PERSISTENCE OF STABILITY LOSS FOR DYNAMICAL BIFURCATIONS. II

A. I. Neishtadt

UDC 517.928.4

We consider differential-equation systems with fast and slow variables [1]. The system with fast variables, when the slow variables are fixed, has an equilibrium position. When the slow variables drift, there is a dynamical bifurcation: a pair of conjugate eigenvalues of this equilibrium cross the imaginary axis from left to right and the remaining eigenvalues remain in the left half-plane. If the system is analytic, then the stability loss persists: phase points remain near the unstable equilibrium position during a time period of length of order $1/\varepsilon$, where ε is a small parameter characterizing the drift rate. Only then can there be a loss of equilibrium. We establish a lower bound for the persistence time, and obtain the asymptotic properties of the break time in several cases.

The asymptotic behavior of the rupture time is obtained for a model system in [2], where equilibrium-loss persistence was first discussed. The result obtained in [2] was extended in [3, 4] to some special classes of systems. It is proved in [5] that, in analytic systems with fast and slow motion, stability-loss persistence must be accompanied by dynamical bifurcations in which there is a loss of stability of a nondegenerate equilibrium position or a nondegenerate limit cycle.

The results described here were announced in [6]. We use the method employed in [2], i.e., analytic continuation of solutions in the plane of a complex time. For motion along paths in this plane, the system has an adiabatic invariant, and this inhibits departure from an equilibrium position.

1. The Original Equations. We investigate equation systems of the following form, often encountered in vibration theory [1]:

$$\dot{x}=f(x, y, \varepsilon), \quad x\in R^n,$$
$$\dot{y}=\varepsilon g(x, y, \varepsilon), \quad y\in R^m. \tag{1.1}$$

Here ε is a small parameter, $(x, y)\in D\subset R^{n+m}$, and $|\varepsilon|<\varepsilon_1=\text{const}$. Moreover x is the fast and y is the slow variable, and (1.1) is a fast–slow system.

The fast system is the equation for x in (1.1), with y = const and $\varepsilon = 0$. Suppose that, for all y in the projection of D on the y-space, the fast system has an equilibrium position $x = X(y)$ depending continuously on y. Let $\lambda_i(y)$, $i = 1, 2,\ldots,n$ be the eigenvalues of this equilibrium.

The slow system is

$$\dot{y}=\varepsilon g(X(y), y, 0), \quad x=X(y).$$

and its solutions are called slow solutions.

We fix a slow solution $y = Y(\tau)$, $x=X(Y(\tau))$, $\tau=\varepsilon t\in[\tau_0, \tau_1]$, $\varepsilon>0$, and consider the behavior of the eigenvalues on it. Let $\lambda_1(Y(\tau))$ be in the left half-plane for $\tau<\tau_*\in[\tau_0, \tau_1]$, and cross the imaginary axis for $\tau = \tau_*$ with a nonzero velocity, let $\lambda_1(Y(\tau_*))\neq 0$, and let $\lambda_2(Y(\tau_*))=\overline{\lambda_1(Y(\tau_*))}$ (a bar indicates a complex conjugate). Let $\lambda_i(Y(\tau))$, $i = 3,\ldots,n$, be in the left half-plane for all τ. We assume that the right sides in (1.1) can be analytically continued with respect to x and y into a complex neighborhood of the point $X(Y(\tau_*))$, independent of ε, and that $Y(\tau_*)$ is a smooth function of ε in this neighborhood. Then (1.1) has stability-loss persistence [5].

2. A Result Concerning a Bound for the Stability-Loss Persistence Time. To formulate our theorem we use constructions related to the analytic continuation of the solution X, Y.

All-Union Medical-Instrument Research Institute. Translated from Differentsial'nye Uravneniya, Vol. 24, No. 2, pp. 226-233, February, 1988. Original article submitted July 7, 1986.

