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Exchange of Stabilities in Autonomous Systems*

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Equilibrium solutions $y = \phi(x)$ of an autonomous system of differential equations, depending on a parameter x, are considered. Bifurcation of a second family of solutions $y = \psi(x)$ and exchange of stabilities is supposed to occur at $(x, y) = (0, 0)$. Considering x as slowly varying leads to a singularly perturbed initial-value problem whose reduced path encounters a point of bifurcation. Rigorous asymptotic estimates are found for the difference between the (unique) solution of the full problem and that solution of the reduced problem which proceeds along stable segments of the reduced path.

1. Introduction

Consider the system of differential equations

$$\frac{dy}{d\tau} = g(x, y), \tag{1}$$

where y is an n-component vector and x is a parameter with values in some interval $[a, b]$. It is natural, especially if (1) models the behavior of some physical or biological system, to seek equilibrium solutions. Suppose $y = \phi(x)$ is such a solution. Suppose further that progressively larger values of x represent progressively later epochs in the life of the physical or biological system. Then the following heuristic description of the system's evolution may be given: if at an early epoch ($x = a$) the system is in equilibrium, and if x increases so slowly as not to disturb the equilibrium, then the system will evolve through the series of equilibrium states $y = \phi(x)$.

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This description is plausible if the equilibrium states are stable. This will be so if the matrix $g_y(x, y)$ is stable for $y = \phi(x)$, $a \leqslant x \leqslant b$, i.e., if all its eigenvalues have negative real parts. Suppose, on the other hand, that $g_y(x, \phi(x))$ is stable only over part of the interval, say for $a \leqslant x < c$, and unstable if $c < x \leqslant b$. Suppose, further, that the point $(c, \phi(c))$ is a bifurcation point of equilibrium solutions of (1)—in particular, that the equation $g(x, y) = 0$ has a second solution $y = \psi(x)$ on $[a, b]$ with $\psi(c) = \phi(c)$, but $\psi(x) \neq \phi(x)$ if $x \neq c$. Finally, suppose this second series of equilibrium states is *unstable* on $[a, c)$, *stable* on $(c, b]$. Then an exchange *of stabilities* is said to occur at the bifurcation point. It is natural to suppose that the evolution of the system proceeds along the path $(x, \phi(x))$ until $x = c$, and along $(x, \psi(x))$ thereafter.

The purpose of the present article is to examine the validity of this supposition. That an exchange of stabilities is to be expected at a bifurcation point seems first to have been recognized by Poincaré [1]; it is a commonly encountered feature in stability problems containing a parameter (cf. [2]). Furthermore, the notion that a parameter in a physical or biological problem should change on a time scale long compared to the natural time scales of system (1) underlies the formulation of many such problems (including the problem to which Poincaré's memoir [1] was addressed; cf. also [3], [4]).

The description above of the evolution of the system through a series of equilibrium states is a contradiction in terms. In order to make sense out of it, we must modify the system (1) so that x is no longer a parameter, but a slowly varying function of time. The way to do this that conforms with the naturally arising problems referred to above is to augment the system (1) as follows:

$$\frac{dx}{d\tau} = \varepsilon f(x, y), \qquad \frac{dy}{d\tau} = g(x, y). \tag{2}$$

Here ε is a number whose smallness ensures that x will change slowly. Its introduction is not arbitrary, but is related to the occurrence of two time scales, the short time scale T_1 and the long time scale T_2, by $\varepsilon = T_1/T_2$. The "parameter" x will from now on be a vector of arbitrary dimension m.

In order to put the system (2) above in a form appropriate to studying evolutionary changes (on the time scale T_2) rather than rapid changes (on the time scale T_1), we make the change of variable $t = \varepsilon\tau$, and consider the initial-value problem

$$\frac{dx}{dt} = f(x, y), \quad x(0) = a, \qquad \varepsilon\frac{dy}{dt} = g(x, y), \quad y(0) = b. \tag{3}$$

We will be interested in asymptotic solutions as $\varepsilon \to 0+$. In this, an important role is played by the *reduced problem*

$$\frac{dx}{dt} = f(x, y), \quad x(0) = a, \qquad 0 = g(x, y), \tag{4}$$

where we have dropped the initial condition on y because of the loss of the y-derivative. The second equation of (4) is that defining equilibrium solutions of (1), and will be assumed to have a solution $y = \phi(x)$ for x in some open set Ω in R^n. Then the reduced problem (4) has a solution $(x^0(t), \phi(x^0(t)))$, where x^0 is the solution of the initial-value problem

$$\frac{dx}{dt} = f(x, \phi(x)), \qquad x(0) = a. \tag{5}$$

Hence the y components of the reduced problem do indeed proceed through equilibrium solutions of (1). What is needed is a theorem asserting that the solution of the full problem (3) approaches that of the reduced problem (4) as $\varepsilon \to 0+$.

This singularly perturbed initial-value problem has been widely studied [5–7]. Hoppensteadt [8] presents a concise formulation along with generalizations in certain directions. Although we are interested here in the autonomous system (3), it will prove useful to have Theorem 1 below formulated more generally. Consider then the system

$$\frac{dx}{dt} = f(t, x, y, \varepsilon), \qquad \varepsilon\frac{dy}{dt} = g(t, x, y, \varepsilon), \tag{6}$$

together with initial conditions $x(t_0) = a$, $y(t_0) = b$; and the corresponding reduced problem

$$\frac{dx}{dt} = f(t, x, y, 0), \qquad 0 = g(t, x, y, 0), \tag{7}$$

together with the initial condition $x(t_0) = a$. The following theorem may be regarded as known (cf. [8]):

THEOREM 1A. *Let f, g have continuous derivatives up to order two in an open set $D \subset R^{n+m+2}$. Let the reduced problem (7) (with $x(t_0) = a$) have a solution $(x^0(t), y^0(t))$ for $t_0 \leqslant t \leqslant t_1$, with $(t, x^0(t), y^0(t), 0)$ in D. Suppose that each eigenvalue λ of the Jacobian matrix $g_y(t, x^0(t), y^0(t), 0)$ satisfies* $\operatorname{Re}\lambda \leqslant -\mu$ *for some $\mu > 0$ and each $t \in [t_0, t_1]$.*

Let $(x(t, \varepsilon), y(t, \varepsilon))$ be the solution of (6) reducing to (a, b) for $t = t_0$. Then if $|b - y^0(t_0)|$ is sufficiently small,

$$x(t, \varepsilon) = x^0(t) + O(\varepsilon) \quad \textit{uniformly on} \quad [t_0, t_1] \textit{ and}$$

$$y(t, \varepsilon) = y^0(t) + O(\varepsilon) \textit{ uniformly on } [t_0 + \delta, t_1] \textit{ for any } \delta \textit{ in } (0, t_1 - t_0).$$

We'll also need a variant of Theorem 1A which we prove in Sec. 3 below, since we have not found an explicit statement or proof of it in the literature, although it is implicit in several of the references already given (cf. [7], [8]). Let $\chi(s)$ be a positive, continuous, decreasing function on some interval $(0, s_0)$ with $s_0 > 0$, and suppose $\chi(s)/s \to +\infty$ as $s \to 0+$. In terms of this function χ, we state the variant as

THEOREM 1B. *Let the first paragraph of Theorem 1A hold, and suppose* $|x(t_0, \varepsilon) - x^0(t_0)| + |y(t_0, \varepsilon) - y^0(t_0)| \leqslant \chi(\varepsilon)$, *for sufficiently small* ε. *Then* $|x(t, \varepsilon) - x^0(t)| + |y(t, \varepsilon) - y^0(t)| = O(\chi(\varepsilon))$ *uniformly on* $[t_0, t_1]$, *for all sufficiently small* ε.

Theorem 1A is just what is needed to justify the heuristic description of evolution based on Eq. (1) above, in the case where the reduced path does not encounter a point of bifurcation; the condition on the eigenvalues implies, among other things, that the reduced path is locally unique.

What is needed in the case where the reduced path *does* encounter a point of bifurcation where an exchange of stabilities occurs, is an analogue of Theorem 1A that exploits the circumstance that a reduced path can be defined on which the eigenvalues all have negative real parts except at a single point (the point of bifurcation), where one eigenvalue is zero. This is the case when the Jacobian matrix g_y, which is singular at the point of bifurcation, has rank $n - 1$ there. A precise formulation of the problem to be solved is given in the following section. Here we simply observe that, of the infinitely many kinds of bifurcation that can occur, we have selected the simplest: two curves, denoted by ϕ and ψ, cross each other with finite slopes (cf. Fig. 1 for the case $n = m = 1$). The principal result of this paper, Theorem 2, justifies the heuristic description of evolution, *provided* a certain criterion is satisfied. Moreover, it's clear that the heuristic description fails if the criterion is violated.

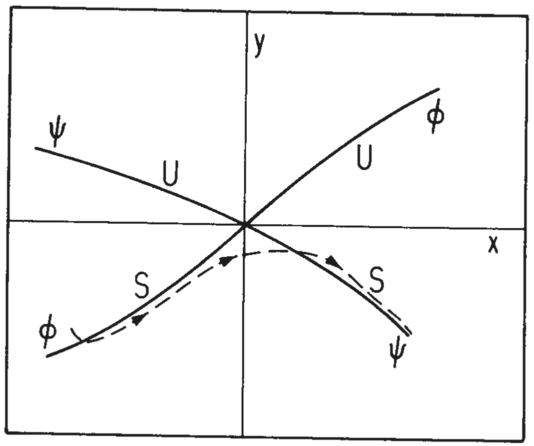

Figure 1. The equilibrium solutions $y = \phi(x)$ and $y = \psi(x)$ are shown intersecting at the origin, the various branches marked stable (s) or unstable (u). The dashed curve represents a solution of the full problem (3) with ε small but not zero.

After formulating the problem in Sec. 2, we devote Sec. 3 to certain preliminaries, including several lemmas and the proof of Theorem 1B. Theorem 2 is proved in three steps in Sec. 4, 5, and 6, which also provide more detailed information about the structure of the solution in the neighborhood of the bifurcation point than is contained in the statement of Theorem 2. The final section, Sec. 7, is devoted to a brief discussion.

2. Formulation

In this section we gather and motivate the hypotheses and give the setting for Theorem 2 below.

The reduced path determined by Eq. (5) above is supposed to encounter a point of bifurcation. We shall suppose that this bifurcation point is the origin in R^{n+m}. It is necessarily the case that g vanishes there. If f vanished there as well, the origin would be a critical point of the system (3), and this would add complications of a kind different from those we wish to study. We therefore suppose that $f(0, 0) \neq 0$. Completing $\pm f(0, 0)$ to a basis, we may suppose

H1: $f_1(0, 0) > 0, \qquad f_k(0, 0) = 0 \qquad (k = 2, 3, \ldots, m).$

Assuming (as we do) that f is continuous at the origin, we may suppose that $f_1 > 0$ in some domain containing the origin, and use x_1 instead of t as the independent variable there. Let $x = (s, \hat{x})$, where $s = x_1$ and $\hat{x}$ represents the remaining components of x. The orbit equation is then

$$\frac{d\hat{x}}{ds} = F(s, \hat{x}, y), \qquad \varepsilon \frac{dy}{ds} = G(s, \hat{x}, y), \tag{8}$$

and the reduced system is

$$\frac{d\hat{x}}{ds} = F(s, \hat{x}, y), \qquad 0 = G(s, \hat{x}, y), \tag{9}$$

where $G = g/f_1$ and $F = \hat{f}/f_1$ ($\hat{f}$ being the last $m - 1$ components of f). From now on we treat only these orbit equations, but we show in Sec. 7 how to transfer the results to the original time-dependent problem. We now suppose

H2: *F and G are of class C^2 and C^4, respectively, in an open set $E \subset R^{n+m}$ containing the origin.*

As we have already indicated, we shall suppose that the singular $n \times n$ matrix $g_y(0, 0)$ has rank $n - 1$. Since $G_y(0, 0, 0) = [f_1(0, 0)]^{-1} g_y(0, 0)$, the same is true of G_y. By appropriate choice of basis in R^n affecting the y variables only, we may suppose

H3: $G_y(0, 0, 0) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix},$

where A is an $(n-1)\times(n-1)$ *matrix whose eigenvalues all have negative real parts.*

This has the effect of distinguishing the first component of y, and we shall write $y = (y_1, \hat{y})$; furthermore, we freely interchange these notations as well as the previous notations $x = (s, \hat{x})$. For example,

$$G(x, y) = G(s, \hat{x}, y) = G(s, \hat{x}, y_1, \hat{y}) = G(x, y_1, \hat{y}).$$

Now consider solving the second of the equations (9). The $n-1$ equations

$$G_2(x, y_1, \hat{y}) = 0, \qquad G_3(x, y_1, \hat{y}) = 0, \ldots, \qquad G_n(x, y_1, \hat{y}) = 0$$

can be solved uniquely for $\hat{y}$ as a function of x and y_1 in a neighborhood of the origin, because the relevant Jacobian matrix is the nonsingular matrix A of H3. Let $w(x, y_1)$ be this unique solution (vanishing with x and y_1); w has continuous derivatives up to order four. The determination of the solutions of the equation $G = 0$ is now completed by studying the function $G_1(x, y)$ and the equation

$$G_1(x, y_1, w(x, y_1)) = 0. \tag{10}$$

We have already assumed that G_1 and $\partial G_1/\partial y_1$ vanish at the origin of R^{n+m}. We now suppose

H4: $\partial^2 G_1/\partial y_1^2 > 0$ *at the origin, and Eq.* (10) *has, in some neighborhood* Ω *of* $x = 0$, *two solutions* $y_1 = \phi_1(x)$ *and* $y_1 = \psi_1(x)$ *vanishing at* $x = 0$ *and of class* C^2 *in* Ω. *Putting* $\alpha_1 = \partial\phi_1/\partial x_1(0)$ *and* $\beta_1 = \partial\psi_1/\partial x_1(0)$, *we have* $\alpha_1 > \beta_1$.

