here are different from those of the Fenichel normal form in the perturbed $0 < \varepsilon \ll 1$ system.

REMARK 5.4. There are many examples of systems which have multiple slow manifolds or slow manifolds whose $\varepsilon = 0$ counterparts are given by multi-valued relations $\mathbf{X}_0(\mathbf{z})$, instead of by functions, as we have assumed in this work. For those problems, one works separately with each connected component on which $\mathbf{X}_0(\mathbf{z})$ is single-valued. Interesting dynamical phenomena, such as canards, can arise due to the presence of folds between pairs of adjacent connected components, but that is well beyond the scope of this paper. See, for example, Chapter 10 of Diener and Diener [1995] and the references therein, as well as the report in Diener [1984].

REMARK 5.5. The works of Takens [1971] and Ilyashenko [1997] develop a full theory for nonhyperbolic fixed points in singular systems. Also, a useful extension of the geometric singular perturbation theory to systems in which invariant manifolds lose their normal hyperbolicity at one (or more) points is given in Krupa and Szmolyan [1998] and in other, forthcoming, works by Szmolyan.

5.4. Regular perturbation theory for finding locally invariant slow manifolds $\mathcal{M}_\varepsilon$. As we have seen in Sections 3 and 4, Fenichel theory enables us to turn the problem for explicitly finding functions $\mathbf{X}_\varepsilon$ whose graphs are locally invariant slow manifolds $\mathcal{M}_\varepsilon$ of (1.1) into a regular perturbation problem. Invariance of the manifold $\mathcal{M}_\varepsilon$ implies that $\mathbf{X}(\mathbf{z}, \varepsilon)$ satisfies:

$$\varepsilon D_{\mathbf{z}}\mathbf{X}(\mathbf{z}, \varepsilon) \cdot \mathbf{g}(\mathbf{X}(\mathbf{z}, \varepsilon), \mathbf{z}, \varepsilon) = \mathbf{f}(\mathbf{X}(\mathbf{z}, \varepsilon), \mathbf{z}, \varepsilon). \tag{5.5}$$

This is the vector version of (3.7) valid for general systems (1.1). Then, one plugs the perturbation expansion: $\mathbf{X}(\mathbf{z}, \varepsilon) = \mathbf{X}_0(\mathbf{z}) + \varepsilon\mathbf{X}_1(\mathbf{z}) + \mathcal{O}(\varepsilon^2)$ into (5.5) to solve order by order for $\mathbf{X}(\mathbf{z}, \varepsilon)$. At order ε^0, one finds the equation: $\mathbf{0} = \mathbf{f}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0)$, which defines $\mathbf{X}_0(\mathbf{z})$ due to the invertibility of $D_{\mathbf{x}}\mathbf{f}$ and the Implicit Function Theorem. Then, at the next order, $\mathcal{O}(\varepsilon)$, one obtains:

$$D_{\mathbf{z}}\mathbf{X}_0(\mathbf{z}) \cdot \mathbf{g}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0) = D_{\mathbf{x}}\mathbf{f}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0) \cdot \mathbf{X}_1(\mathbf{z}) + \frac{\partial \mathbf{f}}{\partial \varepsilon}(\mathbf{X}_0(\mathbf{z}), \mathbf{z}, 0),$$

which yields $\mathbf{X}_1(\mathbf{z})$, and so forth. For application of this technique to systems, see: Fenichel [1979] and Sakamoto [1991] for general systems (1.1), Kath [1985] and subsection 4.3 of this work for third-order examples, and Nipp [1985] for problems with asymptotically stable slow manifolds. Also, this procedure is similar to the procedure commonly used to locate center manifolds of nonhyperbolic fixed points, in which invariance and an expansion are also used. See Carr [1981] .

6. Two-point boundary value problems: matching and transversality

In this section, we study the simple, linear, two point boundary value problem for $y = y(x)$ on $[0, 1]$:

$$\varepsilon\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \quad \text{with} \quad 0 < \varepsilon \ll 1, \quad y(0) = 1 \quad \text{and} \quad y(1) = -1. \tag{6.1}$$

It is a standard example of the use of matched asymptotic expansions, and the solution exhibits a boundary layer at $x = 0$, see Figure 12. There should not be any confusion with the use of x here for the independent variable (and as a dependent variable in Sections 1–5).

Our goal in this section is to use this elementary problem to illustrate how geometric singular perturbation theory complements the matched asymptotic approach. This example and a more general theory for nonlinear differential equations are developed in Hayes, *et al.* [1998], which is joint work with M. Hayes, N. Kopell, and K. Ono.

6.1. Solution of (6.1) by the method of matched asymptotic expansions. Expanding the solution of (6.1) in a power series:

$$y(x; \varepsilon) = y_0(x) + \varepsilon y_1(x) + \mathcal{O}(\varepsilon^2), \tag{6.2}$$

one finds the following equation for $y_0(x)$:

$$\frac{dy_0}{dx} + y_0 = 0. \tag{6.3}$$

This equation is known as the leading order outer equation. Its solution is $y_0(x) = Ke^{-x}$, for some constant K. However, one readily observes that it is impossible to choose K such that a single function $y_0(x)$ satisfies the boundary conditions at both $x = 0$ and $x = 1$. At best, one can ask that $y_0(x)$ satisfies one of these two.

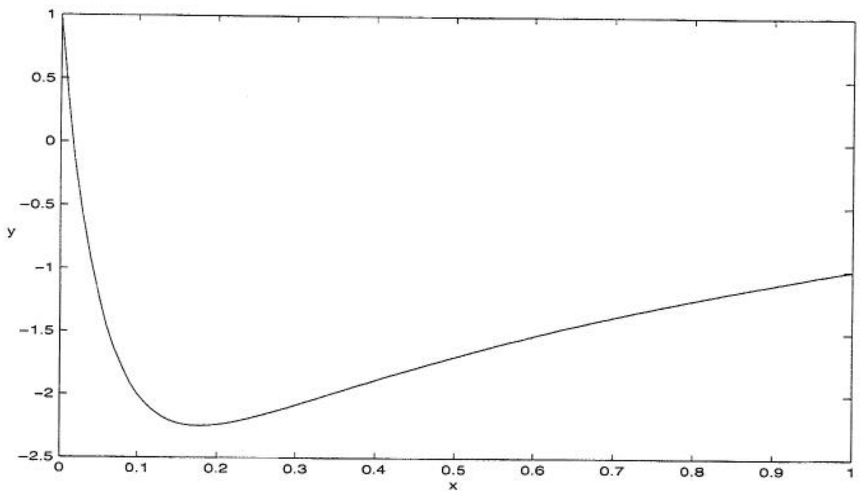

FIGURE 12. Solution of (6.1) with $\varepsilon = 0.05$, showing a fast (inner) solution (boundary layer at $x = 0$) followed by a slow (outer) solution.