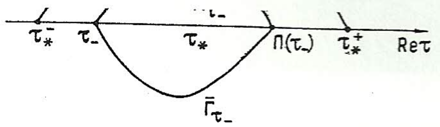

Fig. 1

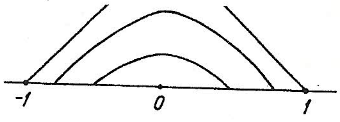

Fig. 2

For definiteness let $\operatorname{Im}\lambda_1(Y(\tau_*))<0$, and for $\tau\in[\tau_0, \tau_1]$ consider the complex phase $\Psi(\tau)=\int_{\tau_*}^{\tau}\lambda_1(Y(\vartheta))d\vartheta$. For $\tau = \tau_*$ the function $\operatorname{Re}\Psi$ has a minimum because at this point $\operatorname{Re}\lambda_1(Y(\tau))$ changes sign from negative to positive. On a sufficient short interval of the real axis, to the left of and adjacent to τ_*, we define the function $\Pi(\cdot)$ mapping a slow time $\tau_- < \tau_*$ into the slow time $\Pi(\tau_-) > \tau_*$ such that $\operatorname{Re}\Psi(\tau_-)=\operatorname{Re}\Psi(\Pi(\tau_-))$. The slow solution is analytically continued into the neighborhood of the point τ_* of the plane of the complex variable τ. The function Ψ is continued into the same neighborhood. In the upper half-plane the points τ_- and $\Pi(\tau_-)$ are joined by an arc Γ_{τ_-} of the level curve $\operatorname{Re}\Psi(\tau) = \text{const}$. If τ_- is sufficiently close to τ_*, then the domain K_{τ_-} bounded by Γ_{τ_-} and its mirror reflection $\bar{\Gamma}_{\tau_-}$ in the real axis (Fig. 1) satisfies the following conditions: 1) the slow solution X, Y is analytic, and the right sides in (1.1) are analytic at points of the slow solution; 2) $\lambda_{1,2}(Y)\neq 0$; 3) $\lambda_1(Y)\neq\lambda_2(Y)$ and $\lambda_i(Y)\neq\lambda_j(Y)$, i = 1, 2; j = 3,...,n; 4) $\operatorname{Re}\lambda_j(Y)<0$, j = 3,...,n, for real τ; the fast solution linearized about the equilibrium position, along any curve $\operatorname{Re}\Psi(\tau) = \text{const}$ for $\operatorname{Im}\tau > 0$, has n - 2 eigenvalues with negative real parts, corresponding to the eigenvalues $\lambda_3,\ldots,\lambda_n$; 5) the tangents to the curves $\operatorname{Re}\Psi(\tau) = \text{const}$ are not vertical. We write τ_*^- for the lower bound of values of τ_- for which conditions 1)-5) are satisfied in K_{τ_-}, and we use the notation $\tau_*^+=\sup_{\tau_*^-<\tau<\tau_*}\Pi(\tau)$.

Let x(t), y(t) be the phase point for the fast–slow system with initial condition $y(t_0) = y_0 = Y(\tau_0)$ for $t_0 = \tau_0/\varepsilon$. If $\tau_0 > \tau_*^-$, then we write β_0, γ_0 for the projection of the vector $x(t_0) - X(y_0)$ onto the eigenvalue subspaces of the matrix $\partial f(X(y_0), y_0, 0)/\partial x$, corresponding to the eigenvalues $\lambda_{1,2}$ and $\lambda_3,\ldots,\lambda_n$, respectively.†

THEOREM. If $\tau_*^- < \tau_0$ and the initial point $[x(t_0), y_0]$ is in a C_1^{-1}-neighborhood of the equilibrium $[X(y_0), y_0]$, then for $\tau_0+C_2\varepsilon|\ln\varepsilon|\leqslant\varepsilon t\leqslant\Pi(\tau_0)-C_3\varepsilon|\ln\varepsilon|$ the phase point $[x(t), y(t)]$ is in the $C_4\varepsilon$-neighborhood of the equilibrium $[X(Y(\varepsilon t)), Y(\varepsilon t)]$. If, further, $|\beta_0|>C_5\varepsilon$ and $|\gamma_0|<C_6\varepsilon$, then for $\varepsilon t=\Pi(\tau_0)+C_7\varepsilon|\ln\varepsilon|$ the phase point is in the C_8^{-1}-neighborhood of the equilibrium indicated.