The solution $y_1 = \phi_1(x)$ generates a solution $y = \phi(x)$ of the equation $G(x, y) = 0$ through the relation $\hat{y} = \hat{\phi}(x) \equiv w(x, \phi_1(x))$, and the solution $y_1 = \psi_1(x)$ similarly generates a solution $y = \psi(x)$. These two solutions provide two solutions to the reduced problem (9), one obtained by integrating the system

$$\frac{d\hat{x}}{ds} = F(s, \hat{x}, \phi(s, \hat{x})), \tag{11}$$

another by integrating the system

$$\frac{d\hat{x}}{ds} = F(s, \hat{x}, \psi(s, \hat{x})). \tag{12}$$

Hypothesis H4 has been so formulated that the reduced path based on Eq. (11) is stable if $s < 0$, whereas that based on Eq. (12) is stable if $s > 0$, in conformity with Fig. 1 above (these statements will be evident from the Taylor series expansions given below). This is useful in making our final hypothesis, regarding initial conditions. Suppose Eq. (11) is solved under initial data imposed at some point $s_1 < 0$, such that the reduced path $(\hat{x}^0(s), \phi(s, \hat{x}^0(s)))$ goes through the origin at $s = 0$ [these data can be found by solving (11) with initial data $x = 0$ at $s = 0$, and then setting $s = s_1 < 0$]. As long as $s_1 \leqslant s \leqslant a_1$, where $a_1 < 0$, the eigenvalue condition of Theorem 1A will be satisfied. Hence, assuming the initial data for the full problem (8) at s_1 are chosen close enough to the initial data for the reduced problem (9), we infer from Theorem 1A that the difference between the solutions of the full and reduced problems is $O(\varepsilon)$ at a_1 for any $a_1 < 0$ (and sufficiently small in absolute value that H2 and H4 are satisfied). We may therefore choose any such a_1 as our initial point. We state this as

H5: *For any* $a_1 < 0$ *and sufficiently small in absolute value, the initial conditions for the system* (8) *are*

$$\hat{x}(a_1) = \hat{x}^0(a_1) + \varepsilon A_1(a_1\varepsilon), \; y(a_1) = \phi(a_1, \hat{x}^0(a_1)) + \varepsilon B_1(a_1, \varepsilon), \tag{13}$$

where, for fixed a_1, A_1 *and* B_1 *are bounded as* $\varepsilon \to 0+$.

While the reduced path $(x^0(s), \phi(s, x^0(s)))$ is the obvious choice for $s < 0$, it clearly will not do for $s > 0$. For this we define a second reduced path $(\hat{x}^1(s), \psi(s, x^1(s)))$, where x^1 is the solution of (12) satisfying the initial condition $x^1(0) = 0$. Then we choose, as a stable reduced path, $(\hat{X}(s), Y(s))$, where

$$(\hat{X}(s), Y(s)) = \begin{cases} (\hat{x}^0(s), \phi(s, \hat{x}^0(s))) & \text{if } s \leqslant 0, \\ (\hat{x}^1(s), \psi(s, x^1(s))) & \text{if } s > 0 \end{cases}. \tag{14}$$

The theorem ensuring that the heuristic description of evolution proceeds as described in Sec. 1, and as illustrated in Fig. 1, is

THEOREM 2. *Suppose* H1–H5 *are satisfied. Then if*

$$\alpha_1 > 0,$$

there exist $a_1 < 0$, $b_1 > 0$, $C > 0$, *and* $\varepsilon^* > 0$ *such that the solution of* (8) *having initial conditions at* a_1 *satisfying* (13) *exists on* $[a_1, b_1]$ *and satisfies*

$$|\hat{x}(s, \varepsilon) - \hat{X}(s)| + |y(s, \varepsilon) - Y(s)| \leqslant C\varepsilon^{1/2}$$

there, provided $0 < \varepsilon < \varepsilon^*$.

This theorem will be proved in a series of steps in Secs. 4, 5, and 6, which provide further information than is contained in the theorem. On the basis of the theorem and Theorem 1A above, we see that the reduced path (14) is within order ε of the true path from some initial point s_1 to the left of a_1, and is within order $\varepsilon^{1/2}$ throughout the interval $[a_1, b_1]$. At b_1, a further application of Theorem 1A shows that the reduced path is again within order ε of the true path immediately thereafter. Hence the effect of the bifurcation is merely to weaken the order of approximation, in a small interval, to $O(\varepsilon^{1/2})$, provided $\alpha_1 > 0$. If $\alpha_1 \leqslant 0$, the conclusion of Theorem 2 is certainly not true in general; this is easily seen by a phase-plane analysis of the case $n = m = 1$, along the lines employed in Sec. 5.

We conclude this section with Taylor formulae for certain of the functions introduced. Superscripts on a remainder term give the order of the term; e.g., $G^{(2)}(x, y) = O((|x| + |y|)^2)$. The following formulae are simple consequences of the assumptions placed on $G(x,y)$ by H2, H3, and H4:

$$\hat{G}(x, y_1, \hat{y}) = \Gamma x + A\hat{y} + \hat{G}^{(2)}(x,y) \tag{15}$$

and

$$w(x, y_1) = Wx + w^{(2)}(x, y_1), \tag{16}$$

where $\hat{G}$ refers as usual to the last $n-1$ components of G, Γ is the constant $(n-1) \times m$ matrix $G_x(0,0)$, and $W = -A^{-1}\Gamma$. In expanding $G_1(x,y) = G_1(x, y_1, \hat{y})$, we observe that this function and *all* its first partial derivatives vanish at the origin: $G_{1,y}(0,0) = 0$ in virtue of H3, and, since $G_1(x, \phi(x)) \equiv 0$ and ϕ is differentiable (cf. H4), we find

$$G_{1,x}(0,0) + G_{1,y}(0,0)\phi_x(0) = 0,$$

implying $G_{1,x}(0,0) = 0$. Hence G_1 begins with quadratic terms in (x,y). One of these quadratic terms is by_1^2, where $b = \frac{1}{2}\partial^2 G_1/\partial y_1^2(0,0) > 0$ by H4. Distinguishing y_1, we find

$$G_1(x, y_1, \hat{y}) = b\{y_1^2 + y_1 L(x,\hat{y}) + Q(x,\hat{y})\} + G_1^{(3)}(x,y) \tag{17}$$

and

$$G_{1,y_1}(x, y_1, \hat{y}) = b\{2y_1 + L(x,\hat{y})\} + G_1^{(2)}(x,y), \tag{18}$$

where L and Q are respectively linear and quadratic in the expressed variables. Substituting $\hat{y} = w(x, y_1)$ in these formulae and using Eq. (16) above, we find

$$G_1(x, y_1, w(x,y_1)) = b\{y_1^2 + (l,x)y_1 + (x,qx)\} + \tilde{G}_1^{(3)}(x,y) \tag{19}$$

and

$$G_{1,y_1}(x, y_1, w(x,y_1)) = b\{2y_1 + (l,x)\} + \tilde{G}_1^{(2)}(x,y),$$

where the vector l and the symmetric matrix q are defined by the equations

$$(l,x) = L(x, Wx), \qquad (x,qx) = Q(x, Wx).$$

We now evaluate G_1 and G_{1,y_1} along the reduced paths satisfying (11) and (12). Since F vanishes at the origin, both $\hat{x}^0(s)$ and $\hat{x}^1(s)$ are $O(s^2)$ as $s \to 0$, and, by (16) above, we easily find

$$\big(\hat{X}(s), Y(s)\big) = \big(\hat{X}(s), Y_1(s), \hat{Y}(s)\big) = \begin{cases} (0, \alpha_1 s, \hat{\alpha}s) + O(s^2) & (s \to 0-) \\ (0, \beta_1 s, \hat{\alpha}s) + O(s^2) & (s \to 0+) \end{cases}, \tag{20}$$

where

$$\hat{\alpha} = W\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and we have represented the paths in question by giving the stable reduced path (14). Referring now to Eq. (19), we find (along $\hat{x}^0$, say)

$$0 = b(\alpha_1^2 + \alpha_1 l_1 + q_{11})s^2 + O(s^3) \qquad (s \to 0),$$

which relates α_1 (and β_1) to l_1 and q_{11}:

$$2\alpha_1 = \left[-l_1 + (l_1^2 - 4q_{11})^{1/2}\right], \qquad 2\beta_1 = \left[-l_1 - (l_1^2 - 4q_{11})^{1/2}\right]. \tag{21}$$

Evaluating $\partial G_1/\partial y_1$ along these paths, we find, first along the reduced path $\hat{x}^0$ belonging to ϕ,

$$\begin{aligned} G_{1,y_1}(s, \hat{x}(s), \phi(s, \hat{x}(s))) &= b(2\alpha_1 s + l_1 s) + O(s^2) \\ &= b(l_1^2 - 4q_{11})^{1/2} s + O(s^2) \\ &= b(\alpha_1 - \beta_1)s + O(s^2); \end{aligned}$$

along the reduced path for ψ we find

$$G_{1,y_1}(s, \hat{x}(s), \psi(s, \hat{x}(s))) = -b(\alpha_1 - \beta_1)s + O(s^2).$$

Since $s < 0$ on the reduced path ϕ and $s > 0$ on the reduced path ψ, G_{1,y_1} is negative near $s = 0$ on either path, and hence, on the overall reduced path $\big(\hat{X}(s), Y(s)\big)$, G_{1,y_1} is negative except at $s = 0$. Choosing $0 < 2\nu < b(\alpha_1 - \beta_1)$, we have, on a suitably restricted s-interval,

$$G_{1,y_1}\big(s, \hat{X}(s), Y(s)\big) < \begin{cases} 2\nu s & (s < 0) \\ -2\nu s & (s > 0) \end{cases}. \tag{22}$$

We may suppose the domain Ω (cf. H4) so chosen that s is restricted to an interval containing the origin on which (22) holds.

3. Preliminaries

In this section we present three lemmas needed in the proof of Theorems 1B and 2, and we prove Theorem 1B. This theorem is implicit in the work of others (cf. [7]). Our proof makes it explicit, and uses, in a relatively simple context, the method used in the more complicated contexts of Secs. 4 and 6.

We collect certain elementary estimates in

LEMMA 1. *Let* $\nu, \varepsilon > 0$.

(a) *If* $a < s < 0$, *then*

$$\int_a^s e^{\nu(\sigma - s)/\varepsilon}\, d\sigma \leqslant \frac{\varepsilon}{\nu}, \qquad \int_a^s e^{\nu(s^2 - \sigma^2)/\varepsilon}\, d\sigma \leqslant \frac{\varepsilon}{2\nu|s|}.$$

(b) *If* $0 < s < b$, *then*

$$\int_s^b e^{-\nu(b-\sigma)/\varepsilon}\, d\sigma \leqslant \frac{\varepsilon}{\nu}, \qquad \int_s^b e^{-\nu(b^2 - \sigma^2)/\varepsilon}\, d\sigma \leqslant \frac{\varepsilon}{2\nu s}.$$

Proof: The first integral of (a) can be integrated explicitly. For the second, note that $\sigma < s < 0$, so $|\sigma| > |s|$, and

$$\int_a^s e^{\nu(s^2-\sigma^2)/\varepsilon}\, d\sigma = \int_a^s \frac{1}{|\sigma|}\cdot|\sigma|e^{-\nu(s^2-\sigma^2)/\varepsilon}\, d\sigma \leqslant \frac{1}{|s|}\int_a^s e^{\nu(s^2-\sigma^2)/\varepsilon}|\sigma|\, d\sigma.$$

The remaining integral can be integrated explicitly, and the result is as stated. The proof of part (b) is similar.

We need estimates on the norms of certain fundamental matrix solutions of differential equations of the form

$$\frac{dx}{dt} = \frac{1}{\varepsilon}Q(t)x. \tag{23}$$

These estimates are provided by the following lemma, which is due to Flatto and Levinson [9].