In fact, in this case, the assumption that the solution can be written in the form (6.2) for all $x \in [0, 1]$ is not valid.

One might expect instead that there is a subinterval of $[0, 1]$ on which the derivative of y becomes large (and one could directly deduce from the exact solution in this simple linear equation). To find this subinterval (without resorting to the exact solution), assume it is centered about some unknown point x_0 and introduce the *stretched* variable: $\xi \equiv (x - x_0)/\varepsilon$. The full equation (6.1) becomes:

$$\frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} + \varepsilon y = 0.$$

Plugging in the expansion

$$y(\xi;\varepsilon) = y_i(\xi) + \varepsilon y^1(\xi) + \mathcal{O}(\varepsilon^2)$$

yields to leading order:

$$\frac{d^2y_i}{d\xi^2} + \frac{dy_i}{d\xi} = 0. \tag{6.4}$$

This equation, referred to as the leading order inner equation, has the solution $y_i(\xi) = Ce^{-\xi} + D$, for constants C and D. We observe that the first term vanishes exponentially as $\varepsilon \to 0$ for $\xi > 0$ (*i.e.*, for $x > x_0$), whereas it grows exponentially without bound as $\varepsilon \to 0$ for $\xi < 0$ (*i.e.*, for $x < x_0$). So, consider a solution that is near $y_i(\xi)$ for x in a neighborhood of x_0 and close to leading order solutions $y_0(x)$ outside of this neighborhood. If $x_0 > 0$, then it is impossible for such a solution to satisfy the boundary condition at $x = 0$, because of the blowup in y_i.

Therefore, it must be the case that $x_0 = 0$: the solution to (6.1) exhibits a boundary layer at $x = 0$ (which may also be spotted from the equation since the coefficients on the first and second terms have the same sign). As a consequence, we can construct, using a concatenation of the inner and outer solutions, a solution of the full problem that is near $y_0(x)$ on $(x, 1]$ for all $x > 0$ and near $y_i(\xi)$ on the small interval $[0, \varepsilon)$. We take the leading order outer solution with $K = -e$ so that the right boundary condition is satisfied, and we take the leading order inner solution with $D = 1 - C$ to satisfy the left boundary condition.

To summarize the results obtained thus far: the leading order outer solution $y_0(x) = -e^{1-x}$, is the asymptotic expansion of the full solution, valid to within $\mathcal{O}(\varepsilon)$, on the interval $(x, 1]$ for any $x > 0$. Also, the function $y_i(\xi) = 1 + C\left(e^{-\xi} - 1\right)$, where C is an as yet unknown constant, is the leading order asymptotic expansion of the full solution over the interval $[0, \varepsilon]$.

The Extension Theorems suggest that the domains on which the above expansions are valid asymptotic expansions of the full solution can be enlarged. In this example, one finds that the outer domain can be extended to any interval $(\eta_o(\varepsilon), 1]$ where $\eta_o(\varepsilon)$ is any function that vanishes as $\varepsilon \to 0$, but that also satisfies $\eta_o(\varepsilon) \gg \varepsilon$. The inner domain can be enlarged to any interval $[0, \eta_i(\varepsilon))$, where $\eta_i(\varepsilon)$ is any function such that $\eta_i(\varepsilon) \ll 1$ as $\varepsilon \to 0$. For example, $\eta_o(\varepsilon)$ may be taken to be ε^α for any $0 < \alpha < 1$; similarly for $\eta_i(\varepsilon)$. Hence, these extended domains overlap for all points $x = \mathcal{O}(\eta)$ with $\varepsilon \ll \eta \ll 1$.

The final step in applying the method of matched asymptotics then involves matching the inner and outer expansions in this overlap interval:

$$\lim_{\xi \to \infty} y_i(\xi) = \lim_{x \to 0^+} y_0(x). \tag{6.5}$$

The unknown constant is therefore $C = 1 + e$. To conclude, the method of matched asymptotic expansions generated asymptotic expansions of the full solution that are valid on different, but overlapping, parts of the interval $[0, 1]$. Also, we remark that a uniformly valid leading order expansion can also be written, and this procedure may be repeated to obtain higher order terms in both expansions.

A partial (nonexhaustive) list of references on the method of matched asymptotic expansions includes: Friedrichs [1943] (see also Chapter 2 of Lagerstrom [1988]), Bender and Orszag [1978], de Jager and Jiang [1996], Eckhaus [1979], Grasman [1987], Holmes [1995], Kevorkian and Cole [1996], Lagerstrom [1988], O'Malley [1974], O'Malley [1991], Smith [1985], and Van Dyke [1964]. There is also a special series of articles in a 1994 issue of *SIAM Review* reviewing some of the history of singular perturbation problems, see for example Eckhaus [1994] and O'Malley [1994]. Various other methods have also been applied to singularly perturbed two-point boundary value problems in addition to those already cited in above. See, for example, Angenent, *et al.* [1986], Chang and Howes [1984], Fife [1974, 1979], Hoppensteadt [1970], Howes [1979], Lin [1990], and Wasow [1956], among others. Finally, for other more recently developed approaches, based on nonstandard analysis, the theory of distributions or the renormalization group method, see for example, Diener and Diener [1995], Estrada and Kanwal [1995] and Chen, *et al.* [1996], respectively.

6.2. Brief outline of the geometric method. Geometric singular perturbation theory relies upon the leading order outer and inner equations (6.3) and (6.4). No explicit information about the solutions of the leading order outer and inner equations is required in this geometric approach. A good set of dependent variables is introduced based on these equations. These new dependent variables then enable us to explicitly display the fast-slow geometry of the problem and to put equation (6.1) into the Fenichel normal form near its slow manifolds (outer solutions). The slow manifolds are asymptotically stable. The Fenichel normal form (or in this case, the Tikhonov-Levinson theory and the work of Nipp [1985], since the slow manifold is asymptotically stable) then enables us to hookup the slow (center) dynamics with the exponential contraction of solutions in forward time along the invariant family of fast, stable fibers that foliate the stable manifolds of the slow manifolds.

REMARK 6.1. In this subsection and in the following subsections, we use the letters τ and t in place of the letters x and ξ to denote the independent variable, since we employ dynamical systems techniques. Also, please note that here the roles of t and τ (and hence also those of the dots and the primes) are reversed as compared to Hayes, *et al.* [1998], to fit into the notational format used here.