Here C_i (and c_i below) are sufficiently large positive constants, i.e., quantities independent of ε. The appearance of C_i and c_i in a relation is equivalent to the statement that there are such constants for which this relation holds for sufficiently small $\varepsilon > 0$.

COROLLARY. If $\tau_*^- > \tau_0$ and the initial point $[x(t_0), y_0]$ is in a C_1^{-1}-neighborhood of the equilibrium $[X(y_0), y_0]$, then the phase point $[x(t), y(t)]$ is in the $C_9\varepsilon$-neighborhood of the equilibrium $[X(Y(\varepsilon t)), Y(\varepsilon t)]$ for $\tau_0+C_2\varepsilon|\ln\varepsilon|\leqslant\varepsilon t\leqslant\tau_*^+-\delta(\varepsilon)$, where $\delta(\varepsilon)\to 0$ for $\varepsilon\to 0$.

The theorem is proved in Sec. 4. The part of the theorem concerning motion for $\tau_0\leqslant\varepsilon t\leqslant\tau_*-C_{10}^{-1}$, is known ([7], p. 55).

Remark 1. Condition 5) can be weakened. It is sufficient to require that the domain K_{τ_-} be covered by a family of nonintersecting smooth curves, symmetric about the real axis and transversal to the curves $\operatorname{Re}\Psi = \text{const}$.

Remark 2. The condition satisfied by γ_0 in the theorem can be weakened to $<C_6^{-1}\sqrt{\varepsilon}$.

3. Discussion and an Example. In the limit (for $\varepsilon\to 0$), the attraction to the equilibrium position and the separation from it on the slow-time axis proceed instantaneously. The corresponding slow-time limit values are called the collapse and separation times [1]. The dependence of the separation time on the collapse time is called the input–output function [8]. The foregoing theorem implies that, under rather general conditions, the input–output function on the interval (τ_*^-, τ_*) is the function Π constructed above. For the system considered in [2-4], the input–output function is constant for $\tau < \tau_*^-$: if the collapse time precedes τ_*^-, then the separation time is τ_*^+.

†If n = 2 then we formally put $\gamma_0 = 0$.

We give a simple example illustrating the construction in Sec. 2 and the behavior of the input–output function.

Example. Consider the inhomogeneous linear system described in complex form as follows:

$$\dot{z}=(y-i)z+\varepsilon h(y),\quad z=x_1+ix_2,\quad \dot{y}=\varepsilon. \tag{3.1}$$

Let h(y) be analytic for $|\operatorname{Im} y|<2$. The slow solution z = 0, y = τ becomes suitable for τ = 0. The eigenvalue $\lambda_1 = \tau - i$ vanishes for τ = i. The level curve $\operatorname{Re}\Psi=1/2(\tau-i)^2+1/2=\text{const}$ passing through the point i consists of two straight lines cutting the real axis at the points $\tau_*^- = -1$ and $\tau_*^+ = 1$. The level curve Re Ψ = const joining the points (−1, 0) and (0, 1) is a segment of a hyperbola (Fig. 2). If the phase point is attracted to the equilibrium near a time $\tau_0\in(-1, 0)$, then the theorem implies that the separation occurs near the time $\Pi(\tau_0)=|\tau_0|$. Here this is easily established directly. It is sufficient to pass from the point τ_0 to the point $|\tau_0|$ along the hyperbola joining them. For z we obtain a linear equation with a purely imaginary eigenvalue and smoothly varying parameters. On the path considered, |z| is an adiabatic invariant ([9], p. 154): its variation is $O(\varepsilon)$. If $z(\tau_0/\varepsilon) \sim \sqrt{\varepsilon}$, then $z(|\tau_0|/\varepsilon) \sim \sqrt{\varepsilon}$ and separation occurs near $|\tau_0|$. If the phase point is attracted to the equilibrium near the time $\tau_0 < -1$, then $z(-1/\varepsilon)=O(\varepsilon)$. We pass from the point −1 to the point 1 in the τ-plane on the segments [−1, i] and [i, 1]. For z we again have a linear equation with a purely imaginary eigenvalue which, however, vanishes for τ = i. Far from the point i, the quantity |z| varies by an amount $O(\varepsilon)$. The main variation of |z| takes place in the neighborhood of the point i, and is calculated by the stationary-phase method. This yields $|z(1/\varepsilon)|=|\sqrt{2\pi\varepsilon}| \times |h(i)|+O(\varepsilon)$. If h(i) ≠ 0, then $\tau_*^+ = 1$ is the separation time.