Lemma 2. *Let $Q(t)$ be a continuous $N \times N$ matrix on $[a, b]$. Suppose the real parts of the eigenvalues of Q are all less than -2μ for some $\mu > 0$. Let $\phi(t, s, \varepsilon)$ be the fundamental matrix solution of (23) reducing to the identity when $t = s$. Then there exist positive constants K, ε_0 such that*

$$|\phi(t, s, \varepsilon)| \leqslant Ke^{-\mu(t-s)/\varepsilon}, \qquad a \leqslant s \leqslant t \leqslant b, \quad 0 < \varepsilon < \varepsilon_0.$$

The following elementary lemma will be needed.

Lemma 3. *Let $f \in C^1[a, b]$, $g \in C[a, b]$. Suppose $g \geqslant 0$, $f \leqslant 0$, and $f(b) = 0$. Then for any $\varepsilon > 0$*

$$\int_a^b e^{f(s)/\varepsilon} f'(s)\left[\int_a^s g(u)\,du\right] ds \leqslant \varepsilon\int_a^b g(u)\,du. \tag{24}$$

Proof: Integration by parts gives for the left-hand side

$$\varepsilon\left[e^{f(s)/\varepsilon}\int_a^s g(u)\,du\right]_a^b - \varepsilon\int_a^b e^{f(s)/\varepsilon} g(s)\,ds$$

$$= \varepsilon\int_a^b (1 - e^{f(s)/\varepsilon})g(s)\,ds \leqslant \varepsilon\int_a^b g(s)\,ds,$$

since $0 < 1 - e^{f(s)/\varepsilon} \leqslant 1$ and $g > 0$.

We conclude this section with the proof of Theorem 1B. We first establish some notation and definitions. We have not as yet specified what we mean by $|x|$, i.e., we have not defined the norm of x. It will be convenient to use, for vector x and matrix A,

$$|x| = \sum_{i=1}^n |x_i|, \qquad |A| = \sum_{i=1}^n\sum_{j=1}^n |A_{ij}|.$$

The equations (7) above are assumed to have a solution $(x^0(t), y^0(t))$ on an interval $[t_0, t_1]$.

We denote by (ξ, η) the difference $(x - x^0, y - y^0)$ between a solution of the full system (6) and the solution (x^0, y^0) of the reduced system (7). Then (ξ, η) satisfy the system of equations

$$\begin{aligned}\frac{d\xi}{dt} &= f_x(t)\xi + f_y(t)\eta + R(t, \xi, \eta, \varepsilon),\\ \varepsilon\frac{d\eta}{dt} &= g_x(t)\xi + g_y(t)\eta + S(t, \xi, \eta, \varepsilon),\end{aligned} \tag{25}$$

where

$$f_x(t) \equiv f_x(t, x^0(t), y^0(t), 0), \quad \text{etc}, \tag{26}$$

and R and S are remainder terms in the Taylor formulae for f and g, respectively. They are easily seen to satisfy the following estimate for some choice of the positive constants K, δ, ε_0:

$$|R(t, \xi, \eta, \varepsilon)| \leqslant K[\varepsilon + |\xi|^2 + |\eta|^2] \tag{27}$$

if $t_0 \leqslant t \leqslant t_1$, $|\xi| + |\eta| < \delta$, and $\varepsilon < \varepsilon_0$, the same estimate holding for S.

Proof of Theorem 1B: Let $U(t, s)$, $V(t, s, \varepsilon)$ be the fundamental matrix solutions of

$$\frac{dU}{dt} = f_x(t)U, \qquad \frac{dV}{dt} = \varepsilon^{-1}g_y(t)V$$

that reduce to the identities (of the appropriate dimensions m and n) when $t = s$. Then the equations (25) above, together with initial conditions, may be written

$$\xi(t) = U(t, t_0)\xi(0) + \int_{t_0}^t U(t, s)\{R(s, \xi, \eta, \varepsilon) + f_y(s)\eta(s)\}\,ds$$

and

$$\eta(t) = V(t, t_0, \varepsilon)\eta(0) + \varepsilon^{-1}\int_{t_0}^t V(t, s, \varepsilon)$$
$$\times\{S(s, \xi, \eta, \varepsilon) + g_x(s)\xi(s)\}\,ds.$$

We may assume (cf. Lemma 2) that for some $\nu > 0$ and $K > 1$

$$\begin{aligned}&|U(t, s)| \leqslant K, \quad |V(t, s, \varepsilon)| \leqslant Ke^{-\nu(t-s)/\varepsilon} \quad \text{if } t_0 \leqslant s \leqslant t \leqslant t_1,\\ &|f_y(s)| \leqslant K, \quad |g_x(s)| \leqslant K \quad \text{if } t_0 \leqslant s \leqslant t,\end{aligned} \tag{28}$$

as well as (27); we may also write

$$|\xi(0)| \leqslant K\chi(\varepsilon), \qquad |\eta(0)| \leqslant K\chi(\varepsilon).$$

We now obtain from the integral equations for ξ and η

$$|\xi(t)| \leqslant K^2\chi(\varepsilon) + K\int_{t_0}^{t}|R(s, \xi, \eta, \varepsilon)|ds + K^2\int_{t_0}^{t}|\eta(s)|ds$$

and

$$|\eta(t)| \leqslant K^2\chi(\varepsilon) + \frac{K}{\varepsilon}\int_{t_0}^{t}e^{-\nu(t-s)/\varepsilon}|S(s, \xi, \eta, \varepsilon)|ds + \frac{K^2}{\varepsilon}\int_{t_0}^{t}e^{-\nu(t-s)/\varepsilon}|\xi(s)|ds.$$

Now we define $T = t_1 - t_0$ and

$$L = 2\left(1 + T + \frac{1}{\nu}\right)K^2\left\{1 + \left(1 + \frac{K^2}{\nu}\right)(1 + K^2T)e^{K^4T}\right\},$$

and we choose ε_1 so small that for $\varepsilon < \varepsilon_1$ the following inequalities hold:

$$\varepsilon < \varepsilon_0, \qquad L^2\chi(\varepsilon) < 1, \qquad L\chi(\varepsilon) < \delta, \qquad \varepsilon < \chi(\varepsilon).$$

Here ε_0 and δ are the numbers figuring in the estimate (27), and the possibility of so selecting ε_1 follows from the definition of χ. We shall show that

$$|\xi(t)| + |\eta(t)| < L\chi(\varepsilon) \tag{29}$$

for $t_0 \leqslant t \leqslant t_1$, provided $\varepsilon < \varepsilon_1$.

Since $L > 2K$, (29) is satisfied if $t = t_0$. It follows that the system (25) has a solution, with initial conditions satisfying the hypothesis of the theorem, that satisfies (29) on some interval to the right of t_0. If this interval does not extend to t_1, then there is some $t_2 \leqslant t_1$ such that (29) holds on the (maximal) interval $[t_0, t_2)$ and becomes an equality at t_2. Suppose this is so. On the interval $[t_0, t_2)$, (29) holds, and, for $\varepsilon < \varepsilon_1$, $|\xi| + |\eta| < \delta$, and (27) holds as well. It follows that

$$|\xi(t)| \leqslant K^2\left\{\chi(\varepsilon) + \left[\varepsilon + L^2\chi(\varepsilon)^2\right]T\right\} + K^2\int_{t_0}^{t}|\eta(s)|ds$$

and

$$|\eta(t)| \leqslant K^2\left\{\chi(\varepsilon) + \left[\varepsilon + L^2\chi(\varepsilon)^2\right]\frac{1}{\nu}\right\} + \frac{K^2}{\varepsilon}\int_{t_0}^{t}e^{-\nu(t-s)/\varepsilon}|\xi(s)|ds.$$

Let

$$K_1(\varepsilon) = K^2\chi(\varepsilon)\left\{1 + \left(T + \frac{1}{\nu}\right)[1 + L^2\chi(\varepsilon)]\right\} < K^2\chi(\varepsilon)\left\{1 + 2\left(T + \frac{1}{\nu}\right)\right\}.$$

Since $\varepsilon < \chi(\varepsilon)$, K_1 is larger than either of the constant terms in the above inequalities, and we may write

$$|\xi(t)| \leqslant K_1(\varepsilon) + K^2\int_{t_0}^{t}|\eta(s)|ds,$$

$$|\eta(t)| \leqslant K_1(\varepsilon) + \frac{K^2}{\varepsilon}\int_{t_0}^{t}e^{-\nu(t-s)/\varepsilon}|\xi(s)|ds,$$

for $t_0 \leqslant t \leqslant t_2$ (since these inequalities are preserved in the limit $t \to t_2 -$). Substituting from the first of these into the second, we find

$$|\eta(t)| \leqslant K_1(\varepsilon) + \nu^{-1}K^2K_1(\varepsilon) + \frac{K^4}{\varepsilon}\int_{t_0}^{t}e^{-\nu(t-s)/\varepsilon}\left[\int_{t_0}^{s}|\eta(u)|du\right]ds$$

$$\leqslant K_1(\varepsilon)(1 + \nu^{-1}K^2) + K^4\int_{t_0}^{t}|\eta(s)|ds$$

by Lemma 3. It follows from Gronwall's lemma that

$$|\eta(t)| \leqslant K_1(\varepsilon)(1 + \nu^{-1}K^2)e^{K^4T}.$$

The preceding inequality for $|\xi(t)|$ now gives

$$|\xi(t)| \leqslant K_1(\varepsilon) + K^2K_1(\varepsilon)(1 + \nu^{-1}K^2)e^{K^4T}T,$$

so

$$|\xi(t)| + |\eta(t)| \leqslant K_1(\varepsilon)\{1 + (1 + \nu^{-1}K^2)e^{K^4T}(1 + K^2T)\}$$

$$< K^2\chi(\varepsilon)\left[1 + 2\left(T + \frac{1}{\nu}\right)\right]\{1 + (1 + \nu^{-1}K^2)e^{K^4T}(1 + K^2T)\}$$

$$< L\chi(\varepsilon)$$

for $t_0 \leqslant t \leqslant t_2$. This contradicts the assumption that $|\xi(t_2)| + |\eta(t_2)| = L\chi(\varepsilon)$, and proves the theorem.

4. Entering the transition interval

In this section, estimates for $|\hat{x}(s, \varepsilon) - \hat{X}(s)|$, $|y_1(s, \varepsilon) - Y_1(s)|$, and $|\hat{y}(s, \varepsilon) - \hat{Y}(s)|$ are obtained on an interval of the form $[a_1, -\zeta_0\varepsilon^{1/2}]$. Here $\hat{x}(s, \varepsilon)$ and $y(s, \varepsilon) \equiv (y_1(s, \varepsilon), \hat{y}(s, \varepsilon))$ are the solutions of the system (8), while $\hat{X}(s)$ and $Y(s) \equiv (Y_1(s), \hat{Y}(s))$ represent the stable reduced path defined by Eq. (14) above. The left-hand endpoint a_1 is negative and may be small in absolute value; the right-hand endpoint $-\zeta_0\varepsilon^{1/2}$ will serve as the left-hand endpoint of the transition interval $[-\zeta_0\varepsilon^{1/2}, +\zeta_1\varepsilon^{1/2}]$ to be considered in the next section. For this it is important to know that $\zeta_0 > 0$ can be chosen as large as we please.

Defining $\xi(s, \varepsilon) = \hat{x}(s, \varepsilon) - \hat{X}(s)$, $\eta(s, \varepsilon) = y(s, \varepsilon) - Y(s)$, we obtain the system (25) above with s in place of t. The estimate (27) for the error terms continues to hold with $t_0 < 0$, $t_1 > 0$, and both sufficiently small in absolute value. In the present case we must write $\eta = (\eta_1, \hat{\eta})$, obtaining in place of (25) the following system:

$$\frac{d\xi}{ds} = F_{\hat{x}}(s)\xi + F_y(s)\eta + R(s, \xi, \eta, \varepsilon),$$

$$\varepsilon \frac{d\eta_1}{ds} = G_{1,\hat{x}}(s)\xi + G_{1,y_1}(s)\eta_1 + G_{1,\hat{y}}(s)\hat{\eta} + S_1(s, \xi, \eta, \varepsilon),$$

$$\varepsilon \frac{d\hat{\eta}}{ds} = \hat{G}_{\hat{x}}(s)\xi + \hat{G}_{y_1}(s)\eta_1 + \hat{G}_{\hat{y}}(s)\eta + \hat{S}(s, \xi, \eta, \varepsilon).$$

