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SINGULAR HOPF BIFURCATION TO RELAXATION OSCILLATIONS II*

S. M. BAER† AND T. ERNEUX‡

Abstract. This paper is a continuation of our study of singular Hopf bifurcations to relaxation oscillations [*SIAM Journal on Applied Mathematics*, 46 (1986), pp. 721-739]. In our previous analysis, a system of two first-order nonlinear equations is investigated, and it is shown how the bifurcation problem can be reduced to a weakly perturbed conservative system of two first-order equations. In this paper, a method, based on analyzing nearly conservative quantities, was developed to find the bifurcation diagram of the periodic solutions emerging from singular Hopf bifurcation points.

Specifically, the subcritical Hopf bifurcation of the FitzHugh-Nagumo equations is analyzed. The extension from the earlier bifurcation analysis is achieved by performing a perturbation analysis from the first integral of the leading-order nonlinear oscillator. At a critical value of a parameter, the first integral exists up to an additional order in the perturbed equation. This condition permits the resolution of features determined by higher-order terms and reveals the "kneebend" in the bifurcation curve.

Key words. higher-order analysis, FitzHugh-Nagumo equations

AMS(MOS) subject classifications. 34C15, 34E15, 58F14, 58F21

1. Introduction. Strongly nonlinear oscillations occur in many problems and are called relaxation oscillations when they depend on at least two different timescales. Available techniques for studying these oscillations are either numerical or based on singular perturbation methods [11], [7] that exploit timescale disparities. The singular perturbation approach captures the large amplitude and pulsating behavior of the relaxation oscillators; however, analytical solutions are generally limited to second- or third-order nonlinear differential equations. In contrast, bifurcation methods apply to systems of an arbitrary number of equations, but the analysis is limited to a description of the small amplitude harmonic oscillations near the bifurcation point. In problems where both singular and bifurcation methods are applicable, solutions exhibiting harmonic oscillations become pulsating as the bifurcation parameter is tuned away from the bifurcation point. By combining both techniques [1], we propose a method to analytically study relaxation oscillator problems.

In [1] we considered equations of the form

$$\frac{dX}{dT} = F(X, Y, \lambda, \varepsilon), \qquad \frac{dY}{dT} = \varepsilon G(X, Y, \lambda, \varepsilon), \tag{1.1}$$

where T is time, λ is the bifurcation parameter, and $\varepsilon \ll 1$ is a fixed parameter. We have assumed that $\lambda = \lambda_0(\varepsilon)$ corresponds to a Hopf bifurcation. By investigating this Hopf bifurcation in the limit $\varepsilon \to 0$, we have shown how the harmonic oscillations for λ near λ_0 becomes pulsating oscillations. As $\varepsilon \to 0$, the regular Hopf bifurcation becomes singular, and the mathematical problem is to investigate the perturbation of a double zero eigenvalue. It appears that this problem is quite different from other double zero eigenvalue bifurcation problems, which results from the coalescence of a Hopf bifurcation point and a steady bifurcation or limit point [12], [10], [8], [6]. In [1] we analyzed

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the bifurcation problem for (1.1) and discovered that it eventually reduces to the study of the following equations for x and y

$$\frac{dx}{dt}=-y+\varepsilon f(x, y, L, \varepsilon), \qquad \frac{dy}{dt}=x(1+y)+\varepsilon g(x, y, L, \varepsilon), \tag{1.2}$$

where $L=\varepsilon^{-1}(\lambda-\lambda_0(0))=0(1)$ is the new bifurcation parameter.

The reduced problem is obtained by setting $\varepsilon=0$ in (1.2) and is given by

$$\frac{dx}{dt}=-y, \qquad \frac{dy}{dt}=x(1+y). \tag{1.3}$$

This system is conservative and admits a one-parameter family of periodic solutions. Small amplitude solutions are harmonic, while large amplitude solutions are pulsating. By studying the effect of the correction terms in (1.2), we obtained in [1] a solvability condition that relates the amplitude of the periodic solution to the bifurcation parameter L.

In this paper, we propose a different method to analyze the effect of the correction terms in (1.2). This method is particularly useful in cases where successive orders of the perturbation analysis must be investigated. This occurs, for example, when the Hopf bifurcation is vertical for a specific value of a parameter that is different from the bifurcation parameter λ. A vertical bifurcation appears as the direction of bifurcation changes from supercritical to subcritical as this second parameter is changed. We illustrate this method by investigating the subcritical Hopf bifurcation of the FitzHugh–Nagumo equations. The results of the asymptotic analysis are then compared to a numerical solution representing the exact solution. The method is based on analyzing nearly conservative quantities as it is proposed by the method of averaging for strongly nonlinear oscillators [11]. However, the bifurcation diagram does not require the determination of the phase of the oscillations, and we do not need to reformulate the differential equations in standard form.

The paper is organized as follows. Sections 2 and 3 investigate a degenerate problem that appears in the study of a subcritical Hopf bifurcation. We develop a method that leads to a more complete description of the periodic solutions. Section 4 investigates the accuracy of our method by comparing the numerical and asymptotic solutions.

2. Formulation of the FitzHugh–Nagumo bifurcation problem. The FitzHugh–Nagumo equations consist of two ordinary differential equations given by

$$\frac{dv}{dt}=-F(v)-w+I, \tag{2.1}$$

$$\frac{dw}{dt}=\varepsilon(v-\gamma w), \tag{2.2}$$

where $F(v)$ is a cubic nonlinear function defined by

$$F(v)=v(v-1)(v-a)=v^3-(a+1)v^2+av, \qquad (0<a<\tfrac{1}{2}). \tag{2.3}$$

The steady-state solution $(u, v)=(\bar{u}(I), \bar{w}(I))$ is defined by the implicit relation

$$I=F(\bar{v})+\bar{v}/\gamma, \qquad \bar{w}=\bar{v}/\gamma. \tag{2.4}$$

From (2.4), we find that the steady state is unique if the condition

$$F'(\bar{v})+1/\gamma>0 \tag{2.5}$$

is satisfied. In this paper, we consider the case of a unique steady state because our main interest is to study the Hopf bifurcation. Expression (2.5) is a quadratic function of $\bar{v}$, and the inequality implies that

$$\gamma < 3(a^2 - a + 1)^{-1}. \tag{2.6}$$

We now introduce the deviations from the steady state

$$X = v - \bar{v} \quad \text{and} \quad Y = w - \bar{w} \tag{2.7}$$

and write (2.1) and (2.2) in terms of X and Y as follows:

$$\frac{dX}{dt} = -F'(\bar{v})X - Y - F''(\bar{v})\frac{X^2}{2} - X^3, \tag{2.8}$$

$$\frac{dY}{dt} = \varepsilon(X - \gamma Y). \tag{2.9}$$

We investigate the limit $\varepsilon \to 0$ by reformulating these equations as a conservative system of equations with constant coefficients plus a small $O(\varepsilon^{1/2})$ term, which is a function of all the parameters. To this end, we first write (2.8) and (2.9) as a second-order equation for X, determine the Hopf bifurcation, and then introduce the new variables and the new bifurcation parameter.