Introduction of these slow and fast coordinates will imply that the boundary conditions are lines in the $\tau = 0$ and $\tau = 1$ planes in the three-dimensional extended phase space, *i.e.*, with the independent variable τ included. The solutions satisfying the boundary conditions at $\tau = 0$ and $\tau = 1$ are flowed (forward and backward, respectively) into the interior of the extended three-dimensional phase-space, yielding a pair of two-dimensional manifolds. The image of the line corresponding to $y(0) = 1$ is tracked forward under the flow up to the point $t = \frac{1}{\sqrt{\varepsilon}}$ $(\tau = \sqrt{\varepsilon})$. In addition, the image of the line corresponding to $y(\tau = 1) = -1$ is tracked backward to the same point. Then, it will follow from the tracking results that these images intersect transversely at $\tau = \sqrt{\varepsilon}$, and hence that, in the three-dimensional extended phase space, the tracked manifolds intersect transversely in an orbit that is precisely the desired solution of the boundary value problem.

In subsection 6.3, we introduce the good coordinates. The tracking of the boundary sets is carried out in subsections 6.4 and 6.5. We also comment on the relation between the geometric and matched asymptotic expansion approaches. Finally, in subsection 6.6, we discuss some of the other types of problems on which the geometric method has been applied.

6.3. Good coordinates and the Fenichel normal form of (6.1). The structure of the leading order outer and inner equations (6.3) and (6.4), respectively, for this linear boundary value problem motivates introducing new variables:

$$u \equiv \varepsilon \left(\frac{dy}{dx} + y \right) \qquad \text{and} \qquad v \equiv \varepsilon \frac{dy}{dx} + y. \tag{6.6}$$

With this choice of new dependent variables, (i) $u = 0$ is the leading order outer equation, (ii) $\frac{dv}{dx} = 0$ is the leading order inner equation, (iii) u is scaled so that it is finite inside the boundary layer and $\mathcal{O}(\varepsilon)$ in the outer regime, and (iv) u is a fast variable, while v is a slow variable. Finally, it is useful to record how y and $\frac{dy}{dx}$ are expressed in terms of u and v:

$$y = \left(\frac{-1}{1-\varepsilon} \right) u + \left(\frac{1}{1-\varepsilon} \right) v \qquad \text{and} \qquad \frac{dy}{dx} = \left(\frac{1}{\varepsilon(1-\varepsilon)} \right) u + \left(\frac{-1}{1-\varepsilon} \right) v. \tag{6.7}$$

We now determine $\dot{u}$ $(\equiv \frac{du}{dx} = \frac{du}{d\tau})$ and $\dot{v}$ by differentiating (6.6), then using (6.1) to replace second derivatives, and finally by using (6.7):

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon) & -\varepsilon + \mathcal{O}(\varepsilon^2) \\ 1 + \mathcal{O}(\varepsilon) & -1 + \mathcal{O}(\varepsilon) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{6.8}$$

where $\dot{} \equiv d/d\tau$. Equivalently, (6.8) may be written in terms of the fast time variable $t \equiv \tau/\varepsilon$ (and $' \equiv d/dt$):

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1+\varepsilon+\mathcal{O}(\varepsilon^2) & -\varepsilon^2+\mathcal{O}(\varepsilon^3) \\ \varepsilon+\mathcal{O}(\varepsilon^2) & -\varepsilon+\mathcal{O}(\varepsilon^2) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6.9}$$

Equation (6.9) is of the form (1.1) to which Fenichel theory applies, with $f(u,v;\varepsilon) = A_\varepsilon u + \varepsilon^2 V_\varepsilon v$ (where $A_\varepsilon = -1 + \frac{\varepsilon}{1-\varepsilon} = -1+\varepsilon+\mathcal{O}(\varepsilon^2)$ and $V_\varepsilon = -\frac{1}{1-\varepsilon}$) and $g(u,v;\varepsilon) = \frac{1}{1-\varepsilon}(u-v)$. When $\varepsilon = 0$, equation (6.9) possesses a normally hyperbolic invariant manifold $\mathcal{M}_0 \equiv \{(u,v) | f(u,v;0) = -u = 0\}$. Restricting our consideration to that part of $\mathcal{M}_0$ in which the v coordinate lies in a closed, sufficiently large interval of length $\mathcal{O}(1)$ about the origin, the invariant manifold is compact. Hence, for $0 < \varepsilon \ll 1$, (6.9) satisfies the conditions (H1)–(H3); and, by Fenichel's Persistence Theorem, there exists a (nonunique) function $m_\varepsilon(v)$ satisfying $m_\varepsilon(0) = 0$ whose graph $\mathcal{M}_\varepsilon \equiv \{(u,v)|u = m_\varepsilon(v)\}$ is a (nonunique) slow (or center) manifold for the system (6.9) that is tangent to the v-axis at $(0,0)$. Moreover, the stable manifold of the slow manifold exists and is C^r $\mathcal{O}(\varepsilon)$ close to its $\varepsilon = 0$ counterpart, as well as being of the same dimension.

To conclude this subsection, we derive a useful normal form for the vector field (6.8). We use the function $m_\varepsilon(v)$ to change to Fenichel coordinates $b \equiv u - m_\varepsilon(v)$, and v (see subsection 5.4, where v plays the role of z_1, b is a scalar, and the a variable is nonexistent since there are no unstable directions). The system (6.9) then has the Fenichel normal form:

$$\begin{aligned} b' &= B_\varepsilon(b,v)b \\ v' &= \varepsilon\tilde{g}(b,v;\varepsilon), \end{aligned} \tag{6.10}$$

where:

$$\begin{aligned} B_\varepsilon(b,v) &= A_\varepsilon + \mathcal{O}(\varepsilon), \\ \tilde{g}(b,v;\varepsilon) &\equiv g(b+m_\varepsilon(v),v;\varepsilon) = (1+\mathcal{O}(\varepsilon))\,b - (1+\mathcal{O}(\varepsilon))\,v. \end{aligned} \tag{6.11}$$

We will drop the tilde from now on.

6.4. Tracking forward solutions good at the left boundary. The boundary conditions in (6.1) can be transformed into u, v variables using (6.7):

$$\frac{u(\tau=0)-v(\tau=0)}{\varepsilon-1} = 1 \quad \text{and} \quad \frac{u(\tau=1/\varepsilon)-v(\tau=1/\varepsilon)}{\varepsilon-1} = -1. \tag{6.12}$$

As remarked above, these are now the equations for lines in the (u,v) planes at $t=0$ $(\tau=0)$ and $t=1/\varepsilon$ $(\tau=1)$, respectively. We restrict v to lie in a closed interval of length $\mathcal{O}(1)$, and we label these line segments $\mathcal{L}$ and $\mathcal{R}$, respectively. In this subsection, we track $\mathcal{L}$ forward from $t=0$ to $t=1/\sqrt{\varepsilon}$; and, in the next subsection, we track $\mathcal{R}$ backward from $t=1/\varepsilon$ to $t=1/\sqrt{\varepsilon}$.