Let $U(s, \sigma)$, $V_1(s, \sigma, \varepsilon)$, $\hat{V}(s, \sigma, \varepsilon)$ be the fundamental matrix solutions of

$$\frac{du}{ds} = F_{\hat{x}}(s)u, \qquad \frac{dv_1}{ds} = \varepsilon^{-1}G_{1,y_1}(s)v_1, \qquad \frac{d\hat{v}}{ds} = \varepsilon^{-1}\hat{G}_{\hat{y}}(s)\hat{v}$$

reducing to the unit matrices of dimensions $n - 1$, 1, $m - 1$, respectively, when $s = \sigma$. Then [cf. Eq. (22)]

$$V_1(s, \sigma, \varepsilon) = \exp\left(\varepsilon^{-1}\int_\sigma^s G_{1,y_1}(u)\,du\right) < \exp\left(\varepsilon^{-1}\int_\sigma^s 2\nu u\,du\right),$$

or

$$V_1(s, \sigma, \varepsilon) < \exp[\nu(s^2 - \sigma^2)/\varepsilon], \qquad t_0 < \sigma < s < 0 \tag{31}$$

provided $|t_0|$ is small enough, for some $\nu > 0$. Since $\hat{G}_{\hat{y}}(0) = A$ has only eigenvalues with negative real parts, the same will be true of $\hat{G}_{\hat{y}}(s)$ on $[t_0, 0]$ (in fact, on $[t_0, t_1]$ for $t_1 > 0$). Lemma 2 applies, and $\hat{V}$ satisfies the estimate satisfied by V in (28) above; we may take the coefficients ν in the estimates for V_1 and $\hat{V}$ to be the same. We may write in place of (30)

$$\xi(s) = U(s, a_1)\xi(a_1) + \int_{a_1}^s U(s, \sigma)\{R(\sigma, \xi, \eta, \varepsilon) + F_y(\sigma)\eta(\sigma)\}\,d\sigma,$$

$$\eta_1(s) = V_1(s, a_1, \varepsilon)\eta_1(a_1) + \varepsilon^{-1}\int_{a_1}^s V_1(s, \sigma, \varepsilon)\{S_1(\sigma, \xi, \eta, \varepsilon) + G_{1,\hat{x}}(\sigma)\xi(\sigma) + G_{1,\hat{y}}(\sigma)\hat{\eta}(\sigma)\}\,d\sigma \tag{32}$$

$$\hat{\eta}(s) = \hat{V}(s, a_1, \varepsilon)\hat{\eta}(a_1) + \varepsilon^{-1}\int_{a_1}^s \hat{V}(s, \sigma, \varepsilon)\{S(\sigma, \xi, \eta, \varepsilon) + \hat{G}_{\hat{x}}(\sigma)\xi(\sigma) + \hat{G}_{y_1}(\sigma)\eta_1(\sigma)\}\,d\sigma.$$

Of the quantities appearing in this system, we have already mentioned estimates for R, S_1, S, V_1, and V. We need estimates for the rest, analogous to the estimates (28). The coefficients of the linear terms are, of course, bounded as in (28), but we require more. Recall (H3) that $G_{y_1}(x, y)$ and $G_{1,y}$ vanish at the origin, and likewise $G_{1,x}(x, y)$ [cf. Eq. (18) above]. Then since $\left(\hat{X}(s), \phi(s, \hat{X}(s))\right) = O(s)$ as $s \to 0$, we find that the coefficients $G_{1,\hat{x}}(s)$, $G_{1,\hat{y}}(s)$, and $\hat{G}_{y_1}(s)$ are all $O(s)$ as $s \to 0$. We collect the required estimates here. There exist constants $\varepsilon_0 > 0$, $K > 1$, and $t_0 < 0$ (with $|t_0|$ sufficiently small) so that

$$|F_y(s)| \leq K, \qquad |\hat{G}_{\hat{x}}(s)| \leq K, \qquad |G_{1,\hat{x}}(s)| \leq K|s|,$$

$$|G_{1,\hat{y}}(s)| \leq K|s|, \qquad |\hat{G}_{y_1}(s)| \leq K|s| \qquad \text{if } t_0 \leq s \leq 0;$$

$$|U(s, \sigma)| \leq K, \qquad |V_1(s, \sigma, \varepsilon)| \leq Ke^{\nu(s^2-\sigma^2)/\varepsilon},$$

$$|\hat{V}(s, \sigma, \varepsilon)| \leq Ke^{-\nu(s-\sigma)/\varepsilon} \qquad \text{if } t_0 \leq s \leq \sigma \leq 0 \text{ and } 0 < \varepsilon < \varepsilon_0; \tag{33}$$

$$|R(s, \xi, \eta, \varepsilon)| \leq K(\varepsilon + |\xi|^2 + |\eta|^2) \quad \left(\text{and similarly for } S_1, \hat{S}\right)$$

$$\text{if } t_0 \leq s \leq 0,\ |\xi| + |\eta| < \delta, \text{ and } 0 < \varepsilon < \varepsilon_0.$$

The common constant K appearing in these estimates does not depend on the initial point a_1 as long as $t_0 \leq a_1 < 0$. On the other hand, H5 provides the estimates

$$|\xi(a_1)| \leq C(a_1)\varepsilon, \qquad |\eta(a_1)| \leq C(a_1)\varepsilon, \tag{34}$$

with a constant C that depends, in general, on a_1.

Any solution of the system (30) must satisfy the following system of inequalities for $s > a_1$:

$$|\xi(s)| \leq KC\varepsilon + K\int_{a_1}^s \{|R(\sigma, \xi, \eta, \varepsilon)| + K|\eta(\sigma)|\}\,d\sigma,$$

$$|\eta_1(s)| \leq K|\eta_1(a_1)| + \varepsilon^{-1}K\int_{a_1}^s e^{\nu(s^2-\sigma^2)/\varepsilon}\{|S_1(\sigma, \xi, \eta, \varepsilon)| + K|\sigma|\,|\xi(\sigma)| + K|\sigma|\,|\hat{\eta}(\sigma)|\}\,d\sigma, \tag{35}$$

$$|\hat{\eta}(s)| \leq K|\hat{\eta}(a_1)| + \varepsilon^{-1}K\int_{a_1}^s e^{-\nu(s-\sigma)/\varepsilon}\{|\hat{S}(\sigma, \xi, \eta, \varepsilon)| + K|\xi(\sigma)| + K|\sigma|\,|\eta_1(\sigma)|\}\,d\sigma.$$

The required estimates are now stated as a theorem.

THEOREM 3. *Suppose inequalities* (33) *hold. Then there exist* $a_1 < 0$ *and positive constants* L, M, $\hat{M}$, ζ_0, and ε_1 such that the solution $(\xi, \eta_1, \hat{\eta})$ of (30) with initial data at a_1 satisfying (34) exists on the interval $[a_1, -\zeta_0\varepsilon^{1/2}]$ and satisfies there the inequalities

$$|\xi(s)| < L\varepsilon \ln(|s|^{-1}), \qquad |\hat{\eta}(s)| < \hat{M}\varepsilon \ln(|s|^{-1}), \quad \text{and} \quad |\eta|(s)| < M\varepsilon|s|^{-1} \tag{36}$$

for all ε *in the interval* $(0, \varepsilon_1)$. *The constant* ζ_0 *may be chosen arbitrarily provided it exceeds some minimal value, the choice of* ε_1 *depending on that of* ζ_0.

Proof: Choose $a_1 < 0$ so that the following inequalities hold:

$$|a_1| < |t_0|, \qquad |a_1| < \frac{1}{e}, \qquad e^{\frac{3}{2}(K^4/\nu)|a_1|}\left\{\frac{K^2}{\nu}|a_1| + \frac{K^6}{\nu^2}|a_1|\ln\left(\frac{1}{|a_1|}\right)\right\} < \frac{1}{2}. \tag{37}$$

With a_1 so chosen, the constant C appearing in (34) is fixed. Define

$$\begin{aligned} K_1 &= KC + 2K^2|a_1|, \\ K_2 &= KC + (2 + \tfrac{3}{2}K_1)K^2\nu^{-1}, \\ M &= 2e^{\frac{3}{2}(K^4/\nu)|a_1|}\left\{|a_1|K_2 + \frac{K^2}{\nu} + \frac{K^2}{2\nu}K_2|a_1|\ln\left(\frac{1}{|a_1|}\right)\right\}, \\ \hat{M} &= K_2 + 2K^4M\nu^{-1}, \\ L &= K_1 + K^2M, \end{aligned} \tag{38}$$

$$\zeta_0 \geqslant \sqrt{L^2 + M^2},$$

$$\varepsilon_1 = \min\left(\varepsilon_0, \left(\frac{\delta\zeta_0}{L+M}\right)^2, \left(\frac{a_1}{\zeta_0}\right)^2\right),$$

where ε_0 and δ figure in the estimates (33). Note that, since $|a_1| < 1/e$, $\ln(1/|a_1|) > 1$, whereas for $|s| \leqslant |a_1|$, $|s|\ln(1/|s|) < 1$, i.e., $\ln(1/|s|) < 1/|s|$.

It is easy to verify that the definitions of L, M, $\hat{M}$ imply that (36) hold for $s = a_1$. The system (30) therefore has a solution satisfying (36) on some interval to the right of a_1. We wish to show that this interval extends to $-\zeta_0\varepsilon^{1/2}$ for $\varepsilon < \varepsilon_1$. The proof is by contradiction.

Suppose the solution satisfying (36) does not exist on $[a_1, -\zeta_0\varepsilon^{1/2}]$. Then there is a maximal interval of existence $[a_1, t_2)$ with $a_1 < t_2 \leqslant -\zeta_0\varepsilon^{1/2}$ where (36) is satisfied. At t_2 the boundary of the region defined by the inequalities (36) is attained, i.e., at least one of them becomes an equality. On $[a_1, t_2]$,

$$|\xi| + |\eta| \leqslant \varepsilon\left(M\frac{1}{|s|} + L\ln\frac{1}{|s|}\right)$$

$$\leqslant \frac{M+L}{|s|}\varepsilon \leqslant \frac{M+L}{|t_2|}\varepsilon < \frac{M+L}{\zeta_0\varepsilon^{1/2}}\varepsilon < \left(\frac{M+L}{\zeta_0}\right)\varepsilon_1^{1/2} < \delta,$$

so the estimates given in (33) for R, S_1, and S hold. It follows that [again we use the inequality $\ln(|s|^{-1}) < |s|^{-1}$]

$$|R(s, \xi(s), \eta(s), \varepsilon)| \leqslant K(\varepsilon + |\xi|^2 + |\eta|^2)$$

$$\leqslant K\left(\varepsilon + \frac{L^2+M^2}{|s|^2}\varepsilon^2\right) \leqslant K\left(\varepsilon + \frac{L^2+M^2}{\zeta_0^2\varepsilon}\varepsilon^2\right) \leqslant 2K\varepsilon,$$

the same estimate holding for S_1 and $\hat{S}$, on $[a_1, t_2]$. Using the last inequality and Lemma 1, we find in place of (35) the following system of inequalities for $a_1 \leqslant s \leqslant t_2$:

$$\begin{aligned} |\xi(s)| &\leqslant K_1\varepsilon + K^2\int_{a_1}^{s}|\eta(\sigma)|d\sigma, \\ |\eta_1(s)| &\leqslant K|\eta_1(a_1)| + \nu^{-1}K^2\varepsilon|s|^{-1} \\ &\quad + K^2\varepsilon^{-1}\int_{a_1}^{s}e^{\nu(s^2-\sigma^2)/\varepsilon}|\sigma|\{|\xi(\sigma)| + |\eta(\sigma)|\}d\sigma, \\ |\hat{\eta}(s)| &\leqslant K|\hat{\eta}(a_1)| + 2\nu^{-1}K^2\varepsilon \\ &\quad + K^2\varepsilon^{-1}\int_{a_1}^{s}e^{\nu(s-\sigma)/\varepsilon}\{|\xi(\sigma)| + |\sigma|\,|\eta_1(\sigma)|\}d\sigma. \end{aligned} \tag{39}$$

Substituting for $|\xi(\sigma)|$ in the inequalities for $|\eta_1|$ and $|\hat{\eta}|$, we have, in view of Lemma 3,

$$\begin{aligned} |\eta_1(s)| &\leqslant K|\eta_1(a_1)| + \nu^{-1}K^2\varepsilon|s|^{-1} + K^2K_1\varepsilon(2\nu)^{-1} \\ &\quad + K^2\varepsilon^{-1}\int_{a_1}^{s}|\sigma|e^{\nu(s^2-\sigma^2)/\varepsilon}|\hat{\eta}(\sigma)|d\sigma + K^4(2\nu)^{-1}\int_{a_1}^{s}|\eta(\sigma)|d\sigma \\ \text{and} \\ |\hat{\eta}(s)| &\leqslant K|\hat{\eta}(a_1)| + (2 + K_1)K^2\nu^{-1}\varepsilon \\ &\quad + K^2\varepsilon^{-1}\int_{a_1}^{s}|\sigma|e^{-\nu(s-\sigma)/\varepsilon}|\eta_1(\sigma)|d\sigma + K^4\nu^{-1}\int_{a_1}^{s}|\eta(\sigma)|d\sigma. \end{aligned} \tag{40}$$

The inequalities (36) hold on $[a_1, t_2)$, so we find

$$\int_{a_1}^{s}|\sigma|e^{\nu(s^2-\sigma^2)/\varepsilon}|\hat{\eta}(\sigma)|d\sigma \leqslant \frac{\hat{M}\varepsilon^2}{2\nu}\ln(|s|^{-1})$$

and

$$\int_{a_1}^{s}|\sigma|e^{-\nu(s-\sigma)/\varepsilon}|\eta_1(\sigma)|d\sigma \leqslant M\varepsilon^2\nu^{-1}.$$