After eliminating Y and grouping linear and nonlinear terms, (2.8) and (2.9) are equivalent to

$$\begin{aligned} &\frac{d^2X}{dt^2} + \frac{dX}{dt}[F'(\bar{v}) + \varepsilon\gamma] + X\varepsilon[1 + \gamma F'(\bar{v})] + \varepsilon\gamma F''(\bar{v})\frac{X^2}{2} + \varepsilon\gamma X^3 \\ &\quad + F''(\bar{v})X\frac{dX}{dt} + 3X^2\frac{dX}{dt} = 0. \end{aligned} \tag{2.10}$$

In (2.10), $\bar{v}$ represents our bifurcation parameter. For each value of $\bar{v}$, we can determine a unique value of I by using (2.4). From (2.10), we also find that if

$$\omega^2 = \varepsilon(1 - \varepsilon\gamma^2) > 0, \tag{2.11}$$

or, equivalently, if

$$\varepsilon < \gamma^{-2}, \tag{2.12}$$

there exists a Hopf bifurcation point given by $(X, \bar{v}) = (0, v_H)$. ω is the frequency of the oscillations at the Hopf point. The critical point $\bar{v} = v_H$ satisfies the following equation:

$$F'(v_H) + \varepsilon\gamma = 0 \quad \text{or} \quad 3v_H^2 - 2(a+1)v_H + a + \varepsilon\gamma = 0. \tag{2.13}$$

Equation (2.13) is a quadratic equation in v_H. If $a^2 - a + 1 - 3\varepsilon\gamma > 0$, there exist two real positive solutions given by

$$v_H(\varepsilon) = v_\pm(\varepsilon) = \tfrac{1}{3}[a + 1 \pm (a^2 - a + 1 - 3\varepsilon\gamma)^{1/2}]. \tag{2.14}$$

From the linearized theory, we conclude that the steady-state solution $X = 0$ is unstable if $v_- < \bar{v} < v_+$.

According to the bifurcation analysis given in [1], the Hopf bifurcation is singular if $\bar{v} - v_H = O(\varepsilon)$. This analysis also shows that $X = O(\varepsilon^{1/2})$ in this critical regime. Moreover, we note from (2.11) that the frequency ω is a small $O(\varepsilon^{1/2})$ quantity. These scalings motivate the introduction of the new dependent and independent variables x, τ, and L, given by

$$x = -\varepsilon^{-1/2} F'' X, \tag{2.15}$$

$$\tau = \varepsilon^{1/2} t, \tag{2.16}$$

$$L = \varepsilon^{-1}(\bar{v} - v^*), \tag{2.17}$$

where $\bar{v} = v^*$ satisfies the condition

$$F'(v^*) = 0 \quad \text{or} \quad 3v^{*2} - 2(a+1)v^* + a = 0, \tag{2.18}$$

and we denote by F'' the expression $F''(v^*) = 6v^* - 2(a+1)$. The factor F'' in (2.15) has been introduced for convenience. Thus the critical point $\bar{v} = v^*$ corresponds to the Hopf bifurcation $v_H(\varepsilon)$ evaluated at $\varepsilon = 0$. Inserting (2.15)-(2.18) into (2.10) and defining $y = -dx/d\tau$, we obtain the following equations for x and y:

$$\begin{aligned} \frac{dx}{d\tau} &= -y, \\ \frac{dy}{d\tau} &= x(1+y) + \varepsilon^{1/2}\left[-y(F''L + \gamma) - \frac{\gamma}{2}x^2 - \frac{3}{F''^2}x^2 y\right] \\ &\quad + \varepsilon\left[\gamma F'' L x + \frac{\gamma}{F''^2}x^3 + \frac{6L}{F''}xy\right] + \varepsilon^{3/2}\left[-3L^2 y - \frac{3L\gamma}{F''}x^2\right] \\ &\quad + \varepsilon^2 3\gamma L^2 x. \end{aligned} \tag{2.19}$$

Equations (2.19) correspond to the starting point of our analysis. In [1] we analyzed these equations in the limit $\varepsilon \to 0$. We now reexamine the problem by using a different method. This method simplifies the higher-order analysis. We first investigate the reduced problem obtained by setting $\varepsilon = 0$ in (2.19). This problem for $(x, y) = (a(\tau), b(\tau))$ is given by

$$\begin{aligned} \frac{da}{d\tau} &= -b, \\ \frac{db}{d\tau} &= a(1+b). \end{aligned} \tag{2.20}$$

By studying the properties of (2.20), we find that this system of equations is conservative and admits a one-parameter family of periodic solutions. (See Fig. 1.) A first integral is defined by

$$M = b - \ln|1+b| + \tfrac{1}{2}a^2, \tag{2.21}$$

where M is a constant. By analyzing (2.20) in the phase plane (a, b) and using (2.21), we find a closed orbit surrounding the center $(a, b) = (0, 0)$ for each positive value of M $(0 \leqq M < \infty)$. These periodic orbits are bounded below by the line $b = -1$. Using (2.21), we also note that the maximum of $a(\tau)$ corresponds to $b = 0$ and is given by $a_M = (2M)^{1/2}$. Thus the parameter M characterizes the amplitude of a specific periodic solution. It also characterizes the period that we denote by T_M $(2\pi \leqq T_M < \infty$ as $0 \leqq M < \infty)$.