In [2] the system

$$\dot{z}=(y-i)z+\varepsilon+\gamma z|z|^2,\quad \dot{y}=\varepsilon, \tag{3.2}$$

is investigated; this system differs from (3.1) for k = 1 by nonlinear terms, which do not have a strong influence on the motion near the equilibrium, and the input–output function remains as for (3.1).

For the systems (3.1) and (3.2), τ_*^- and τ_*^+ are joined by an arc of the curve Re Ψ = const on which $\lambda_1(Y(\tau))$ vanishes at τ_c; the equilibrium position of the fast system X(Y(τ)) is analytic at the point τ_c. There are generalizations of these examples in [3, 4, 10].† This however is not the general case. If $\lambda_1(Y(\tau_c))=0$, then the matrix $\partial f(X(Y(\tau_c)), Y(\tau_c), 0)/\partial x$ is singular and the function X(Y(τ)) has in general a branch of the type $\sqrt{\tau-\tau_c}$. The question of how to construct the input–output function for $\tau < \tau_*^-$, in this case and in general for the general case, remains open. It is plausible to suggest that this function is constant for rather large system classes.

Our problem is related to "duck" theory [8]. In general cases the phase point of a fast–slow system moves for a long period near an unstable equilibrium of the fast system. But a "duck" exists when particularly favorable circumstances prevail simultaneously: in a system depending on a supplementary parameter, a "duck" exists for an exponentially small interval of parameter values. However stability-loss persistence is a common phenomenon for analytic systems. On the other hand "ducks" also occur in systems with finite smoothness, while stability-loss persistence occurs in general only for analytic systems [5].

4. Proof of the Theorem. 4.1. Preliminary Transformations.

Fix $\tau_-\in(\tau_*^-, \tau_0)$, let $\Gamma = \Gamma_{\tau_-}$, let $K = K_{\tau_-}$, and let $\xi=(\xi_1, \xi_2, \ldots, \xi_n)=x-X(y)$. We consider the complex curve $L=\{\xi, y: \xi=0,\ y=Y(\tau),\ \tau\in K\}$, and write $|\cdot|$ for the Hermitian norm.

LEMMA 1. In the c_1^{-1}-neighborhood of L, the real-analytic variable change $\xi, y\mapsto\hat{\xi}, \hat{y}$, which differs by $O(\varepsilon)$ with respect to ξ from a linear transformation and is linear with respect to y, transforms the rapid-slow system (1.1) into the following form (we omit the hats over the new variables):

$$\begin{aligned}&\dot{\xi}=A(y, \varepsilon)\xi+O(|\xi|^2)+O(\varepsilon^3),\\&\dot{y}=\varepsilon F(y, \varepsilon)+\varepsilon O(|\xi|^2)+O(\varepsilon^3),\quad F=g(X(y), y, 0)+O(\varepsilon).\end{aligned} \tag{4.1}$$

The matrix A is block-diagonal, has the eigenvalues $\Lambda_i(y, \varepsilon)=\lambda_i(y)+O(\varepsilon)$, i = 1, 2,...,n, and contains two blocks of dimension 2 × 2 and (n − 2) × (n − 2), respectively. The first block is of the form $\frac{1}{2}\begin{pmatrix}\Lambda_1+\Lambda_2 & i(\Lambda_1-\Lambda_2)\\ i(\Lambda_2-\Lambda_1) & \Lambda_1+\Lambda_2\end{pmatrix}$.