We therefore have, in place of (40),

$$|\eta_1(s)| \leqslant K|\eta_1(a_1)| + K^2K_1\varepsilon(2\nu)^{-1} + K^2\hat{M}(2\nu)^{-1}\varepsilon \ln(|s|^{-1})$$
$$+ K^2\nu^{-1}\varepsilon|s|^{-1} + K^4(2\nu)^{-1}\int_{a_1}^{s}|\eta(\sigma)|\,d\sigma \tag{41}$$

and

$$|\hat{\eta}(s)| \leqslant K|\hat{\eta}(a_1)| + K^2(2 + K_1 + M)\varepsilon\nu^{-1} + K^4\nu^{-1}\int_{a_1}^{s}|\eta(\sigma)|\,d\sigma.$$

Adding and recalling that $|\eta_1(a_1)| + |\hat{\eta}(a_1)| = |\eta(a_1)| < C\varepsilon$, we obtain

$$|\eta(s)| \leqslant Q(s, \varepsilon) + 3K^4(2\nu)^{-1}\int_{a_1}^{s}|\eta(\sigma)|\,d\sigma$$

where

$$Q(s, \varepsilon) = KC\varepsilon + K^2(2 + \tfrac{3}{2}K_1 + M)\varepsilon\nu^{-1}$$
$$+ K^2\nu^{-1}\varepsilon|s|^{-1} + K^2\hat{M}(2\nu)^{-1}\varepsilon \ln(|s|^{-1}).$$

An application of the generalized Gronwall's lemma now gives

$$|\eta(s)| \leqslant Q(s, \varepsilon) + 3K^4(2\nu)^{-1}\int_{a_1}^{s}Q(u, \varepsilon)e^{(3K^4/2\nu)(s-u)}\,du;$$

noting that $Q(u, \varepsilon)$ is increasing on $[a_1, s]$, we find

$$|\eta(s)| \leqslant Q(s, \varepsilon)e^{(3K^4/2\nu)(s-a_1)} < Q(s, \varepsilon)e^{(3K^4/2\nu)|a_1|}, \tag{42}$$

the inequality (42) holding on $[a_1, t_2]$.

As remarked, one at least of the inequalities (36) must be an equality at $s = t_2$. Consider the last of them first. If it becomes an equality at t_2, then

$$M\varepsilon|t_2|^{-1} = |\eta(t_2)| < e^{\frac{3}{2}(K^4/\nu)|a_1|}Q(t_2, \varepsilon),$$

or

$$M < e^{\frac{3}{2}(K^4/\nu)|a_1|}|t_2|\{K_2 + K^2M\nu^{-1} + K^2\nu^{-1}|t_2|^{-1} + K^2\hat{M}(2\nu)^{-1}\ln(|t_2|^{-1})\}$$
$$\leqslant e^{\frac{3}{2}(K^4/\nu)|a_1|}\{|a_1|K_2 + |a_1|K^2M\nu^{-1} + K^2\nu^{-1}$$
$$+ K^2(2\nu)^{-1}|a_1|\ln(|a_1|^{-1})\cdot(K_2 + 2K^4M\nu^{-1})\},$$

where we have used the fact that $x \ln x^{-1}$ is increasing on $(0, |a_1|]$ as well as the

definition of $\hat{M}$ [Eq. (38)]. Collecting the coefficient of M, we find

$$M\left\{1 - e^{\frac{3}{2}(K^4/\nu)|a_1|}\left[|a_1|K^2\nu^{-1} + |a_1|\ln(|a_1|^{-1})K^6\nu^{-2}\right]\right\}$$
$$\leqslant e^{\frac{3}{2}(K^4/\nu)|a_1|}\left[|a_1|K_2 + K^2\nu^{-1} + |a_1|\ln(|a_1|^{-1})K^2K_2(2\nu)^{-1}\right].$$

By (37), the coefficient of M is greater than one-half, so

$$M < 2e^{\frac{3}{2}(K^4/\nu)|a_1|}\left(|a_1|K_2 + \frac{K^2}{\nu} + \frac{K^2}{2\nu}K_2|a_1|\ln\left(\frac{1}{|a_1|}\right)\right),$$

contradicting its definition [Eq. (38)]; hence it *cannot* be the third of the inequalities (36) that becomes an equality at t_2.

Suppose that $|\hat{\eta}(t_2)| = \varepsilon M \ln(|t_2|^{-1})$. Then by the second of the inequalities (41) and the third of the inequalities (36) we have on $[a_1, t_2]$ (recalling $\ln|a_1| < 0$)

$$|\hat{\eta}(s)| \leqslant KC\varepsilon + K^2\nu^{-1}\varepsilon(2 + K_1) + K^2\nu^{-1}\varepsilon M + K^4\nu^{-1}\varepsilon M \ln\frac{|a_1|}{|s|}$$
$$< K_2\varepsilon + \nu^{-1}K^2M\varepsilon\left(1 + K^2\ln\frac{1}{|s|}\right).$$

Hence

$$\hat{M} < \frac{K_2 + \nu^{-1}K^2M}{\ln(|t_2|^{-1})} + \nu^{-1}K^4M.$$

Since $\ln(|t_2|^{-1}) > 1$ and $K^2 < K^4$, we have

$$M < K_2 + 2\nu^{-1}K^4M,$$

contradicting the definition (38).

Hence it must be the first of the inequalities (36) that becomes an equality at t_2. By (39),

$$|\xi(t_2)| = \varepsilon L\ln(|t_2|^{-1}) \leqslant K_1\varepsilon + K^2M\varepsilon\ln(|a_1|/|t_2|)$$
$$< \varepsilon\{K_1 + K^2M\ln(|t_2|^{-1})\},$$

or

$$L < K_1\left[\ln(|t_2|^{-1})\right]^{-1} + K^2M < K_1 + K^2M,$$

again violating the definition of L. This shows that the solution does not arrive at the boundary of the region defined by (36) for any t_2 in $[a_1, -\zeta_0\varepsilon^{1/2}]$, and proves the theorem.

An immediate consequence of this theorem is that $|\eta_1(s)| \leqslant |\eta(s)| \leqslant M\varepsilon|s|^{-1}$ on $[a_1, -\zeta_0\varepsilon^{1/2}]$.

5. The transition interval

Theorem 3 above shows that at $s = -\zeta_0\varepsilon^{1/2}$ the difference between the full and reduced solutions is of order $\varepsilon^{1/2}$. We now wish to show that this estimate persists on an interval $[-\zeta_0\varepsilon^{1/2}, \zeta_1\varepsilon^{1/2}]$. This requires showing that the full solution "switches its allegiance" from the equilibrium branch denoted by ϕ to that denoted by ψ. This is facilitated by the following transformation of variables from (s, x, y) to (σ, u, v):

$$s = \varepsilon^{1/2}\sigma, \qquad \hat{x} = \varepsilon^{1/2}\hat{u}, \qquad y = \varepsilon^{1/2}v. \tag{43}$$

To save writing, we put

$$\mu = \varepsilon^{1/2}; \tag{44}$$

the equations (8) now become

$$\frac{d\hat{u}}{d\sigma} = F(\mu\sigma, \mu\hat{u}, \mu v)$$

$$\mu^2\frac{dv}{d\sigma} = G(\mu\sigma, \mu\hat{u}, \mu v). \tag{45}$$

Since F vanishes at the origin, $F(\mu\sigma, \mu\hat{u}, \mu v) = O(\mu)$ uniformly in $(\sigma, \hat{u}, v)$ when these variables are restricted to a bounded set containing the origin. For G, we use the expansions (16) and (18) above. Then, breaking v up into $(v_1, \hat{v})$, we easily obtain in place of the second equation of (45)

$$\frac{dv_1}{d\sigma} = b\{v_1^2 + L(\sigma, \hat{u}, \hat{v})v_1 + Q(\sigma, \hat{u}, \hat{v})\} + \tilde{G}_1^{(3)}(\sigma, \hat{u}, v, \mu)$$

and

$$\mu\frac{d\hat{v}}{d\sigma} = \Gamma\begin{pmatrix}\sigma\\ \hat{u}\end{pmatrix} + A\hat{v} + \tilde{G}^{(2)}(\sigma, \hat{u}, v, \mu), \tag{46}$$

where $\tilde{G}_1^{(3)}(\sigma, \hat{u}, v, \mu) \equiv \mu^{-2}G_1^{(3)}(\mu\sigma, \mu\hat{u}, \mu v)$ and $\tilde{G}^{(2)}(\sigma, \hat{u}, v, \mu) \equiv \mu^{-1}\hat{G}^{(2)}(\mu\sigma, \mu\hat{u}, \mu v)$ are both $O(\mu)$ for bounded $(\sigma, \hat{u}, v)$.

Initial conditions, at $\sigma = -\zeta_0$, are obtained from Theorem 3 above. For $\hat{u}$ we get

$$|\hat{u}(-\zeta_0) - \mu^{-1}\hat{X}(-\zeta_0\mu)| \leqslant L\mu\ln(1/\zeta_0\mu);$$

since, as previously noted [Eq. (20)], $\hat{X}(s) = O(s^2)$ as $s \to 0$, we find $\mu^{-1}\hat{X}(-\zeta_0\mu) = O(\mu)$ as $\mu \to 0$. Hence

$$\hat{u}(-\zeta_0) = O(\mu\ln\mu^{-1}) \qquad (\mu \to 0). \tag{47}$$

Similarly,

$$|v_1(-\zeta_0) - \mu^{-1}Y_1(-\zeta_0\mu)| \leqslant \frac{M}{\zeta_0},$$

where M is fixed (but ζ_0 may be chosen large). Now $Y_1(s) = \phi_1(s, \hat{X}(s))$ $= \alpha_1 s + O(s^2)$ as $s \to 0$ [cf. Eq. (20)], so

$$v_1(-\zeta_0) = -\alpha_1\zeta_0 + \frac{M_1}{\zeta_0} + O(\mu) \qquad (\mu \to 0), \tag{48}$$

where M_1 is bounded as $\zeta_0 \to \infty$ and as $\mu \to 0$. For $\hat{Y}(s) = \hat{\phi}(s, \hat{X}(s)) = w(s, \hat{X}(s), \phi_1(s, \hat{X}(s)))$, we have [cf. Eq. (20)]

$$\hat{\phi}(s, \hat{X}(s)) = \hat{\alpha}s + O(s^2) \qquad (s \to 0),$$

where

$$\hat{\alpha} = W\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = -A^{-1}\Gamma\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}. \tag{49}$$

Hence, by Theorem 3, we have

$$\hat{v}(-\zeta_0) = -\hat{\alpha}\zeta_0 + O(\mu\ln\mu^{-1}). \tag{50}$$

Collecting equations and initial conditions we have the following initial-value problem:

$$\begin{aligned}
&\frac{d\hat{u}}{d\sigma} = \tilde{F}(\sigma, \hat{u}, v_1, \hat{v}, \mu),\\
&\frac{dv_1}{d\sigma} = b\{v_1^2 + L(\sigma, \hat{u}, \hat{v})v_1 + Q(\sigma, \hat{u}, \hat{v})\} + \tilde{G}_1^{(3)}(\sigma, \hat{u}, v_1, \hat{v}, \mu),\\
&\mu\frac{d\hat{v}}{d\sigma} = \Gamma\begin{pmatrix}\sigma\\ \hat{u}\end{pmatrix} + A\hat{v} + \tilde{G}^{(2)}(\sigma, \hat{u}, v_1, \hat{v}, \mu),\\
&\hat{u}(-\zeta_0) = O(\mu\ln\mu^{-1}),\\
&v_1(-\zeta_0) = -\alpha_1\zeta_0 + \frac{M_1}{\zeta_0} + O(\mu\ln\mu^{-1}) \qquad (\mu \to 0),\\
&\hat{v}(-\zeta_0) = -\hat{\alpha}\zeta_0 + O(\mu\ln\mu^{-1}),
\end{aligned} \tag{51}$$

where $\tilde{F}$, $\tilde{G}_1^{(3)}$, and $\tilde{G}^{(2)}$ are of class C^2 for $\sigma, \hat{u}, v_1, \hat{v}$ bounded and sufficiently small μ, and are all $O(\mu)$ uniformly with respect to $\sigma, \hat{u}, v_1, \hat{v}$.

Referring now to Theorem 1B, with $\varepsilon = \mu$, $t = \sigma$, $x = (\hat{u}, v_1)$, $y = \hat{v}$, and $\chi(\mu) = C\mu\ln\mu^{-1}$ [where C is the largest of the implied constants in the initial data of the equations (51)], we see that the solution of (51) is within $O(\chi(\mu))$ of the solution of the *reduced problem*

$$\frac{d\hat{u}}{d\sigma} = 0,$$

$$\frac{dv_1}{d\sigma} = b\{v_1^2 + L(\sigma, \hat{u}, \hat{v})v_1 + Q(\sigma, \hat{u}, \hat{v})\},$$

$$0 = \Gamma\begin{pmatrix}\sigma\\ \hat{u}\end{pmatrix} + A\hat{v}, \tag{52}$$

$$\hat{u}(-\zeta_0) = 0,$$

$$v_1(-\zeta_0) = -\alpha_1\zeta_0 + \frac{M_1}{\zeta_0},$$

$$\hat{v}(-\zeta_0) = -\hat{\alpha}\zeta_0$$

on any interval $[-\zeta_0, \zeta_1]$ where the solution of (52) exists. The crucial condition on the eigenvalues reduces, in the case of (52), to the requirement that the matrix A have only eigenvalues with negative real parts, and this is hypothesis H3.