We now analyze (2.19) with $\varepsilon \neq 0$, but small. We propose to introduce a new variable $N(\tau) = N(x(\tau), y(\tau))$ defined by

$$N(\tau) = y - \ln|1+y| + \tfrac{1}{2}x^2. \tag{2.22}$$

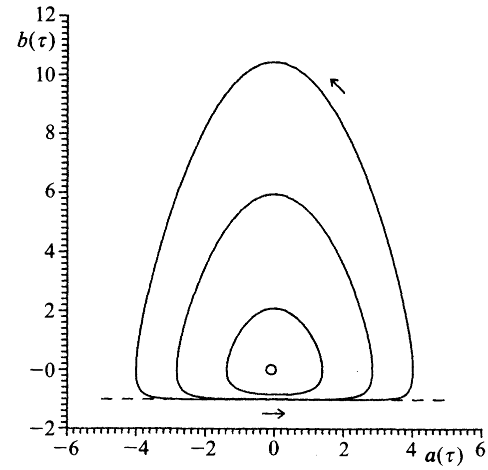

FIG. 1. *Periodic orbits in the phase plane* (a, b). *Three periodic solutions are represented in the phase plane* (a, b) *and satisfy the conservation system of* (2.20). *Each solution is characterized by a different value of the constant M, defined by* (2.21). *The values of M are* $M = 0.8$, $M = 4$ *and* $M = 8$.

The definition of N is motivated by the fact that $N(\tau) = M + O(\varepsilon)$ as $\varepsilon \to 0$ i.e., $N(\tau)$ is a constant in first approximation. Consequently, we expect that the higher-order analysis is simplified if we use the equation for N. We determine this equation for N by differentiating (2.22) with respect to τ and by using (2.19) to eliminate the time derivatives. We then obtain

$$\begin{aligned} \frac{dN}{d\tau} &= R(x, y, \varepsilon^{1/2}) \\ &= \varepsilon^{1/2} \frac{y}{(1+y)} [-y(F''L+\gamma) - \tfrac{1}{2}\gamma x^2 - \tfrac{1}{2}\gamma_0 x^2 y] + O(\varepsilon), \end{aligned} \tag{2.23}$$

where γ_0 is a constant defined by

$$\gamma_0 = 6/F''^2. \tag{2.24}$$

We now seek periodic solutions of (2.19) and (2.23). We assume that these solutions have the form

$$x(T, \varepsilon^{1/2}) = a(T) + \varepsilon^{1/2} x_1(T) + \cdots, \tag{2.25}$$

$$y(T, \varepsilon^{1/2}) = b(T) + \varepsilon^{1/2} y_1(T) + \cdots, \tag{2.26}$$

$$N(T, \varepsilon^{1/2}) = M + \varepsilon^{1/2} N_1(T) + \cdots, \tag{2.27}$$

where the right-hand sides are periodic in T and

$$T = \sigma(\varepsilon^{1/2})\tau = (1 + \varepsilon^{1/2}\sigma_1 + \varepsilon\sigma_2 + \cdots)\tau. \tag{2.28}$$

From (2.22), we see that N must be periodic if x and y are periodic. Thus

$$\frac{dN}{dT} = (1 + \varepsilon^{1/2}\sigma_1 + \varepsilon\sigma_2 + \cdots)\frac{dN}{d\tau} \tag{2.29}$$

must average to zero over a single period for any periodic solution. This implies that

$$\int_0^{T_M} R(x, y, \varepsilon^{1/2})\, dT = 0. \tag{2.30}$$

Since periodic solutions satisfy (2.30), we can discover which solutions are periodic by substituting relatively low-order asymptotic expansions into (2.30). This is the main advantage of our technique, since the more conventional alternative would require us to expand x and y to much higher order. In particular, at $O(\varepsilon^{1/2})$, we have that

$$\int_0^{T_M} R(a, b, 0)\, dT = 0, \tag{2.31}$$

where a and b satisfy (2.20) with d/dT, and T_M is a function of M with M defined by (2.21). Using (2.23), the integral (2.31) is equivalent to

$$\int_0^{T_M} \frac{b}{(1+b)}\left[-b(F''L+\gamma)-\frac{1}{2}\gamma a^2-\frac{1}{2}\gamma_0 a^2 b\right] dT = 0. \tag{2.32}$$

Note that

$$\int_0^{T_M} \frac{b^2a^2}{(1+b)}\, dT = \int_0^{T_M} ba^2\, dT - \int_0^{T_M} \frac{a^2b}{(1+b)}\, dT = -\int_0^{T_M} \frac{a^2b}{(1+b)}\, dT, \tag{2.33}$$

and condition (2.32) can be reformulated as

$$(F''L+\gamma)\int_0^{T_M} \frac{b^2}{(1+b)}\, dT + \frac{1}{2}(\gamma-\gamma_0)\int_0^{T_M} \frac{a^2b}{(1+b)}\, dT = 0. \tag{2.34}$$

By expanding (2.13) to $O(\varepsilon)$ and using (2.18), the coefficient of the first integral in (2.34) can be expressed in terms of the deviation from the Hopf bifurcation point. In terms of L, defined by (2.17), the Hopf bifurcation point corresponds to

$$L = L_H = -\gamma/F'', \tag{2.35}$$

and (2.34) can then be rewritten as

$$F''(L-L_H)\int_0^{T_M} \frac{b^2}{(1+b)}\, dT + \frac{1}{2}(\gamma-\gamma_0)\int_0^{T_M} \frac{a^2b}{(1+b)}\, dT = 0. \tag{2.36}$$

Since the two integrals are functions of M, this equation gives L as a function of M and is called the amplitude or bifurcation equation. In [1] these two integrals have been evaluated numerically for all possible values of M, and the amplitude of the periodic solutions described by $a_M = (2M)^{1/2}$ has been represented as a function of L. For future reference, we briefly investigate (2.36) in the limit of small and large amplitude periodic solutions (i.e., we investigate the limits $M \to 0$ and $M \to \infty$, respectively).