The image of $\mathcal{L}$ tracked forward to $t=1/\sqrt{\varepsilon}$ is a curve which lies exponentially close to the v-axis. This may be seen as follows. Assume that $|v(t)| \le K_1$ for some $K_1 = \mathcal{O}(1)$ for all $t \in [0, 1/\sqrt{\varepsilon}]$. Integration of (6.10)(a) yields:

$$|b(0)|e^{-\frac{1}{\sqrt{\varepsilon}}-c\sqrt{\varepsilon}} \le |b(1/\sqrt{\varepsilon})| \le |b(0)|e^{-\frac{1}{\sqrt{\varepsilon}}+c\sqrt{\varepsilon}}, \tag{6.13}$$

where by definition $b(0) = u(0) - m_\varepsilon(v(0)) = u(0) + \mathcal{O}(\varepsilon^2)$. One sees directly, therefore, that $b(1/\sqrt{\varepsilon})$ is exponentially small for all initial conditions on $\mathcal{L}$. Similarly, under the assumption that $|b(t)| \leq K_2$ for some $K_2 = \mathcal{O}(1)$ for all $t \in [0, 1/\sqrt{\varepsilon}]$, integration of (6.10)(b) shows that v stays close to its initial value:

$$v(0) - c_2\sqrt{\varepsilon} \leq v(1/\sqrt{\varepsilon}) \leq v(0) + c_2\sqrt{\varepsilon}.$$

By integrating (6.10) to a general $t \in [0, 1/\sqrt{\varepsilon}]$ we see that u and v can never differ from their initial values (as determined by the $\mathcal{O}(1)$ boundary conditions at $t = 0$) by more than $\mathcal{O}(1)$ amounts. Hence, both a priori bounds hold over the indicated time interval; and, we have established that $\varphi_{1/\sqrt{\varepsilon}}\mathcal{L}$, the forward image of $\mathcal{L}$ at time $t = 1/\sqrt{\varepsilon}$, lies exponentially close to the $v-$axis.

REMARK 6.2. Analysis of the variational equations of (6.10) shows that the slope of $\varphi_{1/\sqrt{\varepsilon}}\mathcal{L}$ is exponentially large in the (b, v) plane. The magnitude of the slope is $\mathcal{O}(e^{\frac{1}{\sqrt{\varepsilon}}})$ as $\varepsilon \to 0$. See Section 3.2 of Hayes, *et al.* [1998].

6.5. Tracking $\mathcal{R}$ backward. In this subsection, we track the image of the right boundary condition $\mathcal{R}$ backward from $\tau = 1$ $(t = 1/\varepsilon)$ to $\tau = \sqrt{\varepsilon}$ $(t = 1/\sqrt{\varepsilon})$. In τ-time, equation (6.10) can be rewritten as:

$$\begin{aligned} \varepsilon\dot{b} &= B_\varepsilon(b, v)b \\ \dot{v} &= g(b, v; \varepsilon). \end{aligned} \tag{6.14}$$

Using (6.11) for $B_\varepsilon(b, v)$ and $g(b, v; \varepsilon)$, (6.14) may be written as:

$$\begin{aligned} \varepsilon\dot{b} &= (-1 + \mathcal{O}(\varepsilon))b \\ \dot{v} &= (1 + \mathcal{O}(\varepsilon))b + (-1 + \mathcal{O}(\varepsilon))v. \end{aligned} \tag{6.15}$$

We will show that, in the (b, v) plane, the image of $\mathcal{R}$ at $\tau = \sqrt{\varepsilon}$ is a nearly-horizontal curve. The b component gets stretched exponentially, while the v coordinate slowly moves an $\mathcal{O}(1)$ distance as τ decreases from 1 to $\sqrt{\varepsilon}$. Then, plotting the images of $\mathcal{L}$ and $\mathcal{R}$ obtained in the previous and present subsections, respectively, in the (b, v) plane at $\tau = \sqrt{\varepsilon}$, we see that they intersect transversely, and this intersection contains the locally unique solution to the boundary value problem (6.1).

Throughout this subsection, we make the a priori assumption that b and v stay $\mathcal{O}(1)$. We will prove that this assumption remains valid as one tracks solutions backward from $\tau = 1$ to $\tau = \sqrt{\varepsilon}$.

We first work with the $\dot{b}$ equation. If $\varepsilon L_1 (> 0)$ is a bound on the $\mathcal{O}(\varepsilon)$ terms $(L_1 = \mathcal{O}(1))$, then we obtain the differential inequalities

$$\left(\frac{-1}{\varepsilon} - L_1\right)|b| \leq \frac{d|b|}{d\tau} \leq \left(\frac{-1}{\varepsilon} + L_1\right)|b|. \tag{6.16}$$

Integrating the first inequality over $[\sqrt{\varepsilon}, 1]$, we obtain an upper bound for $|b(\sqrt{\varepsilon})|$:

$$|b(\sqrt{\varepsilon})| \leq |b(1)|e^{\frac{1}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} + L_1 - L_1\sqrt{\varepsilon}}. \tag{6.17}$$

Meanwhile, the second inequality of (6.16) can be integrated over $[\tau, 1]$ to provide a lower bound for the b terms for all $\tau \in [\sqrt{\varepsilon}, 1]$:

$$|b(\tau)| \geq |b(1)|e^{(\frac{1}{\varepsilon} - L_1)(1-\tau)}. \tag{6.18}$$

for all $\tau \in [\sqrt{\varepsilon}, 1]$. All initial conditions peel away from the center (slow) manifold $b = 0$ exponentially in backward time with a large rate constant. Therefore, from (6.17) and (6.18), we conclude that $\mathcal{R}$ is stretched out exponentially in the b-direction, but $|b(1)|$ may be chosen small enough to remain close as we like to $b = 0$

under backward integration. Moreover, for $|b(\sqrt{\varepsilon})|$ to be $\mathcal{O}(\omega(\varepsilon))$, for any small $\omega(\varepsilon)$, one must have that $|b(1)| = \mathcal{O}(\omega(\varepsilon)e^{-\frac{1}{\varepsilon}+\frac{1}{\sqrt{\varepsilon}}})$. We will of course be interested in $|b(1)|$ small in order that the image of $\mathcal{R}$ intersects $\varphi_{1/\sqrt{\varepsilon}}\mathcal{L}$, which is situated close to the b-axis.