According to (52), $\hat{u} \equiv 0$ and $\hat{v} = \hat{\alpha}\sigma$ [cf. Eq. (49)] all along the reduced path (note that the latter condition conforms to the initial condition on $\hat{v}$). By Eqs. (18) and (20) we now have

$$v_1^2 + L(\sigma, \hat{u}, \hat{v})v_1 + Q(\sigma, \hat{u}, \hat{v}) = v_1^2 + l_1\sigma v_1 + q_{11}\sigma^2$$
$$= (v_1 - \alpha_1\sigma)(v_1 - \beta_1\sigma),$$

by Eq. (21). Hence the solution of the reduced problem exists on $[-\zeta_0, \zeta_1]$ wherever the solution of the initial-value problem

$$\frac{dv_1}{d\sigma} = b(v_1 - \alpha_1\sigma)(v_1 - \beta_1\sigma), \tag{53}$$

$$v_1(-\zeta_0) = -\alpha_1\zeta_0 + \frac{M_1}{\zeta_0} \tag{54}$$

exists. We now investigate this problem. Note that M_1, and hence $v_1(\sigma)$, continues to depend on μ; this does not affect Theorem 1B.

The behavior of the solution of the initial-value problem (53), (54) is given in Lemmas 4 and 5 and in Theorem 4 below; the arguments used are most easily motivated by drawing a diagram in the σv_1-plane.

Lemma 4. *Let $\alpha_1 > 0$ and suppose v_1 satisfies (53) for $\sigma > \sigma_0$. If $v_1(\sigma_0) \leqslant \alpha_1\sigma_0$, then $v_1(\sigma) < \alpha_1\sigma$ for $\sigma > \sigma_0$.*

Proof: Suppose first that $v_1(\sigma_0) < \alpha_1\sigma_0$, but that $v_1(\sigma) \geqslant \alpha_1\sigma$ for some $\sigma > \sigma_0$. Let σ_1 be the first zero of $v_1 - \alpha_1\sigma$ to the right of σ_0, i.e., $v_1 < \alpha_1\sigma$ in (σ_0, σ_1). Then $v_1'(\sigma_1) \geqslant \alpha_1 > 0$; but v_1' must vanish at a zero of $v_1 - \alpha_1\sigma$. This proves the lemma if $v_1(\sigma_0) < \alpha_1\sigma_0$. If $v_1(\sigma_0) = \alpha_1\sigma_0$, then $v_1'(\sigma_0) = 0$, implying $v_1(\sigma) < \alpha_1\sigma$ for $\sigma - \sigma_0$ small and positive, and the reasoning above then applies.

We have not as yet exploited the freedom to take ζ_0 large. We now assume ζ_0 chosen large enough that

$$v_1 < 0 \quad\text{and}\quad v_1 - \beta_1\sigma < 0 \qquad \text{at } \sigma = -\zeta_0. \tag{55}$$

It is clear from (54) that v_1 can be made negative, and $(v_1 - \beta_1\sigma)|_{-\zeta_0} = -(\alpha_1 - \beta_1)\zeta_0 + M_1/\zeta_0$ can also be made negative if $\alpha_1 - \beta_1 > 0$ (cf. H4).

Lemma 5. *Suppose $\alpha_1 > 0$, $\alpha_1 - \beta_1 > 0$, and suppose (55) holds. Then there exists σ_0 in the interval $[-\zeta_0, 0)$ such that $v_1(\sigma_0) \leqslant \alpha_1\sigma_0$, where v_1 is the solution of* (53), (54).

Proof: There is nothing to prove if $v_1(-\zeta_0) \leqslant -\alpha_1\zeta_0$, so suppose $v_1(-\zeta_0) > -\alpha_1\zeta_0$. Then, by (53) and (55), $v_1'(-\zeta_0) < 0$, and the slope remains negative until the solution curve crosses one or the other of the curves $v_1 = \alpha_1\sigma$, $v_1 = \beta_1\sigma$. The solution exists at least until such a crossing takes place, since v_1, while decreasing, is bounded below by the curve $v_1 = \alpha_1\sigma$. We first show that, if and when the first crossing takes place, it *cannot* be with the curve $v_1 = \beta_1\sigma$. There are two cases to consider:

(i) $\beta_1 > 0$. By (55), the conditions of Lemma 4 are satisfied with β_1 in place of α_1. Hence $v_1 < \beta_1\sigma$ as long as the solution can be continued.

(ii) $\beta_1 \leqslant 0$. In this case, $\beta_1\sigma \geqslant 0$ on $[-\zeta_0, 0)$, whereas $v_1 < 0$ until the first crossing. Hence $v_1 - \beta_1\sigma < 0$ there.

The first crossing of the solution $v_1(\sigma)$ on $[-\zeta_0, 0)$ must therefore be with $v_1 = \alpha_1\sigma$ or not at all. Suppose there were none; then $v_1' < 0$ there, so $\lim_{\sigma\to 0} v_1 - \alpha_1\sigma < 0$, whereas $v_1 - \alpha_1\sigma > 0$ at $-\zeta_0$. This contradiction proves the lemma.

Theorem 4. *Suppose $\alpha_1 > 0$, $\alpha_1 - \beta_1 > 0$, and (55) holds. Then the solution of (53), (54) exists for all $\sigma > -\zeta_0$ and satisfies*

$$v_1(\sigma) = \beta_1\sigma + O\left(\frac{1}{\sigma}\right)^{1} \qquad (\sigma \to \infty). \tag{56}$$

Proof: The solution exists as long as it is bounded. It certainly exists to the point σ_0 provided by Lemma 5, and lies below the curve $v_1 = \alpha_1\sigma_0$ for $\sigma > \sigma_0$ (Lemma 4). For $\sigma < 0$, the latter curve lies below $v_1 = \beta_1\sigma$, so v_1', which becomes positive for $\sigma > \sigma_0$, cannot change sign on $(\sigma_0, 0]$. This implies that v_1 is bounded on $(\sigma_0, 0]$ (since it is increasing but bounded above), and hence exists on $[-\zeta_0, 0]$. The existence claim of the theorem will follow if we can infer boundedness similarly for all finite $\sigma > 0$. There are three cases to consider:

(i) $\beta_1 > 0$. Here the conditions of Lemma 4 are satisfied both for α_1 and for β_1. Hence, not only for $\sigma > 0$ but also for all $\sigma > \sigma_0$ (where σ_0 is given by Lemma 5), $v_1(\sigma) < \beta_1\sigma$ and $v_1'(\sigma) > 0$. This implies boundedness on finite intervals.

(ii) $\beta_1 = 0$. In this case $v_1 \equiv 0$ is a solution of (53). The uniqueness theorem and the condition $v_1(-\zeta_0) < 0$ imply $v_1(\sigma) < 0$, while $v_1' > 0$ for $\sigma > \sigma_0$. Hence v_1 exists and is bounded for all $\sigma > 0$.

(iii) $\beta_1 < 0$. For $\sigma > 0$ (indeed for $\sigma > \sigma_0$), $v_1' > 0$ unless an intersection with

[1] The estimate $O\left(\frac{1}{\sigma}\right)$ can easily be shown to be uniform as to μ for $\mu \to 0+$.

the curve $v_1 = \beta_1\sigma$ occurs; no such intersection with the curve $v_1 = \alpha_1\sigma$ can occur, by Lemma 4. Furthermore, a repetition of the reasoning of Lemma 5 shows that an intersection *must* occur for some $\sigma_1 > 0$, and a repetition of the reasoning of Lemma 4 shows that no further crossing of the curve $v_1 = \beta_1\sigma$ can occur thereafter. Hence for $\sigma > \sigma_1$, $v_1' < 0$, but $v_1(\sigma)$ is bounded below by $v_1 = \beta_1\sigma$. This completes the existence claim.

The asymptotic estimate (56) must be proved separately for the three cases distinguished above:

(i) $\beta_1 > 0$. First note that v_1 eventually becomes positive in this case, for if it remained forever negative, we would infer from (53) $v_1'(\sigma) > b\alpha_1\beta_1\sigma^2$, from which a contradiction may be inferred. Moreover, since $v_1' > 0$ for $\sigma > \sigma_0$, once v_1 becomes positive, it remains positive. Taking $\sigma > \sigma_1$ large enough so $v_1 > 0$, and recalling $v_1 - \beta_1\sigma < 0$, we have

$$-\alpha_1\sigma < v_1 - \alpha_1\sigma < (\beta_1 - \alpha_1)\sigma, \tag{57}$$

so, substituting for $v_1 - \alpha_1\sigma$ in (53), we have

$$(\beta_1 - \alpha_1)\sigma(v_1 - \beta_1\sigma) < v_1'(\sigma) < -\alpha_1\sigma(v_1 - \beta_1\sigma).$$

If $v_1(\sigma) = \beta_1\sigma + \chi(\sigma)$, the latter becomes

$$-(\alpha_1 - \beta_1)\sigma\chi - \beta_1 < \chi'(\sigma) < -\alpha_1\sigma\chi - \beta_1.$$

Integrating, for example, the left-hand inequality, we get

$$\chi(\sigma) > \text{const}\, e^{-(\alpha_1-\beta_1)\sigma^2/2} - \beta_1\int_{\sigma_1}^{\sigma} e^{-(\alpha_1-\beta_1)(\sigma^2-\tau^2)/2}\, d\tau.$$

Now, one easily verifies that for $p > 0$

$$2p\sigma e^{-p\sigma^2}\int_{\sigma_1}^{\sigma} e^{p\tau^2}\, d\tau \to 1 \qquad \text{as } \sigma\to\infty,$$

so

$$\chi(\sigma) > C\frac{1}{\sigma} \qquad \text{for large enough } \sigma,$$

where C is as close as we please to $-\beta_1/(\alpha_1 - \beta_1)$. Similarly, from the other inequality for χ', we find $\chi(\sigma) < C'(1/\sigma)$ for large σ, where C' is as close as we please to $-\beta_1/\alpha_1$. Hence $\chi = O(1/\sigma)$, as claimed.

(ii) $\beta_1 = 0$. Here $v_1 < 0$, so $v_1 - \alpha_1\sigma < -\alpha_1\sigma$, so

$$v_1' > b\alpha_1\sigma v_1,$$

and

$$v_1(\sigma) > \text{const}\, e^{-b\alpha_1\sigma^2/2},$$

the constant being negative. Hence $v_1(\sigma) = O(e^{-b\alpha_1\sigma^2/2}) = O(1/\sigma)$.

(iii) $\beta_1 < 0$. In this case, as described above,

$$\beta_1\sigma < v_1 < \alpha_1\sigma \qquad \text{for } \sigma > \sigma_1.$$

Further, $v_1 < 0$, for the intersection of the solution $v_1(\sigma)$ with the curve $v_1 = \beta_1\sigma$ necessarily occurs for $v_1 < 0$, since it must take place when $\sigma > 0$. Since $v_1' < 0$ thereafter, $v_1 < 0$ always. But then

$$(\beta_1 - \alpha_1)\sigma < v_1 - \alpha_1\sigma < -\alpha_1\sigma.$$

This is similar to the estimate (57) of (i), and the remainder of the proof is as given there.

According to Theorem 4, the solution of the reduced problem (52) exists on $[-\zeta_0, \zeta_1]$ for any choice of ζ_1, and satisfies, for large σ, $(\hat{u}(\sigma), v_1(\sigma), \hat{v}(\sigma)) = (0, \beta_1\sigma + O(1/\sigma), \hat{\alpha}\sigma)$. This puts it near the reduced path ψ [since $\psi(s, \hat{X}(s)) = \hat{\alpha}s + O(s^2)$]. The solution of the full problem (51) differs from this by $O(\mu \ln \mu^{-1}) = O(\varepsilon^{1/2} \ln \varepsilon^{-1})$. Restoring the original variables [Eq. (43)] and taking $\sigma = \zeta_1$ (or $s = \varepsilon^{1/2}\zeta_1$), we find for the solution of the full problem, for any sufficiently large ζ_1,

$$\hat{x}(\varepsilon^{1/2}\zeta_1) = O(\varepsilon \ln \varepsilon^{-1}),$$

$$y_1(\varepsilon^{1/2}\zeta_1) = \varepsilon^{1/2}\left[\beta_1\zeta_1 + O\left(\frac{1}{\zeta_1}\right)\right] + O(\varepsilon \ln \varepsilon^{-1}), \tag{58}$$

$$\hat{y}(\varepsilon^{1/2}\zeta_1) = \alpha_1\zeta_1\varepsilon^{1/2} + O(\varepsilon \ln \varepsilon^{-1}).$$

These formulas represent the principal result of this section, and provide initial conditions for the final interval of the form $[\varepsilon^{1/2}\zeta_1, b_1]$.