2.1. Small amplitude periodic solutions. By applying the Poincaré-Lindstedt method, we construct from (2.20) with time derivative d/dT the small amplitude periodic solution. We find that

$$a(\theta, \eta) = \eta(e^{i\theta}+\text{c.c.}) - \eta^2\left(\frac{i}{3}e^{2i\theta}+\text{c.c.}\right) + O(\eta^3), \tag{2.37}$$

$$b(\theta, \eta) = -\eta(i\,e^{i\theta}+\text{c.c.}) - \eta^2(\tfrac{2}{3}e^{2i\theta}+\text{c.c.}) + O(\eta^3), \tag{2.38}$$

where $\theta = (1+O(\eta^2))T$, c.c. means complex-conjugate, and the amplitude η is defined by

$$\eta = (2\pi)^{-1}\int_0^{2\pi} a(\theta, \eta)\, e^{-i\theta}\, d\theta. \tag{2.39}$$

With (2.37) and (2.38), we compute the leading approximations of the two integrals in (2.36) and obtain

$$4\pi\eta^2[F''(L-L_H)-\tfrac{1}{2}(\gamma-\gamma_0)\eta^2](1+O(\eta^2))=0. \tag{2.40}$$

Equation (2.40) admits the following two solutions:

(2.41) the trivial solution $\eta=0$; and

(2.42) and the nontrivial solution $\eta=[2F''(L-L_H)/(\gamma-\gamma_0)]^{1/2}+O(L-L_H)$.

Solution (2.41) corresponds to the steady-state solution $X=0$. This solution is unstable if $v_-(0)<v^*<v_+(0)$. Thus the bifurcation is supercritical if $L>L_H$ when $v^*=v_-(0)$ and if $L<L_H$ when $v^*=v_+(0)$. Noting then that $F''>0$ when $v^*=v_-(0)$ and $F''<0$ when $v^*=v_+(0)$, we conclude from (2.42) that the *bifurcation is supercritical if* $\gamma>\gamma_0$ and, according to Hopf theorem [9], the bifurcating periodic solutions are stable. On the contrary, *if* $\gamma<\gamma_0$, *the bifurcation is subcritical* ($L<L_H$ when $v^*=v_-(0)$ and $L>L_H$ when $v^*=v_+(0)$) and the bifurcating periodic solutions are unstable.

2.2. Large amplitude periodic solutions. As $M\to\infty$, the periodic solutions of (2.20) are staying mostly near the line $b=-1$. Using (2.21), we find an approximation of the periodic orbit near $b=-1$ (the outer solution). It is given by

$$b\approx -1+\exp[\tfrac{1}{2}(a^2-a_M^2)-1], \qquad -a_M<a<a_M, \tag{2.43}$$

where $a_M=(2M)^{1/2}$ denotes the maximum value of a. Evaluating then the two integrals in (2.36) with respect to a (from the first equation in (2.20) with $b=-1$, we note that $d\tau=da$), we obtain

$$(2\pi)^{1/2}\exp(M+1)[F''(L-L_H)-\tfrac{1}{2}(\gamma-\gamma_0)]\approx 0 \tag{2.44}$$

or, equivalently,

$$L\to L_L=L_H+\frac{1}{2F''}(\gamma-\gamma_0) \quad \text{as } M\to\infty. \tag{2.45}$$

The critical point defined by $L=L_L$ represents a limit point for the periodic solutions. As $|L-L_L|\to 0$, the periodic solutions become unbounded because $M\to\infty$ (see [1, Figs. 1 and 5]). We have found that the critical value of the bifurcation parameter $\bar{v}_c(\varepsilon)=v^*+\varepsilon L_L+O(\varepsilon^2)$ corresponds to the value for a so-called "French duck solution" [3], [14], [5]. The duck solution is defined as an exceptional trajectory in the (X, Y) phase plane, which follows the vicinity of the slow manifold $Y=-F'(\bar{v})X-\tfrac{1}{2}F''(\bar{v})X^2-X^3$. The technique given in [3] to obtain this solution leads to $v_c(\varepsilon)$.

As $\gamma\to\gamma_0$ and $L_L\to L_H$, we note from the leading-order solution that the branch of periodic solutions becomes vertical. To resolve this degenerate case, we must analyze the higher-order problems. To this end, we introduce deviations from $\gamma=\gamma_0$ and $L=L_H=-\gamma/F''$ defined by

$$\delta=\varepsilon^{-1}(\gamma-\gamma_0) \quad \text{and} \quad \lambda=\varepsilon^{-1}(L+\gamma/F''). \tag{2.46}$$

The fact that $\lambda=O(\delta)$ as $\delta\to 0$ is suggested by the amplitude equation (2.36). The specific scaling $\delta=O(\varepsilon)$ is chosen so that the expansions of the two parameters $\bar{v}(\varepsilon)$ and $\gamma(\varepsilon)$ are power series in ε as the Hopf bifurcation point $\bar{v}=v_H(\varepsilon)=v^*+\varepsilon L_H+O(\varepsilon^2)$ and the critical value for a vertical bifurcation $\gamma=\gamma_c(\varepsilon)=\gamma_0+O(\varepsilon)$. The $O(\varepsilon)$ scaling is not a requirement of the perturbation method, but is more convenient when we compare the asymptotic and numerical solutions in § 4. We have verified that the result with $\delta=O(\varepsilon^{1/2})$ can be obtained from our analysis with $\delta=O(\varepsilon)$

by taking the limit δ large. Inserting (2.46) into (2.19) and using the fact that $3/F''^2 = \gamma_0/2$ gives

$$\frac{dx}{d\tau} = -y,$$

$$\frac{dy}{d\tau} = x(1+y) + \varepsilon^{1/2}\left[-\frac{\gamma_0}{2}x^2(1+y)\right] + \varepsilon\left[-\gamma_0^2 x + \frac{\gamma_0}{F''^2}x^3 - \frac{6\gamma_0}{F''^2}xy\right] \tag{2.47}$$

$$+\varepsilon^{3/2}\left[-F''\lambda y - \frac{\delta}{2}x^2 - \frac{3\gamma_0^2}{F''^2}y + \frac{3\gamma_0^2}{F''^2}x^2\right] + O(\varepsilon^2).$$

In § 3 we analyze (2.47) in detail.