Introducing the integrating factor $e^{(1+\mathcal{O}(\varepsilon))\tau}$ in the $\dot{v}$ equation in (6.15) and using (6.18) to integrate over $[\sqrt{\varepsilon}, 1]$, yields an upper bound on the growth of v as $\mathcal{R}$ is tracked backward:

$$|v(\sqrt{\varepsilon})| \leq |v(1)|e^{1-\sqrt{\varepsilon}+\mathcal{O}(\varepsilon)} + \varepsilon\, \Xi\, |b(1)| \left(e^{1-\sqrt{\varepsilon}+\mathcal{O}(\varepsilon)} - e^{\frac{1}{\varepsilon}-\frac{1}{\sqrt{\varepsilon}}-L_1+\mathcal{O}(\sqrt{\varepsilon})}\right),$$

where $\Xi = -1 + \mathcal{O}(\varepsilon)$. Again, if $|b(1)|$ is sufficiently small ($\mathcal{O}(e^{-\frac{1}{\varepsilon}+\frac{1}{\sqrt{\varepsilon}}})$), then v changes at most an $\mathcal{O}(1)$ amount when $\mathcal{R}$ is tracked backward:

$$|v(\sqrt{\varepsilon})| \leq |v(1)|e^{1+\mathcal{O}(\sqrt{\varepsilon})} + \mathcal{O}(\varepsilon). \tag{6.19}$$

Therefore, in the v-direction, motion is limited to $\mathcal{O}(1)$ drift along the center manifold. We also notice that the a priori estimates that v and b stay $\mathcal{O}(1)$ hold.

Hence, we have obtained the desired result that the image of $\mathcal{R}$ at $\tau = \sqrt{\varepsilon}$ is a nearly-horizontal curve in the (b, v) plane. Moreover, as already stated at the beginning of this subsection, the results from this subsection and from subsection 6.4 combine to demonstrate that the images of $\mathcal{L}$ and $\mathcal{R}$ intersect transversely in the (b, v) plane at $\tau = \sqrt{\varepsilon}$, and this intersection contains the locally unique solution to the boundary value problem (6.1).

REMARK 6.3. Using the equation of variations of (6.14), one can readily show that the backward image of $\mathcal{R}$ at $\tau = \sqrt{\varepsilon}$ is actually C^1 $\mathcal{O}(\varepsilon)$ small, *i.e.*, the slope of the backward image of $\mathcal{R}$ is $\mathcal{O}(\varepsilon)$ small. See Section 3.3 of Hayes, *et al.* [1998].

In example (5.1), the matching was accomplished because $\lim_{\xi\to\infty} y_i(\xi)$ varies (with exponentially large derivative) with $\frac{dy}{dx}(0)$. In the geometric method, this is manifested by the fact that the forward image of $\mathcal{L}$ is splayed out over the backward image of $\mathcal{R}$, creating the transverse intersection that contains the solution of the boundary value problem.

To conclude this section, we comment briefly on the role of an overlap domain. The asymptotic matching for a second order equation produces presumptive solutions in the outer and inner regions that have the same value of y at the point of matching. In each region, the value of $\frac{dy}{dx}$ at the matching point can be computed (within exponentially small terms) from these solutions. In particular, if there is an overlap domain, then the fact that the outer and inner solutions satisfy the same asymptotic expansions in that domain means that the matching of y values automatically implies the matching of $\frac{dy}{dx}$ values (up to exponentially small errors). The geometric method, by contrast, directly follows $\frac{dy}{dx}$, as well as y, and does not depend on an overlap domain. In problems without overlap domains, the common point to which the manifolds are tracked (and transversality is checked) would be precisely at the boundary of the two domains of validity.

6.6. Brief references to geometric approaches for more general singularly perturbed problems. In Hayes, *et al.* [1998], the same method as used on this linear example is employed to three classes of boundary value problems: (i) linear $\varepsilon y'' + \alpha y' + \beta y = 0$; (ii) semilinear $\varepsilon y'' + \alpha y' + f(y) = 0$; and (iii) nonlinear $\varepsilon y'' + g(y)y' + f(y) = 0$; on $[0, 1]$, each with $y(0) = A$ and $y(1) = B$.

More complicated boundary value problems – involving inner layers and/or multiple layers, with strongly nonlinear terms, or on unbounded domains – can possess more elaborate geometric structures, such as multiple slow manifolds, unstable fibers in addition to stable ones, turning points, and a host of other complicating features. Geometric singular perturbation theory has already been used successfully on many of these. We refer the reader to Jones [1994] for many examples and references, and confine ourselves to only citing a few here. The existence of solutions composed of multiple fast layers interspersed with slow segments may be demonstrated using the Exchange Lemma, or any of its variants. See Jones and Kopell [1994], Jones, *et al.* [1996], Tin [1994], and Tin, *et al.* [1994]. See also Brunovsky [1998]. Tin, *et al.* [1994] specifically addresses boundary value problems. The work in Hayes, *et al.* [1998] shows how the transversality hypotheses in Tin, *et al.* [1994] are satisfied for classes (i)-(iii). Finally, Soto-Treviño [1998a-b] develops a complete geometric theory for periodic orbits in systems (1.1). This theory establishes existence and local uniqueness of periodic orbits that consist of fast and slow segments, such as in the FitzHugh-Nagumo system, and it builds on an extension of the exchange lemma.

A nonexhaustive list of applications of the geometric theory to establish the existence of homoclinic or heteroclinic orbits includes: Bose [1995], Camassa, *et al.* [1998], Doelman *et al.* [1997a,b], Haller and Wiggins [1995], Jones, *et al.* [1991], Jones and Kopell [1994], Jones and Rubin [1998], Kaper and Kovačič [1996], Kopell and Landman [1995], Rubin [1996], and Szmolyan [1991]. See also Soto-Trevino [1998a-b] and Soto-Trevino and Kaper [1996] for applications to periodic orbits.

7. Fibers and geometry in the problem of orbits homoclinic to resonance bands in perturbed Hamiltonian systems

In this section, we study a nonlinear system consisting of a Duffing oscillator strongly coupled to an anharmonic oscillator:

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \mu^2 q(I - q^2) - \varepsilon\alpha p \\ \dot{I} &= -\varepsilon I \sin\theta - \varepsilon\beta I - \varepsilon\gamma p^2 \\ \dot{\theta} &= I - 1 - \frac{1}{2}\mu^2 q^2 - \varepsilon\cos\theta. \end{aligned} \tag{7.1}$$

Here, q and p are the canonically conjugate position and momentum variables of the Duffing oscillator, and $I \in \mathbb{R}$ and $\theta \in S^1$ are the action-angle variables of the anharmonic oscillator. The parameters μ, α, β, and γ are positive, and $\varepsilon \ll 1$ is a small positive parameter. The oscillators are strongly coupled, since the magnitude of the force experienced by the Duffing oscillator depends strongly on I. In addition, both oscillators are weakly damped, and the action variable of the anharmonic oscillator decreases weakly at a rate proportional to the Duffing oscillator's kinetic energy. While this example is not of the general form (1.1), we will see that it can be put into that form in the neighborhoods of resonance bands.

This example was studied in Kovačič [1992,1993] and Kaper and Kovačič [1996], and it is a prototype of completely-integrable Hamiltonian systems subject to small amplitude perturbations. When $\varepsilon = 0$, the system (7.1) is a two degree of freedom Hamiltonian system with two functionally independent constants of motion: I and

$$H_0(q,p,I) = \frac{1}{2}p^2 - \frac{1}{2}\mu^2 q^2\left(I - \frac{1}{2}q^2\right) + \frac{1}{2}I^2 - I.$$

Hence, it is completely integrable when $\varepsilon = 0$.