6. Leaving the transition interval

With the help of Eq. (20), we can rewrite (58) as

$$\hat{x}(\varepsilon^{1/2}\zeta_1) = \hat{X}(\varepsilon^{1/2}\zeta_1) + O(\varepsilon \ln \varepsilon^{-1}),$$

$$y_1(\varepsilon^{1/2}\zeta_1) = Y_1(\varepsilon^{1/2}\zeta_1) + \varepsilon^{1/2}\frac{\tilde{m}}{\zeta_1} + O(\varepsilon \ln \varepsilon^{-1}), \tag{59}$$

$$\hat{y}(\varepsilon^{1/2}\zeta_1) = \hat{Y}(\varepsilon^{1/2}\zeta_1) + O(\varepsilon \ln \varepsilon^{-1}),$$

where $\tilde{m}(\zeta_1, \varepsilon)$ is bounded as $\zeta_1 \to \infty$ and as $\varepsilon \to 0$. Introducing ξ, η_1, and $\hat{\eta}$ as in Sec. 4, we find that they satisfy Eq. (30), with initial conditions at $s = \varepsilon^{1/2}\zeta_1$ satisfying inequalities of the form

$$|\xi(\varepsilon^{1/2}\zeta_1)| \leqslant L\varepsilon \ln \varepsilon^{-1}, \qquad |\eta_1(\varepsilon^{1/2}\zeta_1)| \leqslant M_1\varepsilon^{1/2}/\zeta_1 + M_2\varepsilon \ln \varepsilon^{-1}, \tag{60}$$

and

$$|\hat{\eta}(\varepsilon^{1/2}\zeta_1)| \leqslant \hat{M}\varepsilon \ln \varepsilon^{-1},$$

for $\varepsilon < \varepsilon_2$ (some $\varepsilon_2 > 0$) and for some choice of the constants $L, M_1, M_2, \hat{M}$. Our object is to complete the proof of Theorem 2 by showing that $|\xi(s)| + |\eta(s)| = O(\varepsilon^{1/2})$ uniformly on an interval $[\varepsilon^{1/2}\zeta_1, b_1]$ for some $b_1 > 0$ and independent of ε.

Define the fundamental matrix solutions $U(s, \sigma)$, $V_1(s, \sigma, \varepsilon)$, and $\hat{V}(s, \sigma, \varepsilon)$ as in Sec. 4, with the difference that they reduce to the appropriate unit matrices when $s = \sigma \geqslant \varepsilon^{1/2}\zeta_1$. Then [cf. Eq. (22)]

$$V_1(s, \sigma, \varepsilon) = \exp\left\{ \frac{1}{\varepsilon} \int_\sigma^s G_{1,y_1}(u) du \right\} \leqslant \exp\left\{ -\varepsilon^{-1} \int_\sigma^s 2\nu u \, du \right\},$$

or

$$0 < V_1(s, \sigma, \varepsilon) \leqslant e^{-\nu(s^2-\sigma^2)/\varepsilon} \qquad \text{if } \varepsilon^{1/2}\zeta_1 \leqslant \sigma \leqslant s \leqslant t_1, \tag{61}$$

provided t_1 is sufficiently small. The system (30) may be replaced by the system (32) of integral equations with a_1 replaced, wherever it appears, by $\varepsilon^{1/2}\zeta_1$. Estimates analogous to (33) hold on $\varepsilon^{1/2}\zeta \leqslant \sigma \leqslant s \leqslant t_1$. In fact, all estimates are identical with the exception of that for V_1, which is replaced by (61) above. Any solution to the right of $\varepsilon^{1/2}\zeta_1$ must then satisfy the system of inequalities (we assume $K > 1$)

$$|\xi(s)| \leqslant KL\varepsilon \ln \varepsilon^{-1} + K\int_{\varepsilon^{1/2}\zeta_1}^s \{|R(\sigma, \xi, \eta, \varepsilon)| + K|\eta(\sigma)|\}\, d\sigma$$

$$|\eta_1(s)| \leqslant K(M_1\varepsilon^{1/2}/\zeta_1 + M_2\varepsilon \ln \varepsilon^{-1}) + K\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^s e^{-\nu(s^2-\sigma^2)/\varepsilon}$$
$$\times \{|S_1(\sigma, \xi, \eta, \varepsilon)| + K\sigma(|\xi(\sigma)| + |\hat{\eta}(\sigma)|)\}\, d\sigma, \tag{62}$$

$$|\hat{\eta}(s)| \leqslant K\hat{M}\varepsilon \ln \varepsilon^{-1} + K\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^s e^{-\nu(s-\sigma)/\varepsilon}\{|\hat{S}(\sigma, \xi, \eta, \varepsilon)|$$
$$+ K|\xi(\sigma)| + K\sigma|\eta_1(\sigma)|\}\, d\sigma.$$

THEOREM 5. *Suppose the inequalities (33) are satisfied for $\varepsilon^{1/2}\zeta_1 \leqslant \sigma \leqslant s \leqslant t_1$ with the exception of that for V_1, which is given by (61). Then there exist positive constants b_1, ε^*, ζ_1, and $\mathcal{Q}$ such that the solution $(\xi, \eta_1, \hat{\eta})$ of (30) with initial data at $\varepsilon^{1/2}\zeta_1$ satisfying (60) exists on the interval $[\varepsilon^{1/2}\zeta_1, b_1]$ and satisfies there the inequality*

$$|\xi(s)| + |\eta(s)| \leqslant \varepsilon^{1/2}\mathcal{Q} \tag{63}$$

for all ε in the interval $(0, \varepsilon^*)$.

Proof: Instead of proving (63) directly, we provide constants Q, $\hat{Q}$, P such that

$$|\eta(s)| < Q\varepsilon^{1/2}, \qquad |\hat{\eta}(s)| < \hat{Q}\varepsilon^{1/2}, \quad \text{and} \quad |\xi(s)| < P\varepsilon^{1/2}. \tag{64}$$

Then (63) follows with $\mathcal{Q} = P + Q$. We define the constants as follows:

(i) Choose b_1 so small that

$$b_1 < t_1 \quad \text{and} \quad b_1 e^{\frac{3}{2}(K^4/\nu)b_1}\left(\frac{K^2}{\nu} + \frac{K^6}{2\nu^2}\right) < \frac{1}{2};$$

(ii) $Q = 4e^{\frac{3}{2}(K^4/\nu)b_1}$, $\hat{Q} = (3K^4/\nu)Qb_1$, $P = 2K^2Qb_1$, $\mathcal{Q} = P + Q$;

(iii) $\zeta_1 > KM_1 + \dfrac{K^2}{2\nu}(1 + \mathcal{Q}^2)$;

(iv) $\varepsilon^* < \min\{\varepsilon_2, (\delta/\mathcal{Q})^2, (b_1/\zeta_1)^2, e^{-1}\}$, and also so small that $\varepsilon < \varepsilon^*$ implies

$$(\varepsilon^{1/2}\ln \varepsilon^{-1})\left\{K\left(M_2 + \hat{M}\right) + \frac{3K^3L}{2\nu} + \frac{3K^4}{2\nu}b_1(1 + \mathcal{Q}^2) + \frac{K^2}{\nu}(1 + \mathcal{Q}^2)\right\} < 1,$$

$$(\varepsilon^{1/2}\ln \varepsilon^{-1})\left\{K\hat{M} + \frac{K^3L}{\nu} + (1 + \mathcal{Q}^2)\frac{K^2}{\nu}(1 + K^2b_1)\right\} < \frac{K^4}{\nu}b_1Q,$$

$$(\varepsilon^{1/2}\ln \varepsilon^{-1})\{KL + K^2(1 + \mathcal{Q}^2)b_1\} < K^2b_1Q,$$

where ε_2 figures in the estimates (60), and δ in (30).

It is straightforward to verify that at $s = \varepsilon^{1/2}\zeta_1$, the inequalities (60) imply (64). The initial-value problem therefore has a solution on some interval to the right of $\varepsilon^{1/2}\zeta_1$, which we suppose cannot be continued to b_1. Then there exists some least value of s, s_0 say, with $\varepsilon^{1/2}\zeta_1 < s_0 \leqslant b_1$, such that (64) is satisfied on $[\varepsilon^{1/2}\zeta_1, s_0)$, but one at least of these inequalities becomes an equality at s_0.

On $[\zeta_1\varepsilon^{1/2}, s_0]$, $|\xi| + |\eta| \leqslant \mathcal{Q}\varepsilon^{1/2} < \delta$, so the conditions on R, S_1, $\hat{S}$ hold, e.g.,

$$|R(\sigma, \xi, \eta, \varepsilon)| \leqslant K(\varepsilon + |\xi|^2 + |\eta|^2) \leqslant K\varepsilon(1 + \mathcal{Q}^2).$$

The system (62) now becomes

$$|\xi(s)| \leqslant KL\varepsilon \ln \varepsilon^{-1} + K^2\varepsilon(1 + \mathcal{Q}^2)b_1 + K^2\int_{\varepsilon^{1/2}\zeta_1}^s |\eta(\sigma)|\, d\sigma,$$

$$|\eta_1(s)| \leqslant (M_1\zeta_1^{-1}\varepsilon^{1/2} + M_2\varepsilon \ln \varepsilon^{-1})K + K^2(1 + \mathcal{Q}^2)\varepsilon^{1/2}/2\nu\zeta_1$$
$$+ K^2\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^s e^{-\nu(s^2-\sigma^2)/\varepsilon}\sigma\{|\hat{\eta}(\sigma)| + |\xi(\sigma)|\}\, d\sigma, \tag{65}$$

$$|\hat{\eta}(s)| \leqslant K\hat{M}\varepsilon \ln \varepsilon^{-1} + K^2(1 + \mathcal{Q}^2)\varepsilon/\nu$$
$$+ K^2\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^s e^{-\nu(s-\sigma)/\varepsilon}\{|\xi(\sigma)| + \sigma|\eta_1(\sigma)|\}\, d\sigma,$$

where we have used Lemma 1. Provisionally put

$$k(\varepsilon) = KL\varepsilon \ln \varepsilon^{-1} + K^2\varepsilon(1 + \mathcal{Q}^2)b_1, \tag{66}$$

so that

$$|\xi(s)| \leqslant k + K^2 \int_{\varepsilon^{1/2}\zeta_1}^{s} |\eta(\sigma)|\, d\sigma.$$

Then we find with the aid of Lemma 3

$$|\eta_1(s)| \leqslant (M_1\varepsilon^{1/2}\zeta_1^{-1} + M_2\varepsilon \ln \varepsilon^{-1})K + \frac{K^2(1+\mathcal{Q}^2)\varepsilon^{1/2}}{2\nu\zeta_1}$$

$$+ \frac{K^2k}{2\nu} + \frac{K^4}{2\nu}\int_{\varepsilon^{1/2}\zeta_1}^{s}|\eta(\sigma)|\, d\sigma + K^2\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^{s} e^{-\nu(s^2-\sigma^2)/\varepsilon}\sigma|\hat{\eta}(\sigma)|\, d\sigma,$$

$$|\hat{\eta}(s)| \leqslant K\hat{M}\varepsilon \ln \varepsilon^{-1} + \frac{K^2(1+\mathcal{Q}^2)\varepsilon}{\nu} + \frac{K^2k}{\nu} + \frac{K^4}{\nu}\int_{\varepsilon^{1/2}\zeta}^{s}|\eta(\sigma)|\, d\sigma \tag{67}$$

$$+ K^2\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^{s} e^{-\nu(s-\sigma)/\varepsilon}\sigma|\eta_1(\sigma)|\, d\sigma.$$

With the aid of (64), which hold on $[\varepsilon^{1/2}\zeta_1, s_0)$, we find

$$K^2\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^{s} e^{-\nu(s^2-\sigma^2)/\varepsilon}\sigma|\hat{\eta}(\sigma)|\, d\sigma \leqslant \frac{K^2\hat{Q}\varepsilon^{1/2}}{2\nu}$$

and

$$K^2\varepsilon^{-1}\int_{\varepsilon^{1/2}\zeta_1}^{s} e^{-\nu(s-\sigma)/\varepsilon}\sigma|\eta_1(\sigma)|\, d\sigma \leqslant \frac{b_1K^2Q\varepsilon^{1/2}}{\nu}.$$