3. The higher-order analysis. In this section, we analyze (2.47) in the limit $\varepsilon \to 0$. Eliminating the $O(\varepsilon)$ and $O(\varepsilon^{3/2})$ terms, we first consider the following problem for $(x, y) = (x_0, y_0)$:

$$\frac{dx_0}{d\tau} = -y_0,$$

$$\frac{dy_0}{d\tau} = x_0(1+y_0) + \varepsilon^{1/2}[-\tfrac{1}{2}\gamma_0 x_0^2(1+y_0)] \tag{3.1}$$

$$= (1+y_0)(x_0 - \varepsilon^{1/2}\tfrac{1}{2}\gamma_0 x_0^2).$$

System (3.1) is again conservative and has a one-parameter family of periodic solutions. A first integral is given by

$$N_0(\varepsilon^{1/2}) = y_0 - \ln(1+y_0) + \tfrac{1}{2}x_0^2 - \varepsilon^{1/2}\tfrac{1}{6}\gamma_0 x_0^3. \tag{3.2}$$

For each positive value of the constant $N_0(\varepsilon^{1/2})$, there exists a periodic orbit in the phase plane (x_0, y_0). The period is denoted by T_{N_0} and varies from 2π to infinity as N_0 is varied from zero to infinity. Motivated by the analysis of § 2, we expect that the analysis of (2.47) as $\varepsilon \to 0$ will be simpler if we consider a differential equation for the variable $N = N(x(\tau, \varepsilon^{1/2}))$, $y(\tau, \varepsilon^{1/2}))$ defined by

$$N(\tau, \varepsilon^{1/2}) = y - \ln(1+y) + \tfrac{1}{2}x^2 - \varepsilon^{1/2}\tfrac{1}{6}\gamma_0 x^3. \tag{3.3}$$

The choice of the function N is suggested by expression (3.2) and the assumption that $N(\tau, \varepsilon^{1/2})$ approaches $N_0(\varepsilon^{1/2})$ as $\varepsilon \to 0$. The function N characterizes the amplitude of the periodic solutions. For example, we note from the phase plane analysis of (3.1) that the maximum value of x_0 occurs at $y_0 = 0$. Setting $y = 0$ in (3.3) gives $x_M(\tau, \varepsilon^{1/2}) = (2N)^{1/2} + O(\varepsilon^{1/2})$. To obtain a differential equation for N, we differentiate (3.3) with respect to τ and use (2.47) to eliminate the time derivatives. We then obtain

$$\frac{dN}{d\tau} = R(x, y, \varepsilon^{1/2})$$

$$= \varepsilon\frac{y}{1+y}\left[-\gamma_0^2 x + \frac{1}{6}\gamma_0^2 x^3 - \gamma_0^2 xy\right] \tag{3.4}$$

$$+\varepsilon^{3/2}\frac{y}{1+y}\left[-F''\lambda y - \frac{\delta}{2}x^2 - \frac{1}{2}\gamma_0^3 y + \frac{1}{2}\gamma_0^3 x^2\right] + O(\varepsilon^2).$$

We now seek periodic solutions of (2.47) and (3.4) of the form

$$x(T, \varepsilon^{1/2}) = x_0(T, \varepsilon^{1/2}) + \varepsilon x_2(T) + \varepsilon^{3/2}x_3(T) + \cdots, \tag{3.5}$$

$$y(\tau, \varepsilon^{1/2}) = y_0(T, \varepsilon^{1/2}) + \varepsilon y_2(T) + \varepsilon^{3/2}y_3(T) + \cdots, \tag{3.6}$$

$$N(T, \varepsilon^{1/2}) = N_0(\varepsilon^{1/2}) + \varepsilon N_2(T) + \varepsilon^{3/2}N_3(T) + \cdots, \tag{3.7}$$

where T is defined by (2.28), x_0, y_0 satisfy (3.1) with time derivative d/dT, and N_0 is defined by (3.2). Note that (3.5)-(3.7) do not have $O(\varepsilon^{1/2})$ terms. Because x and y are periodic, N is periodic and dN/dT must average zero over a single period. Similarly to (2.30), this leads to the condition

$$\int_0^{T_N} R(x, y, \varepsilon^{1/2})\, dT = 0. \tag{3.8}$$

In Appendix A, we analyze this condition and show that it can be reduced to the following equation:

$$\begin{aligned}\left(-F''\lambda - \frac{1}{2}\gamma_0^3\right)\int_0^{T_M} \frac{b^2}{(1+b)}\, dT + \frac{1}{2}(\gamma_0^3 - \delta)\int_0^{T_M} \frac{ba^2}{(1+b)}\, dT \\ + \frac{5}{36}\gamma_0^3 \int_0^{T_M} \frac{ba^4}{(1+b)}\, dT + O(\varepsilon^{1/2}) = 0,\end{aligned} \tag{3.9}$$

where $a(T)$ and $b(T)$ satisfy (2.20) with time derivative d/dT and M is defined by (2.21). The coefficient of the first integral can be rewritten in terms of the deviation from the Hopf bifurcation point. Expanding (2.13) to $O(\varepsilon^2)$, successively using (2.17), (2.46), and definition (2.24), we find that the Hopf bifurcation point corresponds to

$$\lambda = \lambda_H = -\frac{1}{2F''}\gamma_0^3. \tag{3.10}$$

Using (3.10), (3.9) can then be rewritten as

$$\begin{aligned}F''(\lambda - \lambda_H)\int_0^{T_M} \frac{b^2}{(1+b)}\, dT - \frac{1}{2}(\gamma_0^3 - \delta)\int_0^{T_M} \frac{ba^2}{(1+b)}\, dT \\ - \frac{5}{36}\gamma_0^3 \int_0^{T_M} \frac{ba^4}{(1+b)}\, dT + O(\varepsilon^{1/2}) = 0.\end{aligned} \tag{3.11}$$

The solvability condition (3.11) is the amplitude equation which describes how the amplitude of the periodic solutions (parametrized by M) changes as a function of the deviation from the Hopf bifurcation point ($\lambda - \lambda_H$). We analyze this equation numerically in the next section. However, useful information can be obtained by exploring the small and large amplitude limits.