7.1. Normally hyperbolic manifolds when $\varepsilon = 0$ and their persistence when $0 < \varepsilon \ll 1$. When $\varepsilon = 0$, the system (7.1) simplifies considerably to

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \mu^2 q(I - q^2) \\ \dot{I} &= 0 \\ \dot{\theta} &= I - 1 - \frac{1}{2}\mu^2 q^2. \end{aligned} \tag{7.2}$$

For each $I > 0$, the Duffing oscillator with parameter I has a saddle fixed point at $(0,0)$ which is connected to itself by a pair of homoclinic orbits:

$$\begin{aligned} p &= p^h(t,I) = \mp\sqrt{2}\mu I \operatorname{sech}(\mu\sqrt{I}t)\tanh(\mu\sqrt{I}t) \\ q &= q^h(t,I) = \pm\sqrt{2I}\operatorname{sech}(\mu\sqrt{I}t) \\ I &= I \\ \theta &= \theta^h(t,I) + \theta_0 = (I-1)t - \mu\sqrt{I}\tanh(\mu\sqrt{I}t) + \theta_0, \end{aligned} \tag{7.3}$$

where the superscript h denotes homoclinic, and the $\pm$ correspond to the right and left homoclinic orbits, respectively. These explicit solutions are well known, see for example Chapter 4.5 of Guckenheimer and Holmes [1983]. In addition, since the angle θ decouples, it is determined by quadrature.

In the three-dimensional $q-p-I$ system, the union of the points $(q = 0, p = 0)$ for all I in any open interval (I_1, I_2) constitutes a normally hyperbolic invariant manifold by the definition from subsection 2.2. Moreover, if one views this manifold (now crossed with $\theta \in S^1$) in the full four-dimensional (q, p, I, θ) phase space, then it is still normally hyperbolic, but now one needs the definition for more general systems given in Fenichel [1971], because there is dynamics on it even when $\varepsilon = 0$.

Also, in the four-dimensional phase space, the three-dimensional stable and unstable manifolds of such a manifold are the unions of the individual stable and unstable manifolds from the planar systems with the parameter I fixed over the same I range of values, crossed with $\theta \in S^1$. The branches of these manifolds in $q \geq 0$ ($q \leq 0$, respectively) coincide.

Now, when $0 < \varepsilon \ll 1$, the plane $\{(q,p,I,\theta) | q = 0, p = 0\}$ is again invariant, since the q and p components of the vector field both vanish on it. In addition, it has three-dimensional stable and unstable manifolds by the general persistence theory developed in Fenichel [1971], and there are invariant families of fast stable and unstable fibers with basepoints on the local stable and unstable manifolds that foliate the neighborhood of the plane. These geometric structures can then be used to decompose the system dynamics in terms of the motions along the invariant manifold and motions in the normal directions.

However, before analyzing this decomposition, we focus specifically on a narrow interval (I_1, I_2) about $I = 1$. It is in this interval that the full system can be recast as a fast-slow system of the form (1.1) and that the system exhibits rich behavior such as resonance bands.

When $\varepsilon = 0$, the circle $I = 1$ is a circle of fixed points on the invariant plane, since $\dot{\theta} = 0$ when $q = 0$ and $p = 0$. Also, $\dot{\theta} < 0$ along those periodic orbits with $I < 1$ (inside our circle), and $\dot{\theta} > 0$ along those periodic orbits with $I > 1$ (outside our circle). We label the annulus about $I = 1$ by $\mathcal{M}_0$. It is a normally hyperbolic invariant manifold.

The circle $I = 1$ on $\mathcal{M}_0$ is a delicate structure corresponding to a significant degeneration in the system. Due to the nature of the linear shear in an annular neighborhood of it, this circle gets destroyed when the ε dependent terms are added to the equation. To analyze these dynamics, one needs to 'blow up' (*i.e.*, rescale) the significant degeneration by setting $I = 1 + \sqrt{\varepsilon}h$, where $h \in \mathbb{R}$ is a new dependent variable, and the power on ε follows from the requirement that h and θ vary at the same rate, $\mathcal{O}(\sqrt{\varepsilon})$, with respect to ε in this neighborhood. The full system becomes:

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \mu^2 q(1-q^2) + \sqrt{\varepsilon}\mu^2 qh - \varepsilon\alpha p \\ \dot{h} &= -\sqrt{\varepsilon}\left[\sin\theta + \beta + \gamma p^2\right] - \varepsilon h\left[\sin\theta + \beta\right] \\ \dot{\theta} &= \sqrt{\varepsilon}h - \frac{1}{2}\mu^2 q^2 - \varepsilon\cos\theta, \end{aligned} \tag{7.4}$$

where $\sqrt{\varepsilon}$ is the small parameter, instead of ε.

The plane $\mathcal{M}_\varepsilon \equiv \{(q,p,h,\theta) | q = 0, p = 0\}$ is a slow invariant manifold. In terms of the slow time $\tau = \sqrt{\varepsilon}t$, the slow system on $\mathcal{M}_\varepsilon$ is:

$$h' = -(1+\sqrt{\varepsilon}h)\sin\theta - \beta(1+\sqrt{\varepsilon}h), \qquad \theta' = h - \sqrt{\varepsilon}\cos\theta. \tag{7.5}$$

It is defined on the cylinder $\mathbb{R} \times S^1$, and we label this cylinder by $\mathcal{M}_\varepsilon$, since it is precisely a slow invariant manifold of the type we have studied in these notes.

To leading order, this outer system on $\mathcal{M}_\varepsilon$ is the classical nonlinear pendulum with angle variable θ and momentum variable h, constant torque β, and Hamiltonian with $\mathcal{H}(h,\theta) = \frac{1}{2}h^2 + 1 - \cos\theta + \beta\theta$. The higher-order, $\mathcal{O}(\sqrt{\varepsilon})$, terms represent perturbations, both Hamiltonian and dissipative, that break the homoclinic orbits to the saddle equilibria and that turn centers on $\mathcal{M}_\varepsilon$ into stable spiral fixed points.

REMARK 7.1. The term resonance band is used because one can always write the equation for a nonlinear oscillator, such as the anharmonic oscillator here, in terms of the action angle variables near a center and then study the system from a frame comoving with a periodic orbit whose period satisfies a resonance relation. This change of frame makes the periodic orbit into a circle of fixed points in the comoving frame. Then, the nonlinear pendulum arise as the normal form for a generic resonance.