†The separation time is incorrectly calculated in [10].

term into $O(\varepsilon^2)$. In the resulting system, a similar substitution causes the free term to become $O(\varepsilon^3)$.

B. To within an error $O(\varepsilon^3)$, we reduce the matrix in the right side of the equation for ξ, linearized about $\xi = 0$, to the form indicated in Lemma 1. In a sufficiently small neighborhood of L in which $\lambda_1(y) \neq \lambda_2(y)$ and $\lambda_i(y) \neq \lambda_j(y)$, $i = 1, 2$, $j = 3, \ldots, n$, the matrix $\mathscr{A}(y)$ is similar to a matrix $\Lambda(y)$ of the required form. By means of a similarity transformation, we can arrange that the linear change of the variable ξ and the matrix Λ be real-analytic with respect to y. This change reduces the equation for ξ to a form in which the matrix of the linearized right side is $\Lambda(y) + O(\varepsilon)$. This matrix is similar to a matrix $\hat{\Lambda}(y, \varepsilon) = \Lambda(y) + O(\varepsilon)$ of the required form, and the linear change of the variable ξ in the similarity transformation can be chosen to be real-analytic and differing by $O(\varepsilon)$ from the identical transformation. In the new variables the matrix in the linearized right side is $\hat{\Lambda}(y, \varepsilon) + O(\varepsilon^2)$. After a similar variable change for this matrix, we obtain $A(y, \varepsilon) + O(\varepsilon^3)$ as the matrix of the linearized right side, where $A(y, \varepsilon)$ is of the form indicated in Lemma 1.

C. We annihilate, with accuracy $O(\varepsilon^3)$, the terms in the equation for y linear in ξ. To this end we make the substitution $\hat{y} = y + \varepsilon M(y, \varepsilon)\xi$, where M is a matrix to be determined. The new variable $\hat{y}$ satisfies the following equation (we omit the hats):

$$\dot{y} = \varepsilon G(y) + \varepsilon M A \xi + \varepsilon O(\xi) + O(\varepsilon^2).$$

Since A is nonsingular, M can be selected to eliminate terms linear in ξ of order ε. A similar substitution in the resulting equation eliminates terms linear in ξ and of order ε^2.

LEMMA 2. In the c_2^{-1}-neighborhood of L, there is a real-analytic variable change $\hat{\xi} = \xi + O(|\xi|^2)$ in the system (4.1), eliminating some of terms quadratic in ξ so that, if the new variables ξ_1 and ξ_2 (the hats are omitted) are replaced by the variables $z = \xi_1 + i\xi_2$ and $w = \xi_1 - i\xi_2$, then the right sides of the equations for z and w contain no quadratic monomials in z and w, respectively.

Proof. If $\Lambda_i \neq 0$, $i = 1, 2, \ldots, n$, then the relevant quadratic terms are nonresonant, and can be eliminated by virtue of the Poincaré–Dulac theorem ([9], p. 168).

4.2. Conditions Ensuring That the Imaginary Part Vanishes. Let $\xi(t)$, $y(t)$ be a solution of the system obtained from (4.1) after the transformation in Lemma 2, and let the initial data $\xi(t_0)$ and $y(t_0)$ be real.

LEMMA 3. If, in the plane of the complex time t, a solution $\xi(t)$, $y(t)$ can be continued analytically into a neighborhood U of a segment of the real axis, then it can be continued analytically into $\bar{U}$, and $\xi(\bar{t}) = \overline{\xi(t)}$, $y(\bar{t}) = \overline{y(t)}$ for $t \in U \cup \bar{U}$; a bar over a symbol indicates the complex conjugate.

The proof is obvious.