3.1. Small amplitude periodic solutions. Using (2.37) and (2.38), we evaluate the three integrals in (3.11). We find the following equation for the amplitude η:

$$2\eta^2[F''(\lambda - \lambda_H) + \tfrac{1}{2}(\gamma_0^3 - \delta)\eta^2 + \tfrac{5}{18}\gamma_0^3\eta^4](1 + O(\eta^2)) = 0. \tag{3.12}$$

Equation (3.12) admits the following solutions: the trivial solution

$$\eta = 0, \tag{3.13}$$

which corresponds to the reference steady state $X = 0$, and the nonzero solutions $\eta^2 \neq 0$, which satisfy the implicit relation

$$\lambda = \lambda_H - \eta^2 \frac{1}{2F''}(\gamma_0^3 - \delta) - \frac{5}{18F''}\gamma_0^3\eta^4. \tag{3.14}$$

This is leading approximation as $\eta \to 0$, provided that $\delta - \gamma_0^3 = 0(\eta^2)$ and $\lambda - \lambda_H = O(\eta^4)$. Equivalently, using the definition of λ_H given by (3.10), we may rewrite (3.14) in a more elegant form as

$$\lambda = \lambda_H[1 + (1 - \delta/\gamma_0^3)\eta^2 + \tfrac{5}{9}\eta^4]. \tag{3.15}$$

As $\lambda \to \infty$ and $\delta = O(\lambda) \to \infty$, keeping η fixed, and using definitions (2.46), we find that (3.14) is matching the outer solution (2.42). In addition, we have verified that the small ε limit of the amplitude equation obtained for the regular Hopf bifurcation [2] converges to the two first terms in (3.14). From (3.15), we find that the Hopf bifurcation is subcritical if

$$\delta > \gamma_0^3. \tag{3.16}$$

If condition (3.16) is satisfied, (3.15) describes two branches of periodic solutions connected by a limit point. A different method leading to an amplitude equation similar to (3.15) has been developed by Tu and Reiss [13] for small amplitude harmonic periodic solutions.

3.2. Large amplitude periodic solutions. We now consider the limit $M \to \infty$ of (3.11). Using the approximation (2.43) and evaluating the integrals, we obtain, instead of (2.44), the following expression relating M and L:

$$(2\pi)^{1/2} \exp(M+1)[F''(\lambda - \lambda_H) + \tfrac{1}{2}(\gamma_0^3 - \delta) + \tfrac{5}{12}\gamma_0^3] = 0. \tag{3.17}$$

Equivalently, (3.17) defines a value for $\lambda = \lambda_L$ that corresponds to a limiting value for the large amplitude periodic solutions:

$$\lambda \to \lambda_L = \lambda_H - \frac{1}{2F''}(\gamma_0^3 - \delta) - \frac{5}{12F''}\gamma_0^3 \quad \text{as } M \to \infty. \tag{3.18}$$

In summary, we have constructed periodic solutions of the FitzHugh–Nagumo equations (2.1) and (2.2), which are given by

$$v(\tau, \varepsilon^{1/2}) = \bar{v}(\varepsilon) + \varepsilon^{1/2}\left(-\frac{1}{F''} x_0(\tau, \varepsilon^{1/2})\right) + O(\varepsilon), \tag{3.19}$$

$$w(\tau, \varepsilon^{1/2}) = -F(v) - \frac{dv}{dt} + I, \tag{3.20}$$

where $x_0(\tau, \varepsilon^{1/2})$ and $y_0(\tau, \varepsilon^{1/2}) = -(dx_0/d\tau)$ satisfy the conservative system (3.1) with first integral given by (3.2). The amplitude of the periodic solution is determined by solving the solvability condition (3.11).

4. Numerical analysis. In this section, we study numerically the singular Hopf bifurcation to relaxation oscillations for the FitzHugh–Nagumo equations. We also compare asymptotic and numerical solutions using bifurcation diagrams. We show that our higher-order analysis predicts important features of the global bifurcation structure. All computations were performed in double precision on a VAX 11/8600 using a fourth-order Runge–Kutta method.

In Fig. 2(a), we represent the maximum value of v of the periodic solutions as a function of the deviation $I - I_H$. $I = I_H$ is defined as the lower Hopf bifurcation point. The values of the parameters corresponds to a subcritical bifurcation. Solid and dashed lines correspond to stable and unstable periodic solutions, respectively. There is a discontinuity in the bifurcation diagram that occurs at a critical value $I = I_c$. The deviation $I_c - I_H$ is given by

$$I_c - I_H \approx -5.2792516 \times 10^{-7}. \tag{4.1}$$

This discontinuity means that maximum voltage amplitudes between $v_{\max} \approx 0.2$ and $v_{\max} \approx 0.9$ are not observed. At point A, just to the right of I_c, we observe large amplitude solutions. At point B, just to the left of I_c, we observe small amplitude solutions. It is