Substituting in (67) and using (66) for k, we find

$$|\eta_1(s)| \leqslant \left(M_1\frac{\varepsilon^{1/2}}{\zeta_1} + M_2\varepsilon \ln \varepsilon^{-1}\right)K + \frac{K^2(1+\mathcal{Q}^2)\varepsilon^{1/2}}{2\nu\zeta_1} + \frac{K^2\hat{Q}\varepsilon^{1/2}}{2\nu}$$

$$+ \frac{K^2}{2\nu}\{KL\varepsilon \ln \varepsilon^{-1} + K^2\varepsilon(1+\mathcal{Q}^2)b_1\} + \frac{K^4}{2\nu}\int_{\varepsilon^{1/2}\zeta_1}^{s}|\eta(\sigma)|\, d\sigma \tag{68}$$

$$|\hat{\eta}(s)| \leqslant K\hat{M}\varepsilon \ln \varepsilon^{-1} + \frac{K^2(1+\mathcal{Q}^2)\varepsilon}{\nu} + \frac{b_1K^2Q\varepsilon^{1/2}}{\nu}$$

$$+ \frac{K^2}{\nu}[KL\varepsilon \ln \varepsilon^{-1} + K^2\varepsilon(1+\mathcal{Q}^2)b_1] + \frac{K^4}{\nu}\int_{\varepsilon^{1/2}\zeta_1}^{s}|\eta(\sigma)|\, d\sigma.$$

Since $\varepsilon < e^{-1}$, we have $\ln \varepsilon^{-1} > 1$ and $\varepsilon < \varepsilon \ln \varepsilon^{-1}$. Adding the last two inequalities, replacing ε by $\varepsilon \ln \varepsilon^{-1}$, and using Gronwall's lemma we find, by (ii), (iii), (iv) above,

$$|\eta(s)| \leqslant e^{\frac{3}{2}(K^4/\nu)b_1}\left[\varepsilon^{1/2}\left(\frac{M_1}{\zeta_1} + \frac{K^2(1+\mathcal{Q}^2)}{2\nu\xi_1} + \frac{b_1K^2Q}{\nu} + \frac{K^2\hat{Q}}{2\nu}\right)\right.$$

$$\left. + \varepsilon \ln \varepsilon^{-1}\left(K\left(M_2 + \hat{M}\right) + \frac{K^2}{\nu}(1+\mathcal{Q}^2) + \frac{3K^3}{2\nu}[L + (1+\mathcal{Q}^2)b_1K]\right)\right]$$

$$< e^{\frac{3}{2}(K^4/\nu)b_1}\left\{\varepsilon^{1/2}\left[1 + b_1Q\left(\frac{K^2}{\nu} + \frac{3K^6}{2\nu^2}\right)\right] + \varepsilon^{1/2}\right\}$$

$$= \varepsilon^{1/2}e^{\frac{3}{2}(K^4/\nu)b_1}\left[2 + b_1Q\left(\frac{K^2}{\nu} + \frac{3K^6}{2\nu^2}\right)\right].$$

If $|\eta(s_0)| = Q\varepsilon^{1/2}$, then (i) and the above inequality imply that

$$Q < 4e^{\frac{3}{2}(K^4/\nu)b_1},$$

contradicting its definition. Hence $|\eta(s)|$ does not become equal to $Q\varepsilon^{1/2}$ at s_0, so one of the remaining inequalities (64) becomes an equality. Suppose $|\hat{\eta}(s_0)| = \hat{Q}\varepsilon^{1/2}$. By (68),

$$|\hat{\eta}(s)| \leqslant (b_1K^2Q/\nu + b_1K^4Q/\nu)\varepsilon^{1/2} + \varepsilon \ln \varepsilon^{-1}\{K\hat{M} + K^2(1+\mathcal{Q}^2)/\nu$$

$$+ K^3L/\nu + K^4(1+\mathcal{Q}^2)b_1/\nu\}$$

$$< \varepsilon^{1/2}\left(2\frac{K^4b_1Q}{\nu} + \frac{K^4b_1Q}{\nu}\right) = 3\varepsilon^{1/2}\frac{K^4b_1Q}{\nu},$$

so $|\eta(s_0)| < Q\varepsilon^{1/2}$. This leaves $|\xi(s_0)| = P\varepsilon^{1/2}$, and a similar computation based on (65) above shows this, too, is impossible, proving the theorem.

Theorem 2 is now a consequence of the results of this section and the two preceding it. We merely require of ε^* of Theorem 5 that it also be smaller than ε_1 of Theorem 3; this can be done, since their choices were independent. Then ε^* serves for Theorem 2 as well as for Theorem 5.

7. Discussion

In Sec. 2 we replaced the system (3) by the orbit equations (8), remarking that the results can be transferred to the solutions of (3). Specifically, suppose $(\tilde{x}(t), \tilde{y}(t))$ and $(\tilde{x}^0(t), \tilde{y}^0(t))$ are solutions of (3) and (4) respectively. Then

$$|\tilde{x}(t) - \tilde{x}^0(t)| + |\tilde{y}(t) - \tilde{y}^0(t)| = O(\varepsilon^{1/2}) \tag{69}$$

uniformly on an interval $[0, T_1]$ where $\tilde{x}_1^0(t)$ and $\tilde{x}_1(t)$ increase from negative to positive values. We outline the proof of (69).

Let $(\hat{x}(s), y(s))$ and $(\hat{X}(s), Y(s))$ be the solutions of (8) and (9), respectively, with a_1 and b_1 chosen so that Theorem 2 holds. We can obtain the solution $(\tilde{x}(t), \tilde{y}(t))$ as follows: first let $\tilde{x}_1(t)$ be the solution of the initial-value problem

$$\frac{d\tilde{x}_1}{dt} = f_1(\tilde{x}_1, \hat{x}(\tilde{x}_1), y(\tilde{x}_1)), \qquad \tilde{x}_1(0) = a_1; \tag{70}$$

then put

$$(\tilde{x}(t), \tilde{y}(t)) = (\tilde{x}_1(t), \hat{x}(\tilde{x}_1(t)), y(\tilde{x}_1(t))).$$

Similarly, let $\tilde{x}_1^0(t)$ be the solution of the problem

$$\frac{d\tilde{x}_1^0}{dt} = f_1(\tilde{x}_1^0, \hat{X}(\tilde{x}_1^0), Y(\tilde{x}_1^0)), \qquad \tilde{x}_1^0(0) = a_1, \tag{71}$$

and put

$$(\tilde{x}^0(t), \tilde{y}^0(t)) = (\tilde{x}_1^0(t), \hat{X}(\tilde{x}_1^0(t)), Y(\tilde{x}_1^0(t))).$$

The solution of (71) exists and is monotonic on an interval [0, T] where $\tilde{x}_1^0$ increases from a_1 to b_1 (we assume for definiteness that $f_1 > 0$; cf. H1).

The function f_1 satisfies a Lipschitz condition in some domain containing the graph $(s, \hat{X}(s), Y(s))$ for $s \in [a_1, b_1]$. Combining this with Theorem 2 we infer that

$$|f_1(s, \hat{x}(s), y(s)) - f_1(s, \hat{X}(s), Y(s))| = O(\varepsilon^{1/2}) \tag{72}$$

uniformly on $[a_1, b_1]$. It follows, by continuity with respect to parameters [as expressed by (72)] that the solution $\tilde{x}_1$ of (70) exists on $[0, T]$ and approaches $\tilde{x}_1^0$ as $\varepsilon \to 0+$; in fact (cf. [10], Chapter 6, Theorem 3),

$$|\tilde{x}_1(t) - \tilde{x}_1^0(t)| = O(\varepsilon^{1/2}) \tag{73}$$

uniformly on $[0, T]$. By replacing T if necessary by $T_1 < T$ we can ensure that $\tilde{x}_1(t) \in [a_1, b_1]$ for $t \in [0, T_1]$.

Now, for $k = 2, \ldots, m$,

$$|\tilde{x}_k(t) - \tilde{x}_k^0(t)| = |x_k(\tilde{x}_1(t)) - X_k(\tilde{x}_1^0(t))|$$
$$= |x_k(\tilde{x}_1^0(t)) - X_k(\tilde{x}_1^0(t))| + O(|\tilde{x}_1(t) - \tilde{x}_1^0(t)|)$$

uniformly on $[0, T_1]$, since $dx_k/dt = F_k$ is bounded there. Each of the last two terms is $O(\varepsilon^{1/2})$, the first by Theorem 2, the second by Eq. (73). This takes care of the first term on the left-hand side of (69). To show that the other term is also $O(\varepsilon^{1/2})$, we first observe that $Y(s)$ is uniformly Lipschitz continuous on $[a_1, b_1]$. This is easily shown by verifying it first for $Y_1(s)$ [cf. Eq. (14) above], and then noting that it follows for $\hat{Y}$ from the definition $\hat{Y}(s) = w(s, X(s), Y, (s))$; we

omit details. Then

$$|\tilde{y}(t) - \tilde{y}^0(t)| = |y(\tilde{x}_1(t)) - Y(\tilde{x}_1^0(t))|$$
$$\leqslant |y(\tilde{x}_1(t)) - Y(\tilde{x}_1(t))| + |Y(\tilde{x}_1(t)) - Y(\tilde{x}_1^0(t))|$$
$$\leqslant |y(\tilde{x}_1(t)) - Y(\tilde{x}_1(t))| + L|\tilde{x}_1(t) - \tilde{x}_1^0(t)|,$$

L being the Lipschitz constant. The conclusion now follows from Theorem 2 and (73).

We have not tried to find the weakest possible hypotheses in establishing Theorem 2. For example, the requirement that G be in C^4 (cf. H2) can be lightened to C^3 by proving a different, rather specialized version of Theorem 1B. Requiring that $G(s, \hat{x}, y, \mu)$ be in C^4 implies that $\mu^{-2}G$ is in C^2 and allows a direct application of Theorem 1B.

We have not been able to locate any other work following the evolution through a point of bifurcation where an exchange of stabilities takes place. There is, however, a paper by Pontryagin [11] following the evolution up to a point where the equilibrium solutions of (1) terminate, so that motion on the short time scale must eventually take place. This paper, apparently motivated by problems associated with relaxation oscillations, is devoted to working out the forms of the approximations, and leaves open theoretical questions as to whether the formulae generated are in fact asymptotic solutions. The analogous theoretical questions are completely resolved in the present work, in a manner that can be extended to the problem considered in [11] (see [12] for the case $n = m = 1$).

References

1. H. Poincaré, Sur l'equilibre d'une masse fluide animée d'un mouvement de rotation, *Acta Math.* **7** 259–380, (1885).
2. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1961.
3. N. Lebovitz, On the fission theory of binary stars, *Astrophys. J.* **175** 171–183, (1972).
4. S. Rubinow, *Regional Conference Series in Applied Mathematics* **10**, Lecture 5, SIAM, Philadelphia, 1973.
5. A. Tikhonov, Systems of differential equations containing a small parameter multiplying the highest derivatives, *Mat. NS* (31) **73** 575–585, (1952), (in Russian).
6. R. O'Malley, Jr., On initial value problems for nonlinear systems of differential equations with two small parameters, *Arch. Ration. Mech. Anal.* **40** 209–222, (1971).
7. J. Levin and N. Levinson, Singular perturbations of nonlinear systems of differential equations and associated boundary-layer equation, *J. Ration. Mech. Anal.* **3** 247–270, (1954).
8. F. Hoppensteadt, Properties of solutions of ordinary differential equations with small parameters, *Commun. Pure Appl. Math.* **24** 807–839, (1971).
9. L. Flatto and N. Levinson. Periodic solutions of singularly perturbed systems, *J. Ration. Mech. Anal.* **4** 943–950, (1955).
10. G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, 2nd ed., Blaisdell, Waltham, 1969.

11. L. Pontryagin, Asymptotic behavior of the solutions of systems of differential equations with a small parameter in the higher derivatives, *Izv. Akad. Nauk SSSR, Ser. Mat.* **21**, 605–626 (1959); AMS Transl. Ser. 2, **18** 295–319, (1961).
12. R. Schaar. The singularly perturbed initial-value problem when the reduced path encounters a point of bifurcation, Doctoral Dissertation, University of Chicago, 1974.

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On the Formation of Vortex Rings*

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The production of a vortex ring formed by using a piston to drive fluid through an orifice is considered. A cylindrical vortex sheet is supposed to be formed initially which rolls up into a vortex ring. Energy and momentum are conserved during rollup and determine the speed and size of the ring. It is shown that these quantities are independent of the vorticity distribution in the core of the ring. Reasonable agreement with experimental observations is found. A speculation is made about the criterion for the rings to be laminar or turbulent.

1. The analysis of Taylor

Vortex rings have been studied for over 100 years, but much remains to be learned. One problem is the relation of the ring to the mechanical system by which it is formed.

Taylor [15] addressed this question, considering the vortex ring formed by giving an impulse to a circular disc and then dissolving it away. He showed that a disc of radius c, moving with velocity U, makes a vortex ring of radius R, moving with velocity V', where

$$\frac{R}{c} = 0.816, \qquad \frac{V'}{U} = 0.436. \tag{1}$$

To derive these results, Taylor equated the energy and momentum (hydrodynamic impulse) of the motion produced by the disc,

$$T = \tfrac{4}{3}\rho c^3 U^2, \qquad I = \tfrac{8}{3}\rho c^3 U, \tag{2}$$

respectively, where ρ is the fluid density, with the energy and momentum of a circular vortex ring of radius R, core radius a, and approximately uniform

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