We introduce the notation $z = \xi_1 + i\xi_2$ and $w = \xi_1 - i\xi_2$ and rewrite the system in the variables z, w, $\eta = (\xi_3, \ldots, \xi_n)$, and y. The foregoing implies that, for real-analytic solutions, we have $w(t) = \overline{z(\bar{t})}$. This condition will be used instead of the differential equation for w. Hence

$$\begin{aligned} \dot{z} &= \Lambda_1(y) z + O(|\xi|^2) + O(\varepsilon^3), \\ \dot{\eta} &= B(y)\eta + O(|\xi|^2) + O(\varepsilon^3), \\ \dot{y} &= \varepsilon F(y) + \varepsilon O(|\xi|^2) + O(\varepsilon^3). \end{aligned} \tag{4.3}$$

For brevity we omit the argument n + m in the right sides. Instead of n + m ordinary differential equations, we obtain n + m − 1 differential equations with time-lag for imaginary values of the argument.

4.3. The Initial Part of the Motion. Let $z(t)$, $\eta(t)$, $y(t)$ be a solution of the original system written in the variables z, η, and y of Sec. 4.2. Results in [7] (p. 55) imply that, if the initial point is in the C_1^{-1}-neighborhood of equilibrium, then for some $t_1 = t_0 + O(|\ln\varepsilon|)$ a solution is in an $O(\varepsilon)$-neighborhood of equilibrium. Let the constant C_6 be arbitrary and let C_5 be such that, if $|\gamma_0| < C_6\varepsilon$ and $|\beta_0| > C_5\varepsilon$, then $|z(t_0)| > \varepsilon$. For these β_0 and γ_0 we can take t_1 to be the first time $t > t_0$ for which $|z(t)| = \varepsilon$.

4.4. Preliminary Construction for the Main Part of the Motion. A. We write $Y_\varepsilon(\tau)$ for the solution of the modified slow equation $\dot{y} = \varepsilon F(y)$ with the initial condition $Y_\varepsilon(\varepsilon t_1) = y(t_1)$, introduce the "phase" $\Psi_\varepsilon(\tau) = \int_{\tau_*}^{\tau} \Lambda_1(Y_\varepsilon(\vartheta))\,d\vartheta$, and consider arcs of level curves $\mathrm{Re}\,\Psi_\varepsilon =$ const in the upper half-plane of the complex variable τ with ends on the real axis. Let Γ_ε be the arc with left end εt_1 and right end denoted by $\Pi_\varepsilon(\varepsilon t_1)$. Clearly $\Pi_\varepsilon(\varepsilon t_1) = \Pi(\tau_0) + O(\varepsilon|\ln\varepsilon|)$. Let K_ε be the domain of the τ-plane bounded by Γ_ε and $\bar{\Gamma}_\varepsilon$, and let $S(T)$ be the part (sector) of this domain to the left of the line $\mathrm{Re}\,\tau = T$.

B. In the right side of the equation for η in (4.3) we leave only the first term, and put $y = Y(\tau)$, $\tau \in K$, $\tau =$ const. This yields a linear system for η depending on the parameter τ. For real τ, the eigenvalues of the matrix of the system have negative real parts. In this case it follows from ([1], p. 272) that there is a quadratic Lyapunov function $V_1(\eta, \tau)$ whose derivative, calculated in accordance with the system, is $-|\eta|^2$, and $c_3^{-1}|\eta|^2 \leqslant V_1(\eta, \tau) \leqslant c_3|\eta|^2$, and $|\partial V_1/\partial\tau| = O(|\eta|^2)$. If τ is in the upper half-plane, then we change the independent variable in the resulting linear system as follows: We consider the level curve $\mathrm{Re}\,\Psi =$ const passing through the point τ, assume that the time t varies along this curve, and use the arc length measured along this curve as the new variable. If τ is in the lower half-plane we make a similar variable change using the conjugate level curve. Condition 4) in Sec. 2 implies that the matrices of the systems obtained have eigenvalues with negative real parts. For these systems there are Lyapunov function satisfying the same bounds as V_1.