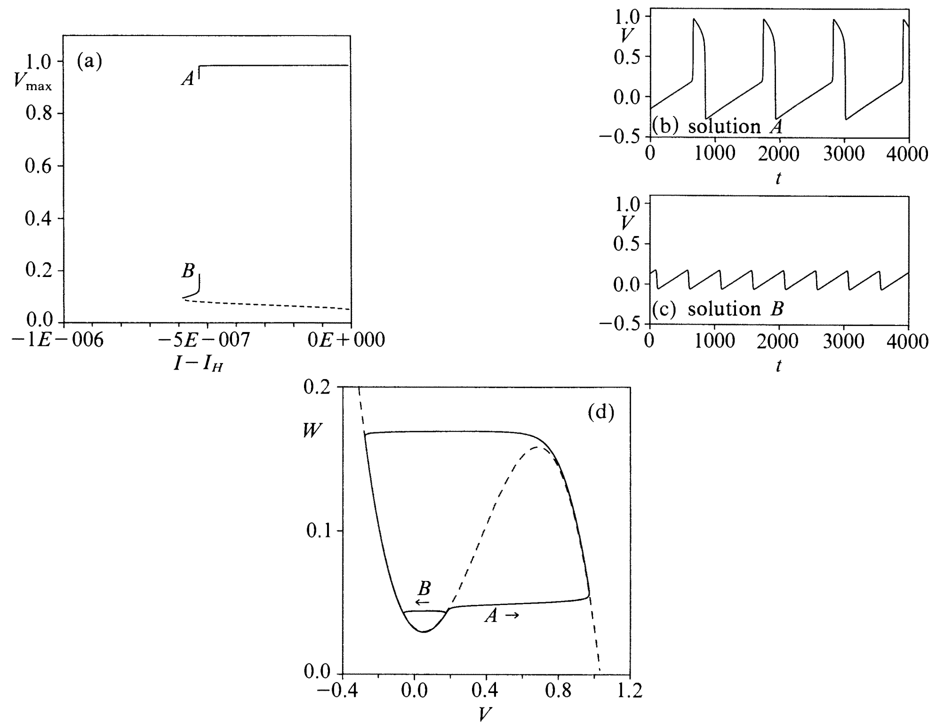

FIG. 2. *Periodic solutions change from moderate to large amplitude relaxation oscillations.* (a) *Periodic solution branches of the FitzHugh–Nagumo equations* (2.1) *and* (2.2); *stable* (*full line*) *and unstable* (*dashed line*). *Location A on the upper branch denotes a large amplitude pulsating solution* ($I > I_c$), *and B a small amplitude pulsating solution* ($I < I_c$). *There is an abrupt jump in the bifurcation curve due to the "canard" phenomenon at* $I_c = I_H - 5.2792516\ 10^{-7}$; (b) *time course of solution A*; (c) *time course of solution B*; (d) *phase plane solutions of A and B. Dashed curves denote w-v nullclines. The values of the parameters are* $a = 0.1$, $\varepsilon = 10^{-3}$, *and* $\gamma = 1.61$.

important to note that computations using continuation methods ([4]) lead to spurious results near $I = I_c$.

Details of the periodic solutions in Fig. 2(a) are given in Figs. 2(b)-2(d). Figures 2(b) and 2(c) represent the time course for the oscillations A and B, respectively. They have the familiar form of relaxation oscillations, i.e., the repetition of a slow depolarization followed by a rapid spike. Note that the small amplitude solution B is pulsating and that small amplitude harmonic oscillations are unstable because the bifurcation is subcritical. Figure 2(d) represents the two periodic orbits in the phase plane (w, v). Orbit A is large and is almost rectangular, while orbit B is a small triangle near $v = 0$. The dotted line is defined as the S-shaped nullcline

$$w = -f(v) + I. \tag{4.2}$$

As $\varepsilon \to 0$, each limit cycle approaches a discontinuous approximation given in part by the smooth curve (4.2), and in part by straight lines corresponding to constant values of w. We note that the main difference between limit-cycle A and limit-cycle B in Fig. 2(d) is in the direction of the jump transition between distinct branches of the curve (4.2). These closed orbits have been called "duck with a head" (solution A) and "duck without a head" (solution B), respectively [3].

An asymptotic analysis predicting $I_c(\varepsilon)$ an the two limit-cycles A and B in an $\exp(-1/\varepsilon)$ neighborhood of $I_c(\varepsilon)$ is possible [5]. In [5] Eckhaus analyzes a class of van der Pol equations with a parameter. Because of the equation for w, his analysis cannot be applied directly to the FitzHugh–Nagumo equations (a modified analysis is possible, but tedious, and is not described here). We conclude from our numerical and asymptotic analysis that the local analysis of the Hopf bifurcation point is able to describe accurately the bifurcation diagram with a limit point until limit-cycle B is approached.

The limit point in Fig. 2(a) (where the stability changes) occurs to the left of the critical value $I = I_c$. Figure 3 is an enlarged view of the bifurcation diagram in the vicinity of the limit point with two approximations. The solid line (denoted by 1) is the numerical result (representing the exact solution) and is the same as in Fig. 2(a). The dashed curves were computed from solvability conditions derived using different asymptotic methods. The dashed curve denoted by 2 corresponds to the singular Hopf approximation. The curve has been obtained from (3.11) by solving the integrals numerically. The dashed curve denoted by 3 corresponds to the regular Hopf approximation and is the simplest approximation of the bifurcation diagram. The curve has been obtained from (2.42). We note that the singular Hopf approximation follows the numerical solution well beyond the limit point, revealing some global features of the bifurcation diagram. In [1] we could only estimate the location of the limit point.

Appendix A. In this appendix, we determine the leading term of the solvability condition (3.8) as $\varepsilon \to 0$. After introducing (3.5) and (3.6) into the right-hand side of (3.4), this condition is given by

$$\begin{aligned}\int_0^{T_N} \frac{y_0}{1+y_0}\Bigg[&-\gamma_0^2 x_0 + \frac{1}{6}\gamma_0^2 x_0^3 - \gamma_0^2 x_0 y_0 \\ &+\varepsilon^{1/2}\left[-F''\lambda y_0 - \frac{\delta}{2}x_0^2 - \frac{1}{2}\gamma_0^3 y_0 + \frac{1}{2}\gamma_0^3 x_0^2\right] + O(\varepsilon)\Bigg]\, dT = 0,\end{aligned} \tag{A.1}$$

where $x_0(T, \varepsilon^{1/2})$ and $y_0(T, \varepsilon^{1/2})$ are periodic functions satisfying (3.1) with d/dT. We now compute the integrals of the three first terms in (A.1). They are given by (A.2)–(A.4),

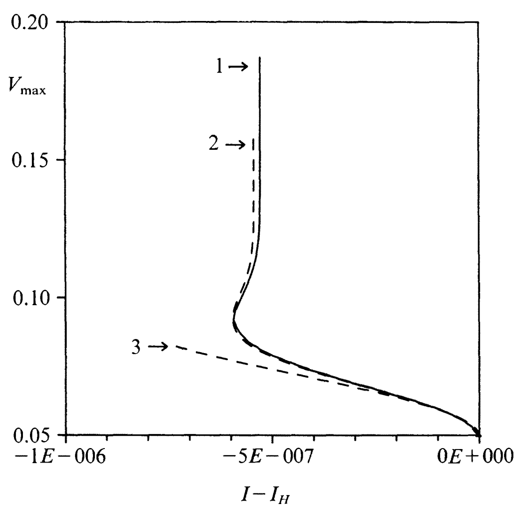

FIG. 3. *The singular Hopf bifurcation method reveals nonlocal features of the bifurcation diagram. The asymptotic solution* 2 *follows closely to the numerical solution* 1 (*solid line*) *and climbs well above the knee. In contrast, the asymptotic solution* 3 *obtained from a regular Hopf bifurcation analysis is accurate only in the vicinity of the Hopf point. Parameter values are the same as in Fig.* 2.