Finally, $\mathcal{M}_\varepsilon$ has local stable and unstable manifolds $W^S_{\text{loc}}(\mathcal{M}_\varepsilon)$ and $W^U_{\text{loc}}(\mathcal{M}_\varepsilon)$ by the Fenichel theory, and these are three-dimensional. This completes our brief discussion of the slow (outer) system that governs the dynamics on $\mathcal{M}_\varepsilon$ (*i.e.*, in the resonance band), and we turn now to the fast (inner) system that governs the dynamics of orbits in the directions normal to $\mathcal{M}_\varepsilon$.

When $I = 1$, system (7.2) simplifies even further to:

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \mu^2 q(1-q^2) \\ \dot{h} &= 0 \\ \dot{\theta} &= -\frac{1}{2}\mu^2 q^2. \end{aligned} \tag{7.6}$$

Every point on the unperturbed cylinder is connected to another point by a heteroclinic orbit. From (7.3)(d), the angle difference $\Delta\theta$ between the end points of any heteroclinic orbit connecting pairs of equilibria on $\mathcal{M}_0$ at $I = 1$ is $\Delta\theta = -2\mu$, which is a direct result of integrating $\dot{\theta}^h(t, I = 1)$ along the entire real-(t)-line. Hence, one can construct singular orbits using these unperturbed fast heteroclinic connections interspersed with orbit segments from the reduced slow system on $\mathcal{M}_0$, and there is an entire two-parameter family of heteroclinics to choose from.

For the case when $0 < \varepsilon \ll 1$, only a few isolated $\varepsilon = 0$ fast heteroclinic orbits, and hence singular orbits, survive the perturbation. In particular, the homoclinic orbits that persist are precisely those that lie in the two-dimensional transverse intersections of the local stable and unstable manifolds, $W^S_{\rm loc}(\mathcal{M}_\varepsilon)$ and $W^U_{\rm loc}(\mathcal{M}_\varepsilon)$, so that they are both backward and forward asymptotic to $\mathcal{M}_\varepsilon$. Since the Melnikov function measures the splitting distance between these manifolds, their intersections can be detected by the simple zeroes of this function:

$$M(1,\theta_0,\mu,\alpha,\beta,\gamma) = 2\sin\mu\sin\theta_0 - \frac{4}{3}\alpha\mu + 2\beta\mu + \frac{8}{15}\gamma\mu^3, \tag{7.7}$$

see Kovačič [1992b,c]. When μ is not a multiple of π, this Melnikov function has simple zeros in θ_0 at some $\theta_0 = \bar\theta_{0,1}$ and $\theta_0 = \bar\theta_{0,2} = \pi - \bar\theta_{0,1}$, provided that

$$\left|\frac{\mu}{\sin\mu}\left(\frac{2}{3}\alpha - \beta - \frac{4}{15}\gamma\mu^2\right)\right| < 1. \tag{7.8}$$

Hence for all admissible α, β, and γ, the stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ intersect transversely along two symmetric pairs of two-dimensional surfaces. In the Hamiltonian case, we find $\bar\theta_{0,1} = 0$ and $\bar\theta_{0,2} = \pi$.

The beauty of the geometric method is that it decomposes the system dynamics into lower-dimensional subsystems, which are individually much easier to analyze than the full system. In this way, one can explicitly determine which inner and outer solutions can be connected to each other, which is hard to do in a four-dimensional system.

7.2. Some multi-bump homoclinic orbits. The existence of several different types of single and multi-bump transition solutions has been shown in Kovačič [1992a-b, 1993, 1995] and Kaper and Kovačič [1996], respectively. Each bump is a fast excursion away from the resonance band (invariant cylinder $\mathcal{M}_\varepsilon$), and the bumps are interspersed with slow segments near a resonance band. The homoclinic orbits, which include multi-bump Šilnikov orbits, connect various equilibria and periodic orbits in the resonance band. The bumps are fast (inner) transition layers, while the long segments near the resonance bands are outer solutions.

Theorem 1 from Kaper and Kovačič [1996] yields the following types of multi-bump homoclinic orbits in this example. Fix $0 < \mu \ll \beta < 1$ and $\tilde\gamma = \mu^2\gamma = \mathcal{O}(1)$. If $5\alpha - 2\tilde\gamma = 0$, then there exist two symmetric pairs of heteroclinic orbits that connect two pairs of points on the separatrix on the $h - \theta$ cylinder $\mathcal{M}_0$, as shown in Figure 20 of Kaper and Kovačič [1996]. These two pairs of points cut the separatrix into five disjoint segments, denoted $O_1, \ldots, O_5$, where the numbers increase in the direction of increasing slow time τ along the separatrix. We denote the pair of heteroclinic orbits connecting the endpoints of the segment O_2 by $\Gamma_1^\pm$, and the pair connecting the endpoints of the segment O_4 by $\Gamma_2^\pm$.

We can now form a countable infinity of singular homoclinic orbits as follows. Consider any finite sequence σ_1 of pairs $\Gamma_1^+ O_2$ and $\Gamma_1^- O_2$, including possibly the empty sequence. Moreover, let σ_2 be either O_4, Γ_2^+, or Γ_2^-. Define the sequence S as $S = O_1 O_2 \sigma_1 O_3 \sigma_2 O_5$, and exclude the sequence $O_1O_2O_3O_4O_5$ from this definition. If we concatenate the orbit segments $O_1, \ldots, O_5$, $\Gamma_1^\pm$, and $\Gamma_2^\pm$ according to the sequence S, we obtain a singular orbit homoclinic to the point s_0, which we denote by Γ_0^S. If we denote by $-S$ the sequence obtained from S by interchanging the $+$ and $-$ signs, then the corresponding singular homoclinic orbit Γ_0^{-S} is the mirror image of the singular orbit Γ_0^S under the transformation $(p,q) \mapsto (-p,-q)$, which preserves equations (7.1). To be precise, Theorem 1 from Kaper and Kovačič [1996] now implies the following:

Proposition 7.1. Fix $0 < \mu \ll \beta < 1$ and $\tilde{\gamma} = \mu^2\gamma = \mathcal{O}(1)$. Let the sequence S be as in the preceding paragraph, and Γ_0^S its corresponding singular orbit homoclinic to the point s_0. Then, there exists a continuous function $\alpha_S(\varepsilon)$ with $5\alpha_S(0) - 2\tilde{\gamma} = 0$, such that for small positive ε and $\alpha = \alpha_S(\varepsilon)$, there exists an orbit Γ_ε^S, homoclinic to the saddle s_ε. The orbit Γ_ε^S is $\mathcal{O}(\delta(\varepsilon))$ close to the singular orbit Γ_0^S, where $\delta(\varepsilon)$ is a function with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, $\alpha_S(\varepsilon) = \alpha_{-S}(\varepsilon)$, and the orbit Γ_ε^{-S} is the mirror image of the orbit Γ_ε^S under the transformation $(p,q) \mapsto (-p,-q)$.