4.5. The Principal Part of the Motion. By virtue of Sec. 4.3, $|z(t_1)| + |\eta(t_1)| < c_4\varepsilon$, and Cauchy's theorem ([12], p. 23) implies that the solution $z(t)$, $\eta(t)$, $y(t)$ can be continued analytically into some disk with center at t_1, and so into a sector $S(T)$ with the conservation of the inequalities

$$|z(t)| < 2c_4\varepsilon,\quad |\eta(t)| < 2c_3^2 c_4\varepsilon,\quad |y(t) - Y_\varepsilon(\varepsilon t)| < \varepsilon. \tag{4.4}$$

We write T_1 for the upper bound of values of T for which the solution can be analytically continued into the sector $S(T)$ with conservation of the inequalities (4.4): $S_1 = S(T_1)$. Our objective is to prove that $\varepsilon T_1 = \Pi_\varepsilon(\varepsilon t_1)$.

We first estimate $y(t)$. It follows from (4.3) and (4.4) that $\dot{y} = \varepsilon F(y) + O(\varepsilon^3)$ for $|y(t)$; hence $Y_\varepsilon(\varepsilon t)| = O(\varepsilon^2) < 1/2\varepsilon$.

Next consider $|\eta(t)|$. Each point of the sector S_1 can be reached from t_1 by passing first along the real axis in the direction of increasing $\mathrm{Re}\,t$, and then along an arc of the level curve $\mathrm{Re}\,\Psi_\varepsilon =$ const in the upper half-plane or along the conjugate arc. On the real axis (4.3), (4.4), and the definition of V_1 imply that $\dot{V}_1 = -|\eta|^2 + O(|\eta|\varepsilon^2)$. Hence $|\eta(t)| < c_3c_4\varepsilon$ for real t. On the arc of the curve $\mathrm{Re}\,\Psi_\varepsilon =$ const or the conjugate curve, we can use the function $V_{2,3}$ to prove similarly that $|y(t)|$, $|\eta(t)|$. By employing V_j we can also prove that $|\eta(t)| = O(\varepsilon^2)$. when $\mathrm{Re}\,t > t_1 + c_5|\ln\varepsilon|$.

Consider $|z(t)|$. From (4.3), (4.4), and the bounds of $|y(t)|$ and $|\eta(t)|$ we conclude that

$$\dot{z} = \Lambda_1(Y_\varepsilon(\varepsilon t))z + O(\alpha), \tag{4.5}$$

for $t \in S_1$, where $\alpha = \varepsilon^2$ for $\mathrm{Re}\,t < t_1 + c_5|\ln\varepsilon|$ and $\alpha = \varepsilon^3$ for $\mathrm{Re}\,t \geqslant t_1 + c_5|\ln\varepsilon|$.

$|z(t)|>c_7\alpha$. Hence $|z(t)|<3/2\varepsilon$ for $t\in S_1$. ..., $|z(t)|$ decreases as long as

Hence, for $t\in S_1$, inequalities (4.4) hold with a margin of safety. If $T_1<\Pi_\varepsilon(\varepsilon t_1)$, then Cauchy's theorem implies that the solution z(t), η(t), y(t) can be continued analytically into a sector S(T), T > T_1, with inequalities (4.4) conserved. This is in contradiction with the definition of T_1, and so $\varepsilon T_1=\Pi_\varepsilon(\varepsilon t_1)=\Pi(\tau_0)+O(\varepsilon|\ln\varepsilon|)$. Using inequalities (4.4) for real t and returning to the original variables, we obtain the first conclusion of the theorem.