as follows:

$$
\begin{aligned}
\int_0^{T_N} \frac{x_0 y_0}{1+y_0}\, dT &= \oint \frac{y_0}{(1+y_0)} \frac{dy_0}{(1+y_0)(1-\varepsilon^{1/2}\gamma_0 x_0/2)} \\
&= \oint \frac{y_0}{(1+y_0)^2}\, dy_0 + \varepsilon^{1/2}\frac{1}{2}\gamma_0 \oint \frac{y_0 x_0}{(1+y_0)^2}\, dy_0 + O(\varepsilon) \\
&= \varepsilon^{1/2}\frac{1}{2}\gamma_0 \int_0^{T_N} \frac{y_0 x_0^2}{1+y_0}\, dT + O(\varepsilon),
\end{aligned} \tag{A.2}
$$

where we have used the equation for y_0 to express dT in terms of dy_0;

$$
\begin{aligned}
\int_0^{T_N} \frac{x_0 y_0^2}{1+y_0}\, dT &= \oint \frac{y_0^2}{(1+y_0)} \frac{dy_0}{(1+y_0)(1-\varepsilon^{1/2}\gamma_0 x_0/2)} \\
&= \oint \frac{y_0^2}{(1+y_0)^2}\, dy_0 + \varepsilon^{1/2}\frac{1}{2}\gamma_0 \oint \frac{y_0^2 x_0}{(1+y_0)^2}\, dy_0 + O(\varepsilon) \\
&= \varepsilon^{1/2}\frac{1}{2}\gamma_0 \int_0^{T_N} \frac{y_0^2 x_0^2}{1+y_0}\, dT + O(\varepsilon);
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
\int_0^{T_N} \frac{x_0^3 y_0}{1+y_0}\, dT &= \oint \frac{y_0 x_0^2}{(1+y_0)} \frac{dy_0}{(1+y_0)(1-\varepsilon^{1/2}\gamma_0 x_0/2)} \\
&= \oint \frac{y_0}{(1+y_0)^2}\left[2N_0 - 2y_0 + 2\ln(1+y_0) + \varepsilon^{1/2}\frac{1}{3}\gamma_0 x_0^3\right] \\
&\qquad \cdot \left[1+\varepsilon^{1/2}\frac{1}{2}\gamma_0 x_0 + O(\varepsilon)\right] dy_0 \\
&= \oint \frac{y_0}{(1+y_0)^2}[2N_0 - 2y_0 + 2\ln(1+y_0)]\, dy_0 \\
&\qquad + \varepsilon^{1/2} \oint \frac{y_0}{(1+y_0)^2}\left\{\frac{1}{3}\gamma_0 x_0^3 + \frac{1}{2}\gamma_0 x_0 \right. \\
&\qquad\qquad \left. \cdot [2N_0 - 2y_0 + 2\ln(1+y_0)]\, dy_0\right\} + O(\varepsilon) \\
&= \varepsilon^{1/2}\gamma_0 \int_0^{T_N} \frac{y_0}{(1+y_0)}\left(\frac{1}{3}x_0^4 + \frac{1}{2}x_0^4\right) dT + O(\varepsilon).
\end{aligned} \tag{A.4}
$$

Using (A.2)-(A.4), condition (A.1) can be reformulated as

$$
\begin{aligned}
\int_0^{T_N} \frac{y_0}{(1+y_0)}\Big[&-\frac{1}{2}\gamma_0^3 x_0^2 + \frac{5}{36}\gamma_0^3 x_0^4 - \frac{1}{2}\gamma_0^3 x_0^2 y_0 - y_0 F''\lambda - \frac{\delta}{2}x_0^2 - \frac{1}{2}\gamma_0^3 y_0 \\
&+ \frac{1}{2}\gamma_0^3 x_0^2 + O(\varepsilon^{1/2})\Big]\, dT = 0.
\end{aligned} \tag{A.5}
$$

Note that this condition can be further simplified if we use

$$
\begin{aligned}
\int_0^{T_N} \frac{y_0}{(1+y_0)} x_0^2 y_0\, dT &= \int_0^{T_N} y_0 x_0^2\, dT - \int_0^{T_N} \frac{y_0}{(1+y_0)} x_0^2\, dT \\
&= -\int_0^{T_N} \frac{y_0}{(1+y_0)} x_0^2\, dT.
\end{aligned} \tag{A.6}
$$

Equation (A.5) then becomes

$$\text{(A.7)}\qquad \begin{aligned}&\frac{1}{2}(\gamma_0^3-\delta)\int_0^{T_N}\frac{y_0x_0^2}{(1+y_0)}\,dT+\frac{5}{36}\gamma_0^3\int_0^{T_N}\frac{y_0x_0^4}{(1+y_0)}\,dT\\&\quad+\left(-F''\lambda-\frac{1}{2}\gamma_0^3\right)\int_0^{T_N}\frac{y_0^2}{(1+y_0)}\,dT+O(\varepsilon^{1/2})=0.\end{aligned}$$

Finally, we note from (3.1) that $x_0(T,\varepsilon^{1/2})=a(T)+O(\varepsilon^{1/2})$ and $y_0(T,\varepsilon^{1/2})=b(T)+O(\varepsilon^{1/2})$, where $a(T)$ and $b(T)$ satisfy (2.20) with d/dT. Thus (A.7) can be rewritten in terms of a and b as

$$\text{(A.8)}\qquad \begin{aligned}&\frac{1}{2}(\gamma_0^3-\delta)\int_0^{T_N}\frac{ba^2}{(1+b)}\,dT+\frac{5}{36}\gamma_0^3\int_0^{T_N}\frac{ba^4}{(1+b)}\,dT\\&\quad+\left(-F''\lambda-\frac{1}{2}\gamma_0^3\right)\int_0^{T_N}\frac{b^2}{(1+b)}\,dT+O(\varepsilon^{1/2})=0.\end{aligned}$$

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