As a second example, we obtain the existence of multi-bump Šilnikov orbits by applying Theorem 2 from Kaper and Kovačič [1996]. The demonstration of this fact proceeds as follows. When $0 < \beta < 1$ and the equation

$$\alpha = \frac{3}{2}\left(1 - \frac{\sin 2\mu}{2\mu}\right)\beta - \frac{3}{2}\frac{(1-\cos 2\mu)}{2\mu}\sqrt{1-\beta^2} + \frac{2}{5}\tilde{\gamma} \tag{7.9}$$

is satisfied, the line $\theta = \bar{\theta}_{0,1} + \mu$ passes through the center c_0. The heteroclinic orbit emerging from c_0 lands inside the region enclosed by the separatrix homoclinic to the saddle s_0, for instance, when $0 < \mu < \frac{\pi}{2} - \arcsin\beta$. When $\mu \ll 1$, equation (7.9) becomes $\alpha = \frac{2}{5}\tilde{\gamma} - \frac{3}{2}\mu\sqrt{1-\beta^2} + \mathcal{O}(\mu^2)$. For small nonzero $\sqrt{\varepsilon}$, this situation gives rise to a pair of single-bump Šilnikov orbits homoclinic to the spiral-saddle c_ε, see Figure 21 of Kaper and Kovačič [1996].

Recall that each slow-time periodic orbit, which lies inside the region enclosed by the separatrix homoclinic to the saddle s_0 and intersects the line $\theta = \bar{\theta}_{0,1} + \mu$, intersects this line in two points, one with positive and one with negative h. Therefore, there exist two pairs of heteroclinic orbits emanating from this periodic orbit, such that the h coordinate along the two orbits of one of these pairs is positive, and the other negative. We denote each of these orbits by one of the four symbols $\Gamma_\pm^\pm$, where the sign in the subscript denotes the sign of the h coordinate along this orbit, and the superscript denotes the intersection surface this orbit lies on.

To obtain the existence of multi-bump Šilnikov orbits, we form the following type of a modified singular transition orbit. Its first segment is a heteroclinic orbit Γ_1 taking off at the center $c_0 \equiv c_1$, and landing inside the region enclosed by the separatrix homoclinic to the saddle s_0 at the point d_2. The second segment is comprised of N_2 revolutions around the slow-time periodic orbit O_2 that passes through the point d_2. The last point of this segment is the point c_2 on the line $\theta = \bar{\theta}_{0,1} + \mu$, where the next segment, the heteroclinic orbit Γ_2, takes off. This orbit lands at the point d_3, which is again assumed to be inside the region enclosed by the separatrix homoclinic to the saddle s_0. This construction can continue for as long as the landing point d_{i+1} of the i-th heteroclinic orbit Γ_i is contained inside the region enclosed by the separatrix homoclinic to the saddle s_0. We denote the largest such i by K. Modified singular transition orbits of the described type with k bumps exist for all integer k with $1 \le k \le K$. Each of the heteroclinic orbits Γ_i can lie on either of the singular intersection surfaces, and, if $i > 1$, can have either positive or negative h coordinate.

The maximal number of bumps, K, of the modified singular transition orbits described in the previous paragraph can be computed as a function of the parameters μ and β. See section 11 of Kaper and Kovačič [1996]. To be precise, Theorem 2 from Kaper and Kovačič [1996]. now implies:

Proposition 7.2. Fix $0 < \beta < 1$, $0 < \mu < \frac{\pi}{2} - \arcsin\beta$, and $\tilde{\gamma} = \mu^2\gamma$. Let the sequence S of length k with $1 \leq k \leq K$ be as in the preceding paragraph, and let Γ_0^S be its corresponding modified singular transition orbit. Then, there exists a continuous function $\alpha_S(\varepsilon)$ with $\alpha_S(0)$ satisfying equation (7.9), such that for small positive ε and $\alpha = \alpha_S(\varepsilon)$, there exists an orbit Γ_ε^S, homoclinic to the spiral-saddle c_ε. Apart from the piece that spirals towards the point c_ε near the $h - \theta$ annulus $\mathcal{M}_\varepsilon$, the orbit Γ_ε^S is $\mathcal{O}(\delta(\varepsilon))$ close to the modified singular transition orbit Γ_0^S, where $\delta(\varepsilon)$ is a function with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, the orbit Γ_ε^S makes N_i full revolutions around the point c_ε between its $(i-1)$-st and i-th bumps. Moreover, $\alpha_S(\varepsilon) = \alpha_{-S}(\varepsilon)$, and the orbit Γ_ε^{-S} is the mirror image of the orbit Γ_ε^S under the transformation $(p, q) \mapsto (-p, -q)$.

A two-bump orbit is shown in Figure 22 of Kaper and Kovačič [1996]. We remark that, if we substitute the right-hand side of equation (7.9) for α in the inequality (7.8), this inequality is always satisfied.

What is the difference between the multi-bump orbits described in propositions Prop. 7.1 and Prop. 7.2 and the frequently discussed type of homoclinic orbits with multiple excursions, the so-called multi-loop orbits, described in Glendinning and Sparrow [1984]? The difference between these multi-loop orbits and the multi-bump orbits given by our propositions lies in the fact that each multi-loop orbit must necessarily return to a small neighborhood of the equilibrium point to which it is homoclinic, while this is not true for any of the multi-bump orbits. In fact, between two bumps, each such multi-bump orbit must stay a finite distance away from the equilibrium point to which it is homoclinic. Finally, the return-map technique of Glendinning and Sparrow [1984] may be used to show the existence of an infinite sequence of multi-loop subsidiary orbits corresponding to each of the multi-bump Šilnikov orbits homoclinic to the spiral-saddle c_ε established here.

REMARK 7.2. A general theory for orbits homoclinic to resonance bands in completely-integrable systems subject to small amplitude Hamiltonian and dissipative perturbations is contained in Kaper and Kovačič [1996]. Such systems have the form

$$\begin{aligned} \dot{x} &= JD_xH(x,I) + \varepsilon g^x(x,I,\theta,\lambda), \\ \dot{I} &= \varepsilon g^I(x,I,\theta,\lambda), \\ \dot{\theta} &= D_IH(x,I) + \varepsilon g^\theta(x,I,\theta,\lambda), \end{aligned} \tag{7.10}$$

where $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, $I \in \mathbb{R}$, and $\theta \in S^1$. Furthermore, D_x denotes the partial derivatives with respect to x, D_I is the partial derivative with respect to I, $\lambda \in \mathbb{R}$ is a real parameter, $\varepsilon \ll 1$ is a small parameter, and $J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ with Id being the $n \times n$ identity matrix. See also Camassa, *et al.* [1998] and Haller and Wiggins [1995] for other types of multiple-pulse homoclinic orbits.

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