4.6. The Separation Region. If $|\beta_0|>C_5\varepsilon$ and $|\gamma_0|<C_6\varepsilon$, then results in Sec. (4.3) imply that $|z(t_1)|=\varepsilon$. Thus, by virtue of Sec. 4.5, for $T_1=\varepsilon^{-1}\Pi_\varepsilon(\varepsilon t_1)$, we have $|z(T_1)|=|z(t_1)|+O(\varepsilon^2|\ln\varepsilon|)>1/2\varepsilon$, $|z(T_1)|=O(\varepsilon)|\eta(t_1)|=O(\varepsilon^2)$, and $|y(T_1)-\dot{Y}(\varepsilon T_1)|=O(\varepsilon)$. It follows from (5.4) that when

$$|z|\leqslant c_7^{-1},\ |z|\geqslant 1/4\varepsilon,\ |\eta|\leqslant c_8^{-1}\sqrt{|z|},\ |y-Y(\varepsilon T_1)|\leqslant c_9^{-1} \tag{4.7}$$

the following relations hold:

$$c_{10}^{-1}|z|^2<d|z|^2/dt<c_{10}|z|^2,\ \dot{V}_1<\sqrt{V_1}|z|^2,\ \dot{y}=O(\varepsilon). \tag{4.8}$$

Hence, if the phase point is in the domain defined by (4.7), then

$$|z(t)|>\frac{1}{2}\varepsilon\exp\left(c_{10}^{-1}(t-T_1)/2\right),\ V_1(\eta(t),\varepsilon t)<c_{11}|z(t)|^4,$$
$$|\eta(t)|<c_{12}|z(t)|^2,\ y(t)=Y(\varepsilon T_1)+O(\varepsilon)(t-T_1+1). \tag{4.9}$$

Let $c_{13}=\max\{2c_7,\ (2c_{12}c_8)^{2/3}\}$. For $|z|<c_{13}^{-1}$ the inequality $c_{12}|z|^2\leqslant\frac{1}{2}c_8^{-1}\sqrt{|z|}$ holds. Let T_2 be the maximum time for which relations (4.7) and the inequality $t\in[T_1,T_2]$ hold when $|z(t)|\leqslant c_{13}^{-1}$. It follows from (4.9) that $T_2<T_1+O(|\ln\varepsilon|)$ and, for t = T_2, the phase point is in the interior of the domain defined by relations (4.7). Hence $|z(T_2)|=c_{13}^{-1}$, and this proves the second conclusion of the theorem.

The author takes this opportunity to thank V. I. Arnol'd for suggesting this subject and D. V. Anosov for his discussion of the work.

LITERATURE CITED

1. E. F. Mishchenko and N. Kh. Rozov, Differential Equations with Small Parameters and Relaxation Oscillations [in Russian], Moscow (1975).
2. M. A. Shishkova, Dokl. Akad. Nauk SSSR, 209, No. 3, 576-579 (1973).
3. S. Karimov, Asymptotic Properties of Solutions of Certain Differential Equations with Small Parameters Multiplying Derivatives, when There Is an Alteration of the Stability of a Stationary Point in the "Fast-Motion" Plane, Doctoral Dissertation, Physicomathematical Sciences [in Russian], Osh (1980).
4. S. Karimov, Differents. Uravn., 21, No. 10, 1698-1701 (1985).
5. A. I. Neishtadt, Differents. Uravn., 23, No. 12, 2060-2067 (1987).
6. A. I. Neishtadt, Usp. Mat. Nauk, 40, No. 5, 300-301 (1985).
7. A. B. Vasil'eva and V. F. Butuzov, Asymptotic Expansions of Solutions of Singularly Perturbed Equations [in Russian], Moscow (1973).
8. A. K. Zvonkin and M. A. Shubin, Usp. Mat. Nauk., 39, No. 2, 77-127 (1984).
9. V. I. Arnol'd, Supplementary Chapters on the Theory of Ordinary Differential Equations [in Russian], Moscow (1978).
10. S. Karimov, Izv. Akad. Nauk KirgSSR, No. 3, 10-15 (1973).
11. R. E. Bellman, Introduction to Matrix Theory [Russian translation], 2nd Ed., Moscow (1976).
12. V. V. Golubev, Lectures on the Analytic Theory of Differential Equations [in Russian], 2nd Ed., Moscow—Leningrad (